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Simon Catterall  
*Syracuse University*

John B. Kogut  
*University of Illinois at Urbana-Champaign*

Ray L. Renken  
*University of Central Florida*

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**Numerical Study of  $c > 1$  Matter Coupled to  
Quantum Gravity**

SIMON M. CATTERALL

*Department of Applied Maths and Theoretical Physics*

*University of Cambridge*

*Silver St. Cambridge CB3 9EW*

*England*

JOHN B. KOGUT

*Loomis Laboratory of Physics*

*University of Illinois at Urbana-Champaign*

*1110 West Green Street*

*Urbana, Illinois 61801*

RAY L. RENKEN

*Department of Physics*

*University of Central Florida*

*Orlando, Florida 32816*

ABSTRACT: We present the results of a numerical simulation aimed at understanding the nature of the ‘ $c = 1$  barrier’ in two dimensional quantum gravity. We study multiple Ising models living on dynamical  $\phi^3$  graphs and analyse the behaviour of moments of the graph loop distribution. We notice a universality at work as the average properties of typical graphs from the ensemble are determined only by the central charge. We further argue that the qualitative nature of these results can be understood from considering the effect of fluctuations about a mean field solution in the Ising sector.

In recent years considerable progress has been made in understanding the nature of some simple systems incorporating two dimensional quantum gravity. Two approaches have been followed which are at first sight quite different. The first of these employs techniques borrowed from conformal field theory to calculate the spectrum of anomalous dimensions in several simple models [1-2]. In the second method a regularisation of the continuum functional integrals is made in terms of random triangulations [3-5]. Where

it has been possible to solve the models analytically the two approaches have been in complete agreement. However, in both cases all attempts to solve the models when the matter central charge  $c$  becomes greater than unity have failed. The lattice regularisation however has the advantage that it allows these ‘strongly coupled’ regions to be explored using computer simulation techniques.

In this letter we present the results of one such study aimed at examining whether the phase transition predicted for  $c > 1$  is a genuine physical phenomena or merely an artefact of our current methods of solution [6]. A single Ising model, at criticality, can be shown to possess  $c = 1/2$ , so a set of  $P$  noninteracting systems would (at least naively) correspond to a central charge  $c = P/2$ . By tuning  $P$  from one through three we can search for any sign of discontinuous behaviour associated with such a transition. The partition function we study can be written as

$$Z(\beta, P) = \sum_{G_N} e^{PF(\beta, G)}$$

$$e^{F(\beta, G)} = \sum_{\sigma_i} e^{\beta G_{ij} \sigma_i \sigma_j} \quad (1)$$

The sum over  $\phi^3$  graphs  $G_N$  is restricted to those with  $N$  nodes and fixed spherical topology. The connectivity matrix  $G_{ij}$  can be taken to be unity when the sites  $i$  and  $j$  are neighbour on the graph and zero otherwise.

We sample the space of all graphs  $G_N$  by random moves which are dual to direct lattice link flips as described in [7], whilst cluster algorithms are used to provide an efficient updating procedure for the Ising sector [8]. On a fixed graph this partition function would simply factorise into  $P$  copies of a single Ising partition function, but one expects that the use of a dynamical lattice induces an effective coupling between different Ising ‘species’. To this end we collected Monte Carlo data for the standard Ising observables such as the specific heat  $C$ , susceptibility  $\chi$ , and generalised spin-spin correlation matrix  $C^{\alpha\beta}(r)$

$$C^{\alpha\beta}(r) = \left\langle \sigma^\alpha(0) \sigma^\beta(r) \right\rangle_C \quad (2)$$

The indices  $\alpha$  and  $\beta$  refer to the species and  $r$  measures the geodesic distance on the graph between the spins. The latter quantity, in particular, could be expected to show signs of this induced coupling by the occurrence of non-vanishing off diagonal entries. However, in spite of relatively good statistics ( $5 \times 10^5$  sweeps for a range of  $\beta$  from 0.3 to 1.2) we were unable (within the statistical errors) to extract any reliable signal that would indicate behaviour inconsistent with that of a single Ising model. As a sample of our data we have listed in table 1. the effective masses extracted by simple exponential fits to the diagonal components of the correlation function both for  $P = 1$  and  $P = 3$ . Except for a possible small additive renormalisation there is no evidence for a qualitative difference between the two cases. There was no observable signal in the off diagonal components. It is possible that our lattices are too small ( $N = 1000$  and  $N = 2000$ ), but this we feel is unlikely since previous calculations for  $P = 1$  with this model (ref. [9-10]) have indicated the onset of scaling behaviour on systems of this size. We thus conclude that the magnitude of any such coupling in the effective spin action is indeed very weak.

It has been argued [6], that the coupling of  $c > 1$  matter to two dimensional gravity creates instabilities in the worldsheet metric so to establish an order parameter for such a transition we should consider the effective gravitational partition function having traced over the spin variables. Associated with this is the question of what purely gravitational observables we should concentrate on. In our discrete model the geometry is determined by the local structure of the  $\phi^3$  graph. This in turn is reflected in the size and frequency of its loops. A large loop corresponds to a highly coordinated vertex in the direct lattice and hence large scalar curvature. Thus, a non trivial quantity to focus on is the probability distribution  $Q(l)$  for the lengths of loops  $l$  in the graph. Moments of this distribution are then simply related to expectation values of moments of the scalar curvature density on

the direct lattice. Thus we are led to consider observables  $r^k(\beta, P)$  defined by

$$r^k(\beta, P) = \sum_l Q(l) (1 - l/6)^k, \quad l = 3, \dots, N \quad (3)$$

One might expect that the large  $k$  moments are more sensitive to the occurrence of large (macroscopic) loops in the lattice. Fig. 1 shows a plot of the  $k = 2$  moment describing the fluctuations in the local scalar curvature density as a function of  $\beta$  for  $P = 0$  (pure gravity),  $P = 1$ ,  $P = 2$  and  $P = 3$  using  $N = 2000$ . On the figure we also mark the result for a  $D = 1$  pure gaussian model on the same lattice. The strong coincidence of the gaussian model result with the two spin curve suggests a large universality at work, the local structure of the graph appears to be insensitive to all but the central charge of the matter and not the details of its fields or action. Clearly the addition of Ising spins enhances the fluctuations in the local curvature peaking strongly at large  $P$  in the vicinity of the Ising critical point ( $\beta_c = 0.77332\dots$ ). Furthermore the height of this peak scales linearly with  $P$  and hence the central charge. This is in qualitative agreement with the results of [11]. We have checked that this observation remains essentially the same if a different measure is adopted for the sum over graphs (e.g the conformal weight determined by adding a term  $-\frac{c}{2} \sum_i \ln(l_i)$  to the action). However there is nothing in this plot signaling any pathology as  $P$  increases through two. These observations support a recent matrix model study by Brezin et al. [12].

One possible explanation for this stems from the observation that any  $c \geq 1$  model is expected to have an infinite number of relevant operators. Any lattice representation of the theory would, in principle, have to include lattice versions of these operators, whose couplings would all have to be tuned to certain values for the system to become critical [13]. Since it is not clear how to construct these lattice operators (a generalised Jordan Wigner construction ?) and even less how to truncate the set to a manageable number whilst preserving the central charge, we chose to work with our simpler model. Since our

susceptibility data when  $P > 1$  are still consistent with the scaling expected at a continuous phase transition, any continuum theory defined in its vicinity is still a candidate for a  $c > 1$  gravity coupled theory.

For the case of pure gravity, the behaviour of  $Q(l)$  at large  $l$  is well known [3].

$$Q(l) \sim e^{-\alpha_0 l}, \quad \alpha_0 = 0.288.. \quad (4)$$

If we make the plausible assumption that this holds for  $P > 0$ , and consider the case of large  $k$ , it is easy to show that

$$\langle r^k(\beta, P) \rangle \sim \frac{k!}{6^k} \left( \frac{1}{\alpha(\beta, P)} \right)^k \quad (5)$$

Hence, if we divide the Ising moments by the corresponding pure gravity ones and consider the quantities

$$M_k(\beta, P) = \ln \frac{\langle r_P^k \rangle}{\langle r_0^k \rangle} \quad (6)$$

We expect

$$M_k(\beta, P) \sim k \ln \left( \frac{\alpha(\beta, 0)}{\alpha(\beta, P)} \right) \quad (7)$$

Thus the parameters of the asymptotic distribution can be probed by fitting  $M$  to  $k$  as both  $\beta$  and  $P$  are varied.

In fig. 2, we plot  $M_k(\beta = 0.7, P = 3)$  as a function of  $k$  for an  $N = 2000$  node graph. Clearly we see good evidence for the expected linear behaviour, a least squares fit yielding a value of  $\ln(\alpha_0/\alpha_3) = 0.18(1)$ ,  $\chi^2 = 0.1$ . Using these fits we can compare the gradients  $\ln(\alpha_0/\alpha_P)$  for  $P = 1$ ,  $P = 2$  and  $P = 3$  from the  $N = 2000$  node ensemble with  $\beta$  in the range  $0.3 - 1.2$  (figs. 3,4,5.). The  $P = 3$  and  $P = 2$  curves now show a prominent peak close to the critical coupling, indicating a rapid decrease in the associated  $\alpha_P$ , whilst the single Ising model exhibits a much weaker structure consistent with a broad plateau in the same region of coupling constant. Furthermore the peak height increases from  $0.17(2)$

to 0.18(2) for three Ising models on going from  $N = 1000$  nodes to  $N = 2000$ , and from 0.09(2) to 0.13(2) for  $P = 2$ . This is in contrast to the single spin case where the peak falls from 0.12(4) to 0.07(2) on going to the larger lattice. Whilst our errors are clearly large it is possible that our results are hinting at a qualitative difference in the thermodynamic behaviour of  $P < 1$  and  $P > 1$ . Clearly high statistics runs on larger lattices will be needed to resolve this issue unambiguously. One possible scenario would be that the exponential behaviour in the tail of  $Q(l)$  for  $P \leq 1$  gets replaced by a power law for  $P > 1$ . This would then ensure that sufficiently high moments of the scalar curvature would formally diverge signaling a breakdown of the model. However at present such ideas are merely speculation of how any possible continuum instability might manifest itself in the lattice models.

It is perhaps surprising that the effect of adding Ising spins on the intrinsic geometry is rather similar to that of gaussian scalar fields. In the latter case the effective gravitational partition function is given by

$$Z' = \sum_{G_N} \det^{-\frac{D}{2}} \left( -\Delta^G \right) \quad (8)$$

Here  $D$  counts the number of bosons (embedding dimension) and the scalar Laplacian  $\Delta_{ij}^G = G_{ij} - q\delta_{ij}$ . Using this representation it has been shown that branched polymers (in the sense of the intrinsic geometry) dominate in the  $D \rightarrow \infty$  limit.

In contrast, the critical region of the single Ising model is usually represented in terms of a Majorana fermion. On integration over this field one would have expected to find a positive power of an appropriate fermion determinant. Thus the dependence on graph  $G_N$  could be expected to be quite different from the bosonic case which is not what we see.

We can gain some insight into this situation by considering the following (standard) representation of the Ising free energy  $F(\beta, G)$  (eqn. 1) in terms of a continuous scalar field  $\phi_i$ .

$$e^F = 2^N \det^{-\frac{1}{2}} \left( \frac{\beta}{2} G \right) \int D\phi e^{-\frac{1}{2\beta} \phi_i G_{ij}^{-1} \phi_j + \sum_i \ln \cosh \phi_i} \quad (9)$$

If the dependence of  $F$  on graph  $G_N$  could be evaluated exactly we could determine which graphs dominated the effective gravitational partition function in the Ising case. Unfortunately an exact solution is not possible and we may only compute  $F(\beta, G)$  in some approximate way. The one loop approximation corresponds to expanding the field  $\phi$  about some stationary configuration and integrating over small fluctuations. The stationary configuration is determined by the usual mean field equation

$$G_{ij}^{-1} \phi_j = \beta \tanh(\phi_i) \quad (10)$$

Although the generic graph  $G_N$  is random it still possesses a constant coordination number  $q = 3$ , so it is consistent to seek a homogeneous solution to this equation

$$\phi^0 / \beta q = \tanh \phi^0 \quad (11)$$

Now expanding the action about this stationary value  $\phi^0$  to quadratic order and doing the resulting gaussian integrals leads to an expression for the free energy to one loop of the form (notice that the ill-defined  $\det(G)$  piece has cancelled)

$$\begin{aligned} F &= F_1 + F_2 \\ F_1 &= N \left( \ln \cosh \phi^0 - \frac{\beta q}{2} \tanh^2 \phi^0 \right) \\ F_2 &= -\frac{1}{2} \text{Tr} \ln \left( 1 - \frac{\beta}{\cosh^2 \phi^0} G \right) \end{aligned} \quad (12)$$

Notice that the mean field piece  $F_1$  does not depend on graph  $G$ . In the limit  $\beta \rightarrow 0$   $F_2$  and  $F_1$  vanish and we recover the pure gravity result. Conversely, as  $\beta \rightarrow \infty$ ,  $F_2$  gives no contribution whilst  $F_1$  contributes the ( $G$  independent) expected piece  $e^{-\beta N q/2}$ . For general  $\beta$  we can recast the  $F_2$  term as

$$F_2 = -\frac{1}{2} \text{Tr} \ln \left( m^2 - \Delta^G \right) \quad (13)$$

where  $\Delta^G$  is the scalar Laplacian on the graph and  $m^2$  is given by

$$m^2 = \frac{\cosh^2 \phi^0}{\beta} - q$$

Thus at  $\beta = \beta_c = 1/q$  the mass  $m$  vanishes and a massless *bosonic* field emerges.

$$e^{F_1+F_2} \sim \det^{-\frac{1}{2}} \left( -\Delta^G \right) \quad (14)$$

In this situation the number of Ising models  $P$  plays a role analogous to the dimension  $D$  in the gaussian case and branched polymers dominate for large  $P$ . However, since we expect fluctuation effects to be very important for two dimensions in determining the exact form of the effective action, the most we can hope for our gaussian approximation is that it indicates the sort of systematic trend we should expect. We feel that it is at least consistent with our numerical results.

In conclusion, we have presented results from a simulation study of noninteracting Ising models on a dynamical random lattice. This was motivated by a desire to try to derive a better understanding of the coupling of  $c \geq 1$  matter to two dimensional quantum gravity. Whilst we find no strong signal for a breakdown of the model in this region, we have perhaps pinpointed the relevant observables which might provide an unequivocal resolution of the question given a larger scale study.

We have discussed some of the problems associated with a lattice formulation of the problem, and demonstrated within the context of a loop expansion an interpretation of at least some of the systematic trends we see in our Monte Carlo data.

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## FIGURE CAPTIONS

[1]  $k = 2$  moment of the loop distribution for  $N = 2000$  versus  $\beta$ .  $P = 1$  by  $\times$ ,  $P = 2$  by  $\diamond$ , and  $P = 3$  by  $\circ$ . Solid lines indicate pure gravity ( $P = 0$ ) result together with the  $D = 1$  gaussian model result.

[2]  $M_k(\beta = 0.7, P = 3)$  versus  $k$ ,  $N = 2000$ .

[3]  $\ln\left(\frac{\alpha_0}{\alpha_1}\right)$  versus  $\beta$ ,  $N = 2000$ .

[4]  $\ln\left(\frac{\alpha_0}{\alpha_2}\right)$  versus  $\beta$ ,  $N = 2000$ .

[5]  $\ln\left(\frac{\alpha_0}{\alpha_3}\right)$  versus  $\beta$ ,  $N = 2000$ .

$\beta$	$m_{P=1}$	$m_{P=3}$
0.50	0.520(4)	0.530(1)
0.60	0.334(2)	0.338(1)
0.65	0.245(2)	0.247(1)
0.70	0.151(1)	0.162(1)
0.75	0.069(1)	0.089(1)
0.80	0.029(1)	0.035(1)

Table 1. Masses obtained from least square fits to simple exponentials, the fitting windows (which are the same for  $P = 1$  and  $P = 3$ ) being determined from the effective mass plots.