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# Topological Gravity on the Lattice

Simon Catterall  
*Syracuse University*

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# Topological gravity on the lattice

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**Simon Catterall**

*Department of Physics, Syracuse University, Syracuse, NY13244, USA*

*E-mail: smc@phy.syr.edu*

ABSTRACT: In this paper we show that a particular twist of  $\mathcal{N} = 4$  super Yang-Mills in three dimensions with gauge group  $SU(2)$  possesses a set of classical vacua corresponding to the space of flat connections of the *complexified* gauge group  $SL(2, C)$ . The theory also contains a set of topological observables corresponding to Wilson loops wrapping non-trivial cycles of the base manifold. This moduli space and set of topological observables is shared with the Chern Simons formulation of three dimensional gravity and we hence conjecture that the Yang-Mills theory gives an equivalent description of the gravitational theory. Unlike the Chern Simons formulation the twisted Yang-Mills theory possesses a supersymmetric and gauge invariant lattice construction which then provides a possible non-perturbative definition of three dimensional gravity.

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## 1. Introduction

The construction of topological field theories from twisted versions of supersymmetric theories has a long history going back to Donaldson-Witten theory and its realization in terms of a twist of  $\mathcal{N} = 2$  super Yang-Mills [1]. More recently these twisted theories have received a lot of attention as the starting point for construction of lattice theories with exact supersymmetry [2]. While these theories are formulated using an arbitrary background metric they typically contain a set of topological operators whose expectation values are independent of this metric and correspond to integrals over the moduli space of the theory.

In this paper we show that a particular twist of  $\mathcal{N} = 4$  super Yang-Mills in three dimensions with gauge group  $SU(2)$  possesses a moduli space corresponding to the space of flat connections of the *complexified* gauge group  $SL(2, C)$ . This moduli space is shared with the Chern-Simons formulation of three dimensional Euclidean gravity [3, 4]. Furthermore, as for the Chern Simons theory, the twisted theory contains a natural set of topological observables, corresponding to Wilson loops of the complexified gauge field, whose expectation values are metric independent.

Unlike the Chern Simons action, the twisted theory has an action that is bounded from below and possesses only a compact  $SU(2)$  gauge symmetry. The path integral defining the quantum theory is thus well defined. Indeed a lattice theory may be constructed which targets this twisted theory in the naive continuum limit while preserving both gauge invariance and the scalar BRST supercharge that arises after twisting. This lattice theory was first constructed using orbifold methods in [5] and can be used to provide a non-perturbative definition of the supersymmetric Yang-Mills theory in flat space. Furthermore, as for the continuum theory, the presence of an exact BRST symmetry means that expectation values of topological observables can be shown to be independent of the coupling constant and the lattice spacing.

In this paper we point out that there is strong evidence to indicate that this lattice theory may also serve as a non-perturbative regulator for the Chern Simons formulation of three dimensional gravity.

The outline of this paper is as follows; first we describe the continuum twisted theory with gauge group  $SU(N)$  formulated on an arbitrary background geometry. We then construct a lattice model which targets the flat space theory in the naive continuum limit and which is both gauge invariant and invariant under a twisted scalar supercharge. We then review the usual construction of three dimensional gravity as a Chern-Simons theory. We show that the moduli space of this Chern-Simons theory is precisely the same as that of the twisted Yang-Mills model with gauge group  $SU(2)$ . Furthermore, both theories possess a set of topological operators corresponding to Wilson loops of the complexified theory which wrap the boundaries, whose vacuum expectation values are independent of any background metric. Indeed, we argue that they may be computed exactly in the associated lattice theory.

## 2. Twisted $\mathcal{N} = 4$ gauge theory in three dimensions

The twist of  $\mathcal{N} = 4$  super Yang-Mills that we are interested in can be most succinctly written in the form

$$S = \beta(S_1 + S_2) \tag{2.1}$$

where

$$\begin{aligned} S_1 &= \mathcal{Q} \int d^3x \sqrt{h} \left( \chi^{\mu\nu} \mathcal{F}_{\mu\nu} + \eta [\overline{\mathcal{D}}^\mu, \mathcal{D}_\mu] + \frac{1}{2} \eta d \right) \\ S_2 &= \int d^3x \theta_{\mu\nu\lambda} \overline{\mathcal{D}}^\lambda \chi^{\mu\nu} \end{aligned} \tag{2.2}$$

The fermions comprise one Kähler-Dirac multiplet of p-form fields  $(\eta, \psi_\mu, \chi_{\mu\nu}, \theta_{\mu\nu\lambda})^1$  where in three dimensions  $p = 0 \dots 3$ . Notice that a single Kähler-Dirac field possesses eight single component fields as expected for a theory with  $\mathcal{N} = 4$  supersymmetry in three dimensions. The imaginary parts of the complex gauge field  $\mathcal{A}_\mu$   $\mu = 1 \dots 3$  appearing in this construction yield the three scalar fields of the conventional (untwisted) theory. The scalar nilpotent supersymmetry  $\mathcal{Q}$  acts on the twisted fields as follows

$$\begin{aligned} \mathcal{Q}\mathcal{A}_\mu &= \psi_\mu \\ \mathcal{Q}\overline{\mathcal{A}}^\mu &= 0 \\ \mathcal{Q}\psi_\mu &= 0 \\ \mathcal{Q}\chi^{\mu\nu} &= \overline{\mathcal{F}}^{\mu\nu} \\ \mathcal{Q}\eta &= d \\ \mathcal{Q}d &= 0 \\ \mathcal{Q}\theta_{\mu\nu\lambda} &= 0 \end{aligned} \tag{2.3}$$

---

<sup>1</sup>It is common in the continuum literature to replace the 2 and 3 form fields in these expressions by their Hodge duals; a second vector  $\hat{\psi}_\mu$  and scalar  $\hat{\eta}$  see, for example [6]

where the background metric  $h_{\mu\nu}$  is used to raise and lower indices in the usual manner  $\chi^{\mu\nu} = h^{\mu\alpha}h^{\nu\beta}\chi_{\alpha\beta}$  and is a  $\mathcal{Q}$ -singlet. The topological character of the theory follows from the  $\mathcal{Q}$ -exact structure of  $S_1$  together with the explicit metric independence of  $S_2$  which can be recognized as the integral of a 3-form since  $\theta_{\mu\nu\lambda} = \theta\epsilon_{\mu\nu\lambda}$ . Furthermore,  $S_1$  is clearly  $\mathcal{Q}$ -invariant by virtue of the nilpotent property of  $\mathcal{Q}$  while the supersymmetric invariance of the  $\mathcal{Q}$ -closed term relies on the Bianchi identity

$$\epsilon^{\lambda\mu\nu}\mathcal{D}_\lambda\mathcal{F}_{\mu\nu} = 0 \quad (2.4)$$

The complex covariant derivatives appearing in these expressions are defined by

$$\begin{aligned} \mathcal{D}_\mu &= \partial_\mu + \mathcal{A}_\mu = \partial_\mu + A_\mu + iB_\mu \\ \bar{\mathcal{D}}^\mu &= \partial^\mu + \bar{\mathcal{A}}^\mu = \partial^\mu + A^\mu - iB^\mu \end{aligned} \quad (2.5)$$

while all fields take values in the adjoint representation of  $SU(N)^2$ . It should be noted that despite the appearance of a complexified connection and field strength the theory possesses only the usual  $SU(N)$  gauge invariance corresponding to the real part of the gauge field. The structure of this twisted theory is similar to that of the Marcus twist of  $\mathcal{N} = 4$  super Yang-Mills in four dimensions [7, 8, 9] which plays an important role in the Geometric-Langlands program [10].

Doing the  $\mathcal{Q}$ -variation and integrating out the field  $d$  yields

$$S = \frac{1}{g^2} \int d^3x \sqrt{h} (L_1 + L_2) \quad (2.6)$$

where

$$\begin{aligned} L_1 &= \text{Tr} \left( -\bar{\mathcal{F}}^{\mu\nu} \mathcal{F}_{\mu\nu} + \frac{1}{2} [\bar{\mathcal{D}}^\mu, \mathcal{D}_\mu]^2 \right) \\ L_2 &= \text{Tr} \left( -\chi^{\mu\nu} \mathcal{D}_{[\mu} \psi_{\nu]} - \psi_\mu \bar{\mathcal{D}}^\mu \eta - \theta_{\mu\nu\lambda} \bar{\mathcal{D}}^{[\lambda} \chi^{\mu\nu]} \right) \end{aligned} \quad (2.7)$$

The terms appearing in  $L_1$  can then be written

$$\begin{aligned} \bar{\mathcal{F}}^{\mu\nu} \mathcal{F}_{\mu\nu} &= (F_{\mu\nu} - [B_\mu, B_\nu])(F^{\mu\nu} - [B^\mu, B^\nu]) + (D_{[\mu} B_{\nu]})(D^{[\mu} B^{\nu]}) \\ \frac{1}{2} [\bar{\mathcal{D}}^\mu, \mathcal{D}_\mu]^2 &= -2(D^\mu B_\mu)^2 \end{aligned} \quad (2.8)$$

where  $F_{\mu\nu}$  and  $D_\mu$  denote the usual field strength and covariant derivative depending on the real part of the connection  $A_\mu$ . The classical vacua of this theory correspond to solutions of the equations

$$\begin{aligned} F_{\mu\nu} - [B_\mu, B_\nu] &= 0 \\ D_{[\mu} B_{\nu]} &= 0 \\ D^\mu B_\mu &= 0 \end{aligned} \quad (2.9)$$

---

<sup>2</sup>The generators are taken to be *anti-hermitian* matrices satisfying  $\text{Tr}(T^a T^b) = -\delta^{ab}$

The same moduli space arises in the study of the Marcus twist of four dimensional  $\mathcal{N} = 4$  Yang-Mills where it is argued to correspond to the space of flat *complexified* connections modulo *complex* gauge transformations. A simple way to understand this is to recognize that the additional term  $D^\mu B_\mu = 0$  appearing in the vacuum equations 2.9 resembles a partial gauge fixing of a theory with a complexified gauge invariance and associated gauge fields  $A_\mu$  and  $B_\mu$  down to a theory possessing just the usual  $SU(N)$  gauge invariance and gauge field  $A_\mu$ .

More specifically, Marcus showed in [7] that the solutions of eqns. 2.9 modulo  $SU(N)$  gauge transformations are in one to one correspondence with the space of flat complexified  $SU(N)$  connections modulo complex gauge transformations. These arguments should hold in the three dimensional case too.

The topological character of the theory then guarantees that any  $\mathcal{Q}$ -invariant observable such as the partition function can be evaluated *exactly* by considering only Gaussian fluctuations about such vacuum configurations. Furthermore, it is easy to see from eqn. 2.2 that the energy momentum tensor of this theory is  $\mathcal{Q}$ -exact rendering the expectation values of such topological observables independent of smooth deformations of the background metric  $h_{\mu\nu}(x)$ .

Returning to the bosonic action and integrating by parts we find that the term linear in  $F_{\mu\nu}$  cancels and the contribution of  $L_1$  reads

$$L_1 = \text{Tr} (F_{\mu\nu}F^{\mu\nu} + 2B^\mu D^\nu D_\nu B_\mu - [B_\mu, B_\nu][B^\mu, B^\nu] - 2R^{\mu\nu} B_\mu B_\nu) \quad (2.10)$$

where  $R_{\mu\nu}$  is the background Ricci tensor. Thus the bosonic theory possesses the usual Yang-Mills field strength for a real  $SU(N)$  gauge field together with three vectors arising from the imaginary part of the connection. In flat space the theory is fully equivalent to the usual  $\mathcal{N} = 4$  theory as this twisting operation can be regarded merely as an exotic change of variables – the vectors can be regarded as scalar fields since their kinetic term is simply the usual scalar Laplacian and the Kähler-Dirac action is equivalent to the Dirac action for four degenerate Majorana spinors.

Notice that the fact that the complexified connection  $\bar{\mathcal{A}}_\mu$  is a  $\mathcal{Q}$ -singlet allows us to trivially construct a class of topological observables corresponding to the trace of an associated Wilson loop around a non-trivial cycle  $\gamma$  in the background space

$$O(\gamma) = \mathcal{P}e^{\int_\gamma \bar{\mathcal{A}}^\mu \cdot dx_\mu} \quad (2.11)$$

As we will argue later the specialization of this model to the case of  $SU(2)$  is particularly interesting as the resulting moduli space coincides with that arising in three dimensional Chern Simons gravity.

### 3. Lattice construction

The twisted theory described in the previous section may be discretized using the techniques developed in [9, 11, 12]. The resultant lattice theories have the merit of preserving both

gauge invariance and the scalar component of the twisted supersymmetry. Here we show how to derive this lattice theory by direct discretization of the continuum twisted theory. We will start by assuming that the continuum theory is formulated in flat (Euclidean) space with metric  $h_{\mu\nu} = \delta_{\mu\nu}$ . This guarantees that the twisted theory is completely equivalent to the usual Yang-Mills theory and hence that the resulting lattice models target the usual continuum theory in the continuum limit.

Furthermore, in the case of topological observables the choice of metric is unimportant and hence the lattice theory we construct will yield expectation values for topological operators which depend only on the topology of the lattice and not on the coupling, lattice spacing or the fact that we started by discretization of a theory in a flat background.

The transition to the lattice from the continuum theory requires a number of steps. The first, and most important, is to replace the continuum complex gauge field  $\mathcal{A}_\mu(x)$  at every point by an appropriate complexified Wilson link  $\mathcal{U}_\mu(\mathbf{x}) = e^{\mathcal{A}_\mu(\mathbf{x})}$ ,  $\mu = 1 \dots 3$ . These lattice fields are taken to be associated with unit length vectors in the coordinate directions  $\boldsymbol{\mu}$  in an abstract three dimensional hypercubic lattice. By supersymmetry the fermion fields  $\psi_\mu(\mathbf{x})$ ,  $\mu = 1 \dots 3$  lie on the same oriented link as their bosonic superpartners running from  $\mathbf{x} \rightarrow \mathbf{x} + \boldsymbol{\mu}$ . In contrast the scalar fermion  $\eta(\mathbf{x})$  is associated with the site  $\mathbf{x}$  of the lattice and the tensor fermions  $\chi^{\mu\nu}(\mathbf{x})$ ,  $\mu < \nu = 1 \dots 3$  with a set of diagonal face links running from  $\mathbf{x} + \boldsymbol{\mu} + \boldsymbol{\nu} \rightarrow \mathbf{x}$ . The final 3 form field  $\theta_{\mu\nu\lambda}(\mathbf{x})$  is then naturally placed on the body diagonal running from  $\mathbf{x} \rightarrow \mathbf{x} + \boldsymbol{\mu} + \boldsymbol{\nu} + \boldsymbol{\lambda}$ . The construction then posits that all link fields transform as bifundamental fields under gauge transformations

$$\begin{aligned}
\eta(\mathbf{x}) &\rightarrow G(\mathbf{x})\eta(\mathbf{x})G^\dagger(\mathbf{x}) \\
\psi_\mu(\mathbf{x}) &\rightarrow G(\mathbf{x})\psi_\mu(\mathbf{x})G(\mathbf{x} + \boldsymbol{\mu}) \\
\chi^{\mu\nu}(\mathbf{x}) &\rightarrow G(\mathbf{x} + \boldsymbol{\mu} + \boldsymbol{\nu})\chi^{\mu\nu}(\mathbf{x})G^\dagger(\mathbf{x}) \\
\mathcal{U}_\mu(\mathbf{x}) &\rightarrow G(\mathbf{x})\mathcal{U}_\mu(\mathbf{x})G^\dagger(\mathbf{x} + \boldsymbol{\mu}) \\
\bar{\mathcal{U}}^\mu(\mathbf{x}) &\rightarrow G(\mathbf{x} + \boldsymbol{\mu})\bar{\mathcal{U}}^\mu(\mathbf{x})G^\dagger(\mathbf{x})
\end{aligned} \tag{3.1}$$

Notice that we can keep track of the orientation of the lattice field by following its continuum index structure – upper index fields are placed on negatively orientated links, lower index fields live on positively oriented links.

The action of the scalar supersymmetry on these fields is given by the continuum expression in eqn. 2.3 with the one modification that the continuum field  $\mathcal{A}_\mu(x)$  is replaced with the Wilson link  $\mathcal{U}_\mu(x)$  and the lattice field strength being defined as  $\mathcal{F}_{\mu\nu} = D_\mu^{(+)}U_\nu$ . The supersymmetric and gauge invariant lattice action which corresponds to eqn. 2.7 then takes a very similar form to its continuum counterpart

$$\begin{aligned}
S_1 &= \mathcal{Q} \sum_{\mathbf{x}} \left( \chi^{\mu\nu} \mathcal{F}_{\mu\nu} + \eta \left[ \bar{\mathcal{D}}^{(-)\mu} \mathcal{U}_\mu \right] + \frac{1}{2} \eta d \right) \\
S_2 &= \sum_{\mathbf{x}} \theta_{\mu\nu\lambda} \bar{\mathcal{D}}^{(+)\lambda} \chi^{\mu\nu}
\end{aligned} \tag{3.2}$$

The covariant difference operators appearing in these expressions are defined by

$$\begin{aligned}\mathcal{D}_\mu^{(+)} f_\nu(\mathbf{x}) &= \mathcal{U}_\mu(\mathbf{x}) f_\nu(\mathbf{x} + \boldsymbol{\mu}) - f_\nu(\mathbf{x}) \mathcal{U}_\mu(\mathbf{x} + \boldsymbol{\nu}) \\ \overline{\mathcal{D}}^{(-)\mu} f_\mu(\mathbf{x}) &= f_\mu(\mathbf{x}) \overline{\mathcal{U}}^\mu(\mathbf{x}) - \overline{\mathcal{U}}^\mu(\mathbf{x} - \boldsymbol{\mu}) f_\mu(\mathbf{x} - \boldsymbol{\mu})\end{aligned}\tag{3.3}$$

These expressions are determined by the twin requirements that they reduce to the corresponding continuum results for the adjoint covariant derivative in the naive continuum limit  $\mathcal{U}_\mu \rightarrow 1 + \mathcal{A}_\mu + \dots$  and that they transform under gauge transformations like the corresponding lattice link field carrying the same indices. This allows the terms in the action to correspond to gauge invariant closed loops on the lattice. Similarly the difference operator appearing in  $S_2$  takes the form

$$\overline{\mathcal{D}}^{(+)\lambda} \chi^{\mu\nu}(\mathbf{x}) = \chi^{\mu\nu}(\mathbf{x} + \boldsymbol{\lambda}) \overline{\mathcal{U}}^\lambda(\mathbf{x}) - \overline{\mathcal{U}}^\lambda(\mathbf{x} + \boldsymbol{\mu} + \boldsymbol{\nu}) \chi^{\mu\nu}(\mathbf{x})\tag{3.4}$$

This definition allows the lattice term corresponding to  $S_2$  to be both gauge invariant and supersymmetric – the latter property holding because of the remarkable property that the lattice field strength satisfies an exact Bianchi identity as for the continuum [9]. The action can also be shown to be free of fermion doubling problems – see the discussion in [9].

As in the continuum, the presence of an exact  $\mathcal{Q}$ -symmetry allows the definition of a class of supersymmetric Wilson loop corresponding to the trace of the product of  $\overline{\mathcal{U}}_\mu$  links around a closed loop in the lattice.

$$O = \prod_{t=1}^T \overline{\mathcal{U}}^t(\mathbf{x})\tag{3.5}$$

The vacuum expectation value of these operators can be computed exactly by restriction to the moduli space of theory and can probe only topological features of the background space.

#### 4. Chern-Simons formulation of three dimensional gravity

The twisted model we have discussed appears on the face of it to have little connection to gravity. However it has been known for a long time that three dimensional gravity can be reformulated in the language of gauge theory [3, 4]. For a review of three dimensional gravity see [13]. The construction employs a Chern-Simons action and in Euclidean space the local symmetry corresponds to the group  $SO(3,1) \sim SL(2, C)$  – the complexified  $SU(2)$  group. Furthermore the resulting theory is topological and determined by the moduli space of flat  $SL(2, C)$  connections hinting at a close connection to the twisted Yang-Mills theory described above. We now summarize this theory which will allow us to re-interpret the fields in the two color Yang-Mills theory described earlier as geometrical objects in a gravitational theory.

Consider the following three dimensional Chern-Simons action

$$\int d^3x \epsilon_{\mu\nu\lambda} \hat{\text{Tr}} \left( A_\mu F_{\nu\lambda} - \frac{1}{3} A_\mu [A_\nu, A_\lambda] \right)\tag{4.1}$$

Furthermore, assume that the gauge field  $A_\mu$  takes values in the adjoint representation of the group  $SO(3,1)$ . A convenient representation for the six generators of the Lie algebra of this group is then given by commutators of the (Lorentzian) Dirac matrices  $\gamma^{AB} = \frac{1}{4} [\gamma^A, \gamma^B]$  where  $(\gamma^a)^\dagger = \gamma^a$   $a = 1 \dots 3$  and  $(\gamma^4)^\dagger = -\gamma^4$ . This yields an expression for the gauge field of the form

$$A_\mu = \sum_{A < B} A_\mu^{AB} \gamma^{AB} \quad A, B = 1 \dots 4 \quad (4.2)$$

Finally, the group indices are contracted using the invariant tensor  $\epsilon_{ABCD}$  corresponding to a trace of the form

$$\hat{\text{Tr}}(X) = \text{Tr}(\gamma_5 X) \quad (4.3)$$

Notice that this way of contracting the group indices differs from the simple trace that appears in the twisted Yang-Mills theory considered in the previous section. To see explicitly that the resulting theory is just three dimensional gravity we decompose the gauge field and field strength in terms of an  $SO(3)$  subgroup

$$\begin{aligned} A_\mu &= \sum_{a < b} \omega_\mu^{ab} \gamma^{ab} + \frac{1}{l} e_\mu^a \gamma^{4a} \quad a, b = 1 \dots 3 \\ F_{\mu\nu}^{ab} &= \sum_{a < b} \left( R_{\mu\nu}^{ab} + \frac{1}{l^2} e_{[\mu}^a e_{\nu]}^b \right) \gamma^{ab} \\ F_{\mu\nu}^a &= \sum_a D_{[\mu} e_{\nu]}^a \end{aligned} \quad (4.4)$$

The covariant derivative appearing in the field strength contains just the  $SO(3)$  gauge field  $\omega_\mu$  and we have introduced an explicit length scale  $l$  into the definition of the gauge fields  $e_\mu$ . After substituting into the Chern-Simons action one recognizes that it corresponds to three dimensional Einstein-Hilbert gravity including a cosmological constant and written in the first order tetrad-Palatini formalism [3].

$$S_{\text{EH}} = \frac{1}{l} \int \epsilon^{\mu\nu\lambda} \epsilon_{abc} \left( e_\mu^a R_{\nu\lambda}^{bc} - \frac{1}{3l^2} e_\mu^a e_\nu^b e_\lambda^c \right) \quad (4.5)$$

with  $\omega_\mu$  and  $e_\mu$  corresponding to the spin connection and dreibein and  $1/l^2$  playing the role of a cosmological constant. To see that this theory is classically equivalent to the usual metric theory of gravity one merely has to notice that the classical equations of the Chern-Simons theory require that the  $SO(3,1)$  field strength vanish  $F_{\mu\nu}^{AB} = 0$  which sets both the torsion  $T = D_{[\mu} e_{\nu]}$  to zero and requires a constant  $SO(3)$  curvature  $R_{\mu\nu}$  equal to  $-1/l^2$ . Such a solution corresponds (at least locally) to hyperbolic three space  $H^3 \sim SO(3,1)/SO(3)$ .

Finally one can show that the theory restricted to this space of flat connections is also invariant under diffeomorphisms [3]. This result follows from the fact that one can express a general coordinate transformation on  $A_\mu$  with parameter  $-\xi^\nu$  as a gauge transformation with parameter  $\xi^\mu A_\mu$  plus a term which vanishes on flat connections.

$$\delta A_\mu^\xi = -D_\mu(\xi^\nu A_\nu) - \xi^\nu F_{\mu\nu} \quad (4.6)$$

Furthermore, the fact that this action is explicitly independent of any background metric ensures that the theory is topological. The classical solutions correspond to the space of flat  $SO(3, 1)$  connections up to  $SO(3, 1)$  gauge transformations. Since  $SO(3, 1) \sim SL(2, C)$  this is equivalent to the moduli space appearing earlier in the twisted model. Indeed, the topological observables in that theory, corresponding to Wilson loops of the complexified gauge field  $\overline{A}_\mu$ , map into the natural topological observables of the Chern-Simons theory - Wilson loops of an  $SO(3, 1)$  connection<sup>3</sup>.

Indeed, an explicit connection between the Yang-Mills theory and the gravity theory would identify the imaginary parts of the  $SL(2, C)$  connection - the field  $B_\mu$  occurring in the Yang-Mills theory - with the matrix valued field  $e_\mu$  occurring in the tetrad-Palatini action. Notice that the field  $B_\mu$  transforms in the adjoint representation of the  $SU(2)$  gauge group which translates in the gravitational theory to the statement that the dreibein  $e_\mu^a$  transforms as a vector under local Lorentz transformations as it should.

These considerations together with the equivalence of the topological sectors of these two theories leads us to conjecture that the twisted two color Yang-Mills gives an alternative representation of the gravity theory. Furthermore, this alternative representation has some advantages - the path integral is now well defined and indeed may be given a non-perturbative definition as the appropriate limit of a gauge and supersymmetric invariant lattice model.

## 5. Discussion

In this paper we have shown how a twist of  $\mathcal{N} = 4$  super Yang-Mills theory in three dimensions with gauge group  $SU(2)$  shares both a moduli space and a set of topological observables in common with the Chern Simons formulation of three dimensional Euclidean gravity. Indeed, on this basis we conjecture that the topological sector of the twisted Yang-Mills theory is equivalent to the Chern Simons theory. This is particularly interesting in the light of the fact that this twisted theory may be discretized while maintaining the BRST symmetry of the twisted model and hence its topological properties. Indeed, the lattice model can be thought of as supplying a rigorous non-perturbative definition of the twisted Yang-Mills model. By these arguments it thus also defines a lattice theory of topological gravity. This lattice theory may be simulated using standard Monte Carlo techniques similar to those reported from initial simulations of its four dimensional cousin [14].

It has been known for quite some time that topological gravity in odd dimensions can be formulated as a Chern Simons theory. In particular such a formulation exists in five dimensions [15, 16]. It would be very interesting to see whether it is possible to construct a twisted Yang-Mills theory in five dimensions which targets the same moduli space as this Chern Simons theory.

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<sup>3</sup>This mapping between connections is manifest if the Weyl basis is used for the Dirac matrices which leads to generators proportional to  $\sigma$  and  $i\sigma$

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