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On the absence of an exponential bound in four dimensional simplicial gravity

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Abstract

We have studied a model which has been proposed as a regularisation for four dimensional quantum gravity. The partition function is constructed by performing a weighted sum over all triangulations of the four sphere. Using numerical simulation we find that the number of such triangulations containing V simplices grows faster than exponentially with V. This property ensures that the model has no thermodynamic limit.

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Introduction

In the last few years there has been considerable interest generated in a model for quantum gravity in which the functional integral over metrics (ill-defined in the continuum) is replaced by a discrete sum over random triangulations. The initial proposal [1, 2, 3] arose as a natural generalisation of random surface theories in two dimensions. The results of these numerical studies were encouraging and were confirmed by other groups [4, 5]. The most exciting possibility was the observation of a possible phase transition for a critical value of the bare Newton constant. The hope was that a nonperturbative quantum theory for gravity could be recovered in the vicinity of this new fixed point. These observations were rendered more quantitative by the recent work [6] in which a serious finite size scaling study was performed.

The model is defined from the partition function.

$$Z = \sum_{T(S^4)} e^{-\kappa_4 N_4 + \kappa_0 N_0} \tag{1}$$

The sum is restricted to run over all simplicial manifolds (triangulations) with the topology of S^4. The first term in the action N_4 is just the number of four simplices in the triangulation T and this allows us to identify the corresponding coupling κ_4 as a bare cosmological constant. The second term depends only on the number of vertices in the triangulation N_0 and plays the role of the integrated Ricci scalar – the coupling κ_0 is then essentially the inverse bare Newton constant.

This correspondence is clear classically from the usual Regge expression for the curvature associated to any triangle r_{ijk} with the extra constraint that the four simplices are all considered equilateral

$$r_{ijk} = 2\pi - \cos^{-1}\left(\frac{1}{4}\right) n_4^{ijk} \tag{2}$$

Notice that if the volume is bounded, the number of four simplices shared by a given triangle n_4^{ijk} is necessarily also bounded. This automatically ensures that the model is well defined at finite volume – it is a dynamical question as to whether the problems associated to the unboundedness of the continuum action return on taking the large volume limit.

As we have remarked, the analogous model in two dimensions has been studied extensively, see, for example, the review [7]. It seems clear that at least for central charges less than unity, the sum over triangulated graphs correctly mimics the continuum functional integrals over the metric including the conformal anomaly. In four dimensions it is not at all clear that a simple generalisation, such as the one described above, is sufficient to explore the space of metrics. However, it constitutes a simple ansatz which may be studied using numerical simulation.

We may rewrite eqn.1 in the form

$$Z = \sum_{N_4} e^{-\kappa_4 N_4} \Omega\left(N_4, \kappa_0\right) \tag{3}$$

The partial sum $\Omega\left(N_4, \kappa_0\right)$ counts the number of triangulations (weighted by the Ricci term) with volume N_4. The results we discuss here are concerned with the volume dependence of this entropy function $\Omega\left(N_4, \kappa_0\right)$. It is helpful at this point to recall the behaviour of the equivalent two dimensional model.

Two dimensional gravity regulated using dynamical triangulations has a partition analogous to eqn. 1. The number of simplices is now just N_2 with corresponding cosmological constant κ_2. The coupling κ_0 plays no role in two dimensions as the number of vertices N_0 is strictly proportional to the number of two simplices $N_0 = \frac{1}{2}N_2 + \chi$ if the Euler character χ is kept constant (for example S^2). If we sum over all two dimensional triangulations with fixed volume we arrive at a quantity $\omega(N_2)$ analogous to $\Omega(N_4, 0)$ for the four dimensional theory.

There are rigorous proofs [8, 9] that this quantity is exponentially bounded.

$$\omega(N_2) \sim e^{\kappa_2^c N_2} \tag{4}$$

This property is *crucial* for the very existence of the partition function. It implies that for a sufficiently large bare cosmological constant $\kappa_2 > \kappa_2^c$ the partition function will be finite. The thermodynamic limit $N_2 \to \infty$ is then obtained by tuning κ_2 towards this critical value κ_2^c. The mean volume $\langle N_2 \rangle$ then behaves as $\langle N_2 \rangle \sim (\kappa_2 - \kappa_2^c)^{-1}$. If the number of triangulations were to increase faster than exponentially, it would be impossible to tune the bare cosmological constant to approach the large volume limit in a regular fashion – the partition function would be dominated by infinite volume triangulations independent of the bare lattice parameters. Constructing a continuum limit would then be impossible.

Thus, it is absolutely essential for the very existence of these higher dimensional models that there be such a bound. Unfortunately, there are no analytic proofs available for dimension greater than two. If the topology is not fixed it can be shown that the number of triangulations increases factorially with volume even in two dimensions [10]. The situation is made worse by the lack of any topological classification of three and four dimensional manifolds.

Faced with this we have used numerical simulation to estimate the volume of the triangulation space. Whilst the previous studies [1, 2, 4, 5] have claimed evidence for an exponential bound we believe the issue is of such paramount importance that a very detailed study is required. Indeed, the results we shall present favour a very different scenario.

Method

For an entropy function that behaves exponentially with volume we have argued that it is possible to choose the coupling κ_4 to fix the mean volume $\langle N_4 \rangle$. In practice this is a difficult fine tuning problem. Even under the assumption of an exponential bound, the entropy $\Omega(N_4, \kappa_0)$ is of the form

$$\Omega(N_4, \kappa_0) \sim N_4^{a(\kappa_0)} e^{\kappa_4^c(\kappa_0) N_4} \tag{5}$$

We have included the leading power law correction parametrised by $a(\kappa_0)$. In practice the power a is negative, so that the partition function is dominated by small or large volumes depending on the sign of $\Delta\kappa_4 = \kappa_4 - \kappa_4^c(\kappa_0)$.

This problem has been tackled in a variety of ways. We have followed the approach of Migdal et al. [1] and added to the action a small correction term of the form

$$\Delta S = \gamma (N_4 - V)^2 \tag{6}$$

Replacing the sums by integrals and forgetting for the moment any power law corrections it is now simple to obtain a relation between the mean volume $\langle N_4 \rangle$ and the parameters in the action.

$$\langle N_4 \rangle = \frac{1}{2\gamma} \left(\Delta\kappa_4 + 2\gamma V \right) \tag{7}$$

Thus tuning κ_4 to yield an average volume V yields a measurement of the coupling $\kappa_4^c(\kappa_0)$. The auxiliary coupling γ merely controls the magnitude of volume fluctuations. We have set $\gamma = 0.005$. The presence of power law (and other subleading) corrections gives $\kappa_4^c(\kappa_0)$ a volume dependence $\kappa_4^c(\kappa_0) = \kappa_4^c(V, \kappa_0)$. The relation eqn. 7 may be rewritten

$$\kappa_4^c(N_4, \kappa_0) = \kappa_4 + 2\gamma \left(\langle N_4 \rangle - V \right) \tag{8}$$

In practice we iterate the above relation during the thermalisation stage of our simulation and apply it once more at the end of our run to compute our final estimate for κ_4^c.

In this picture the presence of an exponential bound would be signalled by this critical cosmological constant $\kappa_4^c(V, \kappa_0)$ having a finite limit for large volumes V. In contrast $\kappa_4^c(V, \kappa_0)$ would increase logarithmically in a model for which $\Omega(N_4, \kappa_0)$ grew factorially with volume (this just follows from the asymptotic result $(x!)^\delta \sim e^{\delta x \ln x}$).

Notice that it is sufficient to prove an exponential bound for a single value of κ_0 – the following inequality guarantees that there will then be a bound for any other $\kappa_0 > 0$.

$$\Omega(N_4, 0) \leq \Omega(N_4, \kappa_0) \leq \exp(\alpha\kappa_0 N_4)\, \Omega(N_4, 0) \tag{9}$$

We have used a Monte Carlo algorithm to sample the triangulation space of the model – the details are given in [11]. Our code is written in such a way as to make the dependence on dimension d trivial – it enters only as an input parameter to the program.

We have simulated systems from size $V = 500$ to $V = 32000$. Typical runs utilised on the order of 4×10^5 MC sweeps with one sweep corresponding to V trial updates. In addition we performed a series of runs for both the two dimensional and three dimensional models. The results of these simulations could then be contrasted with the equivalent four dimensional data and served as an important test of our code.

Results

Fig. 1 is a plot of the critical cosmological coupling $\kappa_d^c(V, \kappa_0)$ against the logarithm of the volume for the two, three and four dimensional models at $\kappa_0 = 0$ (To improve clarity we plot $\kappa_2^c - 0.5$ and $\kappa_3^c - 1.0$). Clearly, the presence of an exponential bound emerges very clearly in the two dimensional case – $\kappa_2^c(V, 0)$ is statistically consistent with a constant $\kappa_2^c(\infty) = 1.1249(6)$ for volumes $V \geq 2000$.

For three dimensions the situation is rather different. The finite volume dependence of $\kappa_3^c(V, 0)$ is large over the full range of volumes analysed. However, as the plot reveals there is no strong evidence of a logarithmic component – indeed the best fit we could make to the data corresponds to a *convergent* power law (the solid line in the figure) $\kappa_3^c = a + bV^c$. The fit yields $a = 2.01(1)$, $b = -3.2(1)$ and $c = -0.28(1)$ with a χ^2 per

degree of freedom 2.0. Thus, our data in three dimensions favours a bound. Indeed these numbers are consistent with the ones quoted in a previous study by Ambjørn and Varsted [12] who give $a = 2.06$, $b = -3.9$ and $c = -0.32$. Their fit derives from lattice sizes of $V = 14000$ and smaller with lower statistics but it is reassuring to see that we are in pretty good agreement. We are currently extending our dimension three runs to larger lattices to strengthen our confidence in the three dimensional bound.

The situation in four dimensions is radically different. Clearly, the data support the hypothesis that there is a logarithmic component to the critical volume coupling $\kappa_4^c(V,0)$. A fit of all the $d = 4$ data to a simple logarithm $\kappa_4^c = a + b \ln V$ results in a value for $b = 0.0315(3)$ with a χ^2 per degree of freedom 2.7 (solid line shown). Converging power fits simply fail to describe the data.

To test this hypothesis further we looked at the situation for non zero κ_0. Fig. 2 shows a plot of $\kappa_4^c(V,\kappa_0)$ for $\kappa_0 = 0.0$, $\kappa_0 = 0.5$ and $\kappa_0 = 1.0$. The inequality eqn.9 implies that the coefficient of this logarithm should be universal (independent of κ_0). The leading effect of a non zero value for κ_0 is simply a renormalisation of any exponential terms in $\Omega(N_4,0)$. This is confirmed by the data in fig. 2. Although the curves start out with different gradients their large volume behaviour appears to be the same.

However the plot makes it also clear that the onset of this asymptotic regime is dependent on κ_0 – as κ_0 increases the curves start off increasingly flat and the logarithm only manifests itself for large volumes.

We found that very long runs were required to thermalise the four dimensional lattices. The initial configurations were created by employing only the node insertion move which effectively generates lattices corresponding to large values of κ_0. For the largest volumes we employed, $V = 32000$, we found that subsequent relaxation times were of the order of 10^5 sweeps. This difficulty of reaching true equilibrium was the main factor in determining the largest volumes we could reach. It is perhaps a practical demonstration of the results reported in [13] in which the algorithmic unrecognisability of four manifolds is shown to lead to a lack of a reasonable bound on the number of local moves needed to pass from one configuration to another.

Thus our four dimensional data would indicate that the entropy $\Omega(V,0)$ has a leading behaviour

$$\Omega(V,0) \sim (V!)^{\delta} \tag{10}$$

If we fit the $\kappa_0 = 0.0$, 0.5, 1.0 data for the three largest volumes by straight lines we find consistent estimates for the exponent δ. These are $\delta = 0.027(1)$, $0.026(1)$, $0.025(2)$ respectively. We would then assign our best estimate for δ as $\delta = 0.026(5)$.

Outlook

In summary, we have presented results which are consistent with a leading factorial behaviour for the entropy of triangulations of the four sphere $\Omega(V,0)$. Specifically, the number of triangulations of S^4 grows like

$$\Omega(V,0) \sim \exp(aV)\,(V!)^{\delta} \tag{11}$$

Furthermore, we estimate the exponent $\delta = 0.026(5)$. This rapid growth renders it impossible to take the thermodynamic (large volume) limit – the partition function for any κ_0 is dominated by large volumes. This in turn implies there is no continuum limit for the model.

We have argued that the presence of large finite volume effects can obscure this behaviour for large values of the inverse bare Newton constant (κ_0) on lattices that are computationally accessible. It is tempting to speculate that the rather rapid shift of the pseudo critical node coupling reported in [6] is further evidence for the lack of a well-defined continuum limit. The data presented in [6] is not inconsistent with a scenario in which this pseudo-critical coupling diverges as the mean volume approaches infinity, leaving the system in an extremely crumpled, degenerate phase.

It is important to notice also that the term added to help fine tune the cosmological constant κ_4 is now playing a crucial role in defining the partition function. There is now no reason to believe that different methods of doing this are equivalent.

Clearly an extension of this work (with perhaps a more refined method for computing $\kappa_4^c(V, \kappa_0)$) to larger volumes and node couplings would help to confirm these conclusions.

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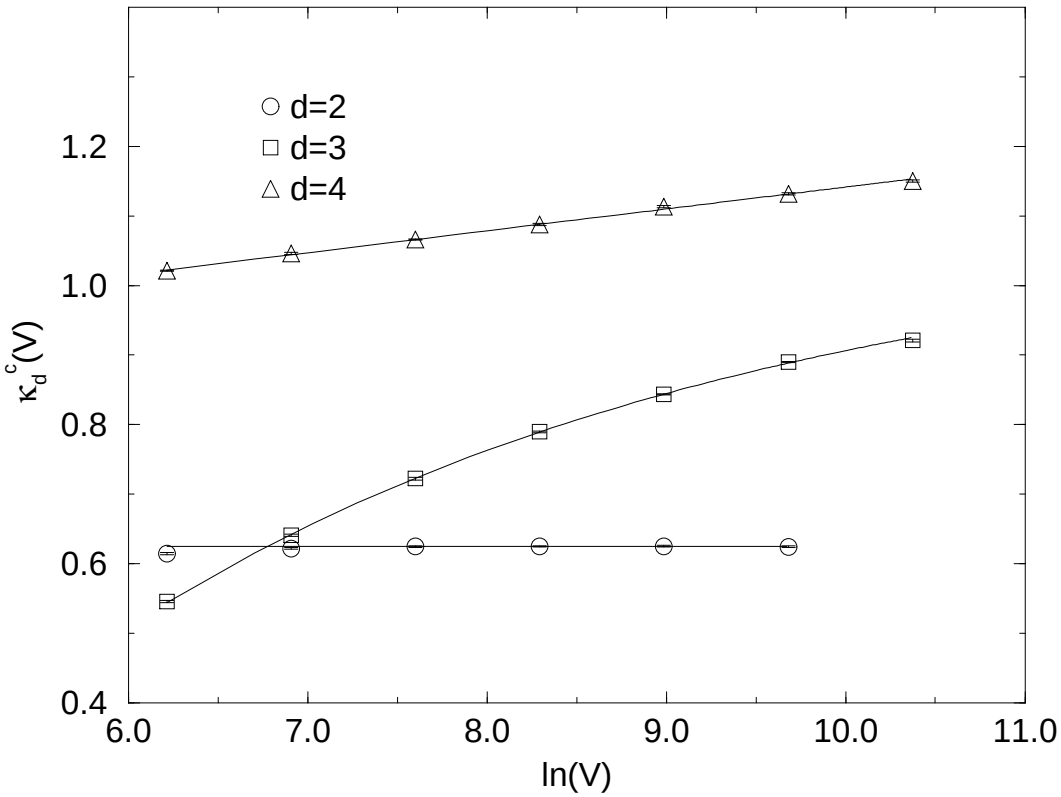

Figure 1: Critical cosmological constant $\kappa_0 = 0$

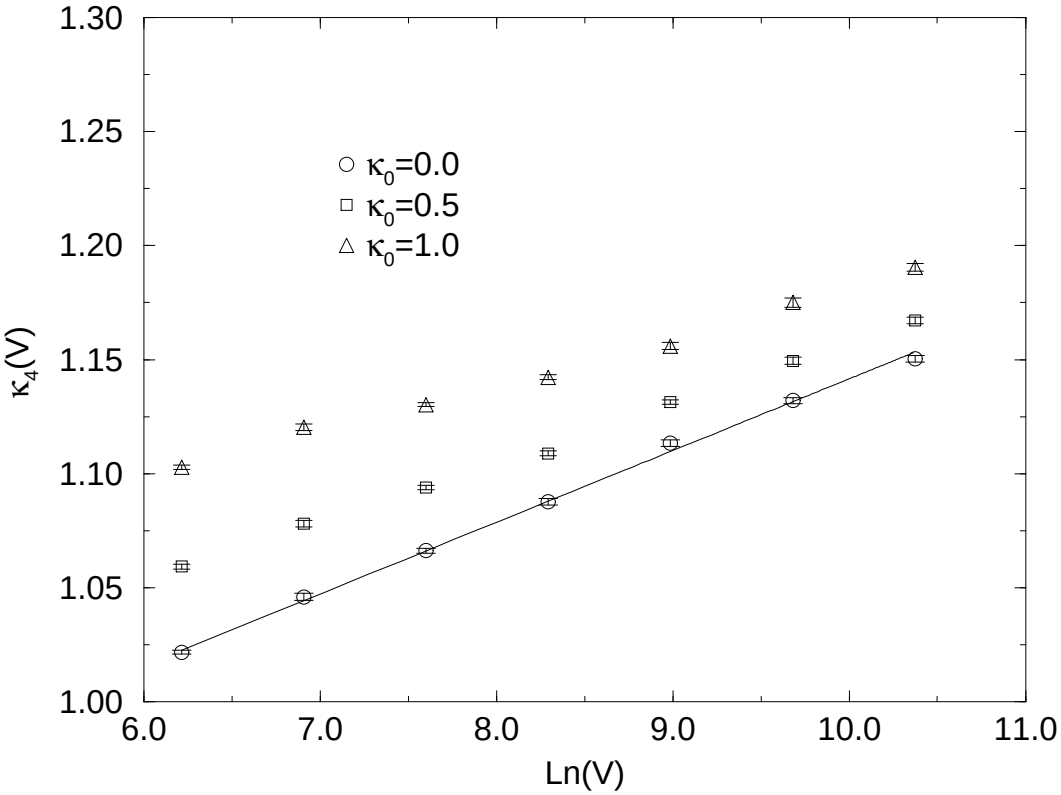

Figure 2: $\kappa_4^c(V)$ $d = 4$