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Living on a dS Brane: Effects of KK modes on Inflation

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We develop a formalism to study non-local higher-dimensional effects in braneworld scenarios from a four-dimensional effective theory point of view and check it against the well-known Garriga-Tanaka result in the appropriate limit. We then use this formalism to study the spectrum of density perturbations during inflation as seen from the lower-dimensional effective theory. In particular, we find that the gravitational potential is greatly enhanced at short wavelengths. The consequences to the curvature perturbations are nonetheless very weak and will lead to no characteristic signatures on the power spectrum.

I. INTRODUCTION

As an alternative to Planck-scale string compactifications, braneworld models have recently received an increased interest from both cosmologists and phenomenologists. One reason is because such theories generically predict consequences of quantum theories of gravity at an energy scale much lower than one would naively expect. As a consequence, this opens the possibility of probing for experimental signatures coming from IR/UV modifications to the low-energy effective theory at upcoming experiments. Moreover, this also allows for model building, which can then attempt to unravel outstanding puzzles in both particle phenomenology (e.g. hierarchy problems), and cosmology (e.g. the origin and nature of inflation and dark energy) by exploring consequences of the fundamental theory.

Within these models, the presence of extra-dimensions and the associated non-local terms are of particular interest. A common approach to treating such models is to utilize the low-energy effective theory, which is expected to be valid at scales comparable or less than the Hubble scale today. In the context of the Randall-Sundrum model [1], this approach has been used extensively and the low-energy effective theory, first developed in [2], gives rise to a scalar-tensor theory of

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gravity.

At higher energies, however, such as those occurring in the early universe, the gravitational behavior of braneworld scenarios is significantly different from any conventional scalar-tensor cosmology. Understanding the behavior of these models during, for instance, inflation can then provide an opportunity to find characteristic signatures of such higher-dimensional models. Motivated by this possibility, effective theories including characteristic high-energy corrections have been developed in [3, 4]. In these models, corrections to the effective theory were considered arising from higher dimensional Kaluza-Klein (KK) modes in a Minkowski background. Although this approach gives an important insight into the higher-dimensional effects, it is more desirable to consider a brane which more closely resembles the observed universe, namely one of the Friedmann-Robertson-Walker type. In this paper, we will develop such an effective theory, allowing us to understand the effect of the entire non-local Kaluza-Klein (KK) continuum at the perturbed level around a single de Sitter (dS)/inflationary brane.

One common approach to brane inflation model building is to consider the inflaton scalar responsible for inflation to result from one of the many moduli of the theory, such as the size or shape of the extra dimensions, or the relative position or orientation of the branes[1]. However in this study, we take an alternative perspective and explore the possibility of having an inflaton scalar field confined to the brane. This approach has been consider previously in [6], where it was argued that the usual constraints on slow-roll inflation models (e.g. slope of the potential) may be partially relaxed due to the modifications of the effective gravity theory from the presence of extra dimensions.

Here we want to extend the analysis performed in [6] to incorporate the effect of the entire KK continuum, which is often neglected in the literature. To this end, we first recall how the KK tower of [7] can be derived using the Gauss-Codacci formalism for the Minkowski brane as shown in [8]. We then extend this analysis to the case of an inflationary brane and argue that within the slow-roll regime this theory remains a successful model of inflation even in the presence of the entire non-local KK tower. This result is certainly not surprising given the anticipated scaling of the corrections to the low-energy effective theory. In fact, we will find this scaling explicitly by finding the influence of the KK tower on the initial state of the scalar perturbations. In particular, we find that the gravitational potential is greatly enhanced at short wavelengths, but explain why this enhancement is not significant enough to produce any noticeable signature during inflation (to

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[1] See e.g. [5] and references within.

leading order in the slow-roll parameters.) We will also discuss how our approach can be used to study the spectrum of gravity waves and the effect of higher dimensional anisotropy on the lower dimensional effective theory.

The paper is organized as follows. In Section II, we present the effective theory for an inflationary brane embedded in an AdS bulk and discuss the main departures from the standard low-energy effective theory. In the next section, we consider as an application the behavior of scalar metric perturbations during the modified gravity inflation. We will consider the short-wavelength modification to both dynamics and initial conditions, showing that the resulting correction to the power spectrum is suppressed. This is our main result. We then conclude in Section IV. In the appendices we provide a more detailed derivation of the effective theory resulting from the inclusion of the KK tower. We first consider the Minkowski case, followed by the case of a dS/inflationary brane. We then compare both results in the Minkowski limit and check the validity of our theory in Appendix B. The short-wavelength limit of this theory is then developed in Appendix C for the special case of scalar perturbations during inflation.

II.   KK TOWER ON A INFLATIONARY BRANE

In this section we first develop a new formalism to make the problem of finding the low-energy effective theory more technically tractable. We will then use the new formalism to study perturbations around an inflating brane.

A.   Formalism

We want to consider the one-brane Randall-Sundrum II model [1], with a positive tension $D3$ brane localized at the $\mathbb{Z}_2$ fixed point in a semi-infinite AdS$_5$ bulk. The bulk cosmological constant is $\Lambda_5 = -6/\kappa_5^2 \ell_{\text{AdS}}^2$, where $\ell_{AdS} = M_{\text{AdS}}^{-1}$ is the AdS$_5$ curvature scale and $\kappa_5^2 = 8\pi G_5$ is the five-dimensional gravitational coupling constant (in what follows the index AdS is omitted.) We note that this was shown to be a consistent solution of the five-dimensional supergravity theory in [9, 10]. Although for such a theory to have a UV completion in quantum gravity (string/M-theory) the presence of additional dimensions and moduli are certainly of concern, for simplicity we will assume that such moduli have been fixed by fluxes, or non-perturbative effects, at a high scale so that they decouple from the five-dimensional scales that we will consider in what follows.

Using diffeomorphism (gauge) invariance we are free to choose coordinates to work in the frame

where the metric takes the form

$$\mathrm{d}s^2 = q_{\mu\nu}(x^\mu, y)\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2, \tag{1}$$

with the brane located at the orbifold fixed point $y \equiv 0$.

The five-dimensional action is

$$S = \int \mathrm{d}^5 x \sqrt{-g} \left(\frac{1}{2\kappa_5^2} {}^{(5)}R - \Lambda \right) + \int_\mathcal{M} \mathrm{d}^4 x \sqrt{-q} \left(\mathscr{L} - \frac{1}{2\kappa_5^2} K \right), \tag{2}$$

where $K_{\mu\nu}$ is the extrinsic curvature of a $y = \mathrm{const}$ hypersurface. In our frame, $K_{\mu\nu}$ is defined by

$$K_{\mu\nu}(y, x) = \frac{1}{2} \partial_y q_{\mu\nu}(y, x). \tag{3}$$

For a brane with positive tension $\lambda = 6/\kappa_5^2 \ell$, and with stress-energy tensor $T_{\mu\nu}$, the Israël matching condition [11] imposes the extrinsic curvature on the brane to be

$$K^\mu_\nu(y \equiv 0) = -\frac{1}{\ell} \delta^\mu_\nu - \frac{\kappa_5^2}{2} \left(T^\mu_\nu - \frac{1}{3} T \delta^\mu_\nu \right). \tag{4}$$

B.   Embedding the inflationary brane

We now consider the stress-energy tensor for an inflaton scalar field φ confined to the brane:

$$T_{\mu\nu} = \partial_\mu\varphi \, \partial_\nu\varphi - \left(\frac{1}{2}(\partial\varphi)^2 + V \right) q_{\mu\nu}. \tag{5}$$

The modified Friedmann equation (Hamiltonian constraint) on the brane is then given by[2]

$$H^2 = \frac{\kappa_5^2}{3\ell} \rho \left(1 + \frac{\kappa_5^2 \ell}{12} \rho \right), \tag{6}$$

where ρ is the energy density on the brane. Assuming slow-roll inflation and working to lowest order we have $\bar{T}^\mu_\nu \approx -V\delta^\mu_\nu$, and the Friedmann equation becomes

$$H^2 = \frac{\kappa_5^2}{3\ell} V \left(1 + \frac{\kappa_5^2 \ell}{12} V \right). \tag{7}$$

We can now focus on the fluctuations to the inflationary brane resulting from the effects of both brane and bulk fields. The perturbed Einstein equations are

$$\delta G^\mu_\nu = \frac{\kappa_5^2 \bar{z}}{\ell} \delta T^\mu_\nu - \delta E^\mu_\nu, \tag{8}$$

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[2] We refer the reader to [12], for a careful derivation of the following cosmological equations.

where $\bar{z} = \sqrt{1 + \ell^2 H^2} = \left(1 + \frac{\kappa_5^2 \ell}{6} V\right)$, δG^μ_ν is the perturbed Einstein tensor, and δE^μ_ν the perturbed Weyl tensor. This result is derived in detail in Appendix A. In particular, there it is shown that the induced value of the Weyl tensor on the brane is given by (A24), *i.e.*

$$\delta E^\mu_\nu = \frac{1}{\ell^2 H^2} \frac{\mathcal{Q}_\nu(\bar{z})}{\mathcal{Q}'_\nu(\bar{z})} \frac{\ell \kappa_5^2}{2} \Box \, \delta \Sigma^\mu_\nu \,, \tag{9}$$

where $\mathcal{Q}$ is the modified Legendre function, $\nu = \frac{1}{2}(-1 + \sqrt{1 + 4(4 - \frac{\Box}{H^2})})$ and $\Box$ is the four-dimensional d'Alembertian on de Sitter. As mentioned in the appendix, the source tensor $\delta \Sigma^\mu_\nu$ is given by

$$\Box \delta \Sigma^\mu_\nu = \left(\Box \delta T^\mu_\nu - \frac{1}{3}\Box \delta T \, \delta^\mu_\nu + \frac{1}{3}\gamma^{\mu\alpha}\delta T_{;\,\alpha\nu}\right) - 4H^2\left(\delta T^\mu_\nu - \frac{1}{4}\delta T \delta^\mu_\nu\right) \,, \tag{10}$$

and is transverse and traceless.

In the Minkowski limit, $H \to 0$, one should recover the result (A10) obtained in [7], which has been rederived in appendix A. The comparison is performed in appendix B and represents a highly non-trivial check of this four-dimensional effective theory.

C. Anisotropy and an unusual trace coupling

We note that there are two main distinctions from standard four-dimensional Einstein gravity arising in the model we have considered[3]. Firstly, the tensor $\delta\Sigma^\mu_\nu$ presents a source of anisotropic stress which takes the form $\gamma^{\mu\alpha}D_\alpha D_\nu \delta T$ and contributes as an effective shear. We note that this shear is completely gravitational in origin and will be present even in the absence of anisotropic matter on the brane. This effective anisotropy from the four-dimensional point of view could have potentially observable consequences, and will be considered in detail in a future publication [13].

In this paper we wish to concentrate on the second main distinction, namely the peculiar coupling to the trace. In any four-dimension theory, gravity (*i.e.* gravitons) couples directly to the stress-energy T^μ_ν, ensuring gravity to be massless. Although the zero mode graviton is still massless in this model, the coupling of matter to gravity in (8) and (9) is modified by an additional term proportional to the trace. In the following section we will study the implications of this unusual trace coupling during inflation, but we first study in what follows the asymptotic behavior of the effective theory at short and long wavelengths.

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[3] These modifications are true regardless of whether the slow-roll approximation holds.

D. UV/IR modifications

As already mentioned, the effective theory (8) is only valid at leading order in the slow-roll parameters. However, it is important to make the distinction between the time evolution of the background quantities and that of perturbations. The slow-roll approximation only requires the background to evolve smoothly and slowly in time (dS approximation), however perturbations are not constrained by the same requirement and are free to oscillate very rapidly (such as at short wavelengths) without affecting the validity of the theory.

At short wavelengths, perturbations evolve much more rapidly than the background and will therefore live on dS up to a very good approximation. In other words, small scales cannot feel the slow rolling of the background and the approximation that the background is dS will be a very good one. Fortunately, this will be precisely the regime we will be interested in, in what follows. The initial conditions for the curvature perturbation are indeed usually set by imposing the scalar field to be in the Bunch-Davies vacuum [14] well inside the horizon. In the following section we will therefore study how the presence of the extra-dimension affects the initial conditions.

Since at short wavelengths, the modes are expected to oscillate rapidly, we will therefore be interested in the limit where $\Box/H^2 \gg 1$. A good approximation for the operator $r \equiv \frac{\mathcal{Q}_\nu(\bar{z})}{\mathcal{Q}'_\nu(\bar{z})}$ appearing in (9) is then

$$r \equiv \frac{\mathcal{Q}_\nu(\bar{z})}{\mathcal{Q}'_\nu(\bar{z})} = -\frac{\ell H}{|\nu|} \quad \text{for } |\nu| \gg 1 \text{ with } \ell H \sim \text{const}. \tag{11}$$

For completeness, one can also simplify the expression for r in the long wavelength regime. In this regime, spatial derivatives may be neglected. Furthermore, the modes are usually expected to freeze while crossing the horizon and their time evolution is also negligible. Note that since the time evolution of these modes is now comparable to that of the background, such modes will be much more sensitive to any slow-roll departure from dS. We can thus expect the effective theory (8) to be less accurate in that regime, unless the departures from dS are negligible. In that regime, the action of the operator $\Box$ on the transverse and traceless tensor $\Box \delta \Sigma^\mu_\nu$ is thus simply $\Box \simeq 8H^2$, i.e. $\nu = \frac{1}{2}(-1 + \sqrt{-15})$.

Using this value of ν, the effective theory can be simplified a step further in both limits $\ell H \gg 1$ and $\ell H \ll 1$ using the relation

$$r \equiv \frac{\mathcal{Q}_\nu(\bar{z})}{\mathcal{Q}'_\nu(\bar{z})} = \begin{cases} -\frac{\ell H}{8} & \text{for } \ell H \gg 1 \\ \ell^2 H^2 \log\left(\frac{\ell H}{2}\right) & \text{for } \ell H \ll 1 \end{cases}. \tag{12}$$

We also note that r is always negative, even outside these two asymptotic regimes. As expected

the corrections will thus vanish as $\ell H \to 0$ which can either be seen as the Minkowski limit $H \to 0$ (with no derivative terms), or as the small extra-dimension limit ($\ell \to 0$). In both cases, the contribution of the Kaluza-Klein tower should indeed vanish.

To summarize, we have reached the important conclusion that if the energy scale of the bulk curvature is comparable or less than the Hubble scale (as can be the case during inflation), the Kaluza-Klein tower can give a contribution of the same order of magnitude as the zero mode and should thus be considered with care.

III. IMPLICATIONS FOR INFLATION

In this section we consider density fluctuations on the dS brane to which the inflaton scalar field is confined.

We will work in longitudinal gauge where the metric for scalar metric perturbations is

$$\mathrm{d}s^2 = a(\tau)^2 \left[-(1 - 2\Phi)\mathrm{d}\tau^2 + (1 + 2\Psi)\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \right] \tag{13}$$

$$\varphi = \varphi_0(\tau) + \delta\varphi(\tau, x^\mu), \tag{14}$$

where we will work using conformal time $d\tau^2 = a^{-2}dt^2$. In the absence of anisotropic stress, the Einstein equations imply $\Phi = \Psi$, so that there is one scalar degree of freedom associated to the metric. As mentioned previously, we do expect anisotropy contributions to be present in this scenario, however they are studied in detail in [13] and we focus in what follows in the other types of corrections arising from the Weyl tensor.

Following the usual four-dimensional prescription [15], it is useful to introduce the gauge invariant curvature perturbations on uniform density hypersurfaces as[4]

$$\zeta = \Psi - aH\frac{\delta\rho}{\dot{\rho}} \tag{15}$$

where a dot represents the derivative with respect to the conformal time τ.

In what follows, we will analyze how the evolution of the curvature perturbation ζ differs from the standard four-dimensional case. In particular, we will show that the dynamics of ζ and $\delta\varphi$ are both governed by the same set of coupled differential equations as in the standard four-dimensional case. We will thus argue that the only possible departure from the usual result can

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[4] Note that at long wavelength, the above definition of the curvature perturbation coincides with that on comoving hypersurfaces $\zeta_\varphi = \Psi - aH\frac{\delta\varphi}{\dot{\varphi}_0}$, and that on uniform effective energy density hypersurfaces $\zeta_{eff} = \Psi - aH\frac{\delta\rho_{\mathrm{eff}}}{\dot{\rho}_{\mathrm{eff}}}$, where ρ_{eff} is the total effective energy density as seen by the metric. Since we are interested in the power spectrum and spectral index at late time, we are free to use any of these three definitions.

only arise from a modification of the initial conditions. We will then study in detail the initial conditions to be imposed at short wavelengths on the curvature perturbations. Although the gravitational potential Ψ is greatly enhanced by the higher curvature effects at short wavelengths, the gravitational backreaction still remains negligible in this regime, unaffecting the initial condition of ζ and thus its time evolution.

A. Dynamics

In what follows we study the dynamics of the perturbations. Given the isotropy of the model, there are only two independent variables, which we will take to be $\delta\varphi$ and ζ. These two variables are coupled via the Einstein equations. We first observe that because the scalar field is confined to the brane, its equation of motion will remain unaffected by the presence of the extra-dimension. Furthermore, the Weyl tensor defined in (9) is traceless. The trace of the Einstein equation (8) will therefore be similarly unaffected by the extra-dimension, outside of "background effects"[5] which have been studied extensively in the literature [6].

Both the scalar field and the trace equations:

$$\delta T^{\mu}_{\nu\,;\,\mu} = 0 \quad \text{and} \quad \delta R = -\frac{\kappa_5^2 \bar{z}}{\ell}\, \delta T \tag{16}$$

will thus provide two dynamical equations for $\delta\varphi$ and ζ which are identical to their usual four-dimensional counterpart in standard inflation. These two dynamical equations govern the entire dynamics of the system and will provide two coupled equations for ζ and $\delta\varphi$. In other words, the fact that the bulk is empty (implying that the Weyl tensor is traceless) and that the scalar field is confined to the brane ensure that the equations of motion for the curvature perturbations and the scalar field will be unaffected by the extra dimension outside of "background effects".

Thus, the only way the extra-dimension can interfere in the evolution of the perturbations is through the setting of the initial conditions. We have here a set of two coupled second order equations for ζ and $\delta\varphi$, or equivalently a fourth order equation for ζ. We will thus require four initial conditions to fix ζ entirely.

Two of these initial conditions can be fixed by the usual requirement that the scalar field is in the Bunch-Davies vacuum [14] well inside the horizon. But to fix the two remaining ones, we should make use of the constraint equations. We emphasize that the constraint equations (the

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[5] By "background effects" we refer to the effect that comes uniquely from the unusual Friedmann equation (7) and are independent of the Weyl tensor.

Hamiltonian constraint G_0^0 and the momentum density constraint G_i^0) are affected by the Weyl tensor and will thus give a different relation than in the standard case. In principle, we expect the initial conditions for the curvature perturbation to be affected at short wavelengths, which might in turn affect its evolution and the observed power spectrum. This possibility is explored in what follows.

B. Constraints and Initial Conditions

1. Standard Chaotic Inflation

We begin by analyzing the constraints and initial conditions for the case of standard chaotic inflation. As mentioned previously, two of the initial conditions are fixed by imposing the scalar field $\delta\varphi$ to be in the Bunch-Davies vacuum at short wavelength, *i.e.* for $k\tau \gg 1$,

$$\delta\varphi_k \sim -\frac{\ell}{a\kappa_5^2}\frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) . \tag{17}$$

In terms of the energy density perturbation, this implies, to leading order in the slow-roll parameters,

$$\frac{\delta\rho_k}{\dot\rho} = \frac{\delta\varphi_k}{\dot\varphi_0} - \frac{\dot{\delta\varphi_k}}{V'} \approx -\frac{\dot{\delta\varphi_k}}{V'} , \tag{18}$$

where the second equality holds at short wavelengths.

In order to fix the two remaining initial conditions, we use the constraints equations G_0^0 and G_i^0 to express the metric perturbation Ψ in terms of $\delta\varphi$. This gives the following relation in the slow-roll regime

$$\Psi_k = \kappa_4^2\frac{\dot\varphi_0}{2k^2}\,\delta\dot\varphi_k ,$$

where $\kappa_4^2 = \kappa_5^2/\ell$ is the four-dimensional gravitational coupling constant. At short wavelengths, the contribution from the gravitational potential Ψ is thus completely negligible compared to the contribution of the scalar field when setting the initial conditions for ζ: $\Psi_k \propto \frac{\delta\dot\varphi_k}{k^2} \ll \frac{aH}{V'}\delta\dot\varphi_k$, so that $\zeta_k \approx \frac{aH}{V'}\delta\dot\varphi_k$. This is a simple consequence of the nature of inflation for which the gravitational backreaction is very weak. On small scales, the perturbations are unable to distinguish between an expansion driven by a scalar field or by a cosmological constant. Since a cosmological constant would have very little effects on scalar perturbations (beside "background effects"), it is no surprise that the gravitational potential Ψ plays little role in setting the initial conditions at small scales.

We now consider how this argument is modified in the context of brane inflation.

2. Inflation on the Brane

As mentioned previously, the constraints equations G^0_0 and G^0_i are affected by the presence of the non-local corrections embedded in the Weyl tensor. They will therefore give rise to a slightly modified relation between Ψ and $\delta\varphi$. Using the Einstein equation (8) in the gauge (13), we indeed have the modified relation:

$$\Psi_k = \frac{\kappa_5^2}{\ell}\frac{\dot\varphi_0}{12a^2H^2k^2}\left(\left(5rk^2 + 6\bar{z}a^2H^2\right)\delta\dot\varphi_k - 6raH\delta\ddot\varphi_k + 3r\delta\dddot\varphi_k\right),\tag{19}$$

which holds at short wavelengths and in the slow-roll approximation. The derivation of this result is detailed in appendix C. For a scalar field in the Bunch-Davies vacuum (17), the behaviour of the gravitational potential is thus modified to

$$\Psi_k \approx r\,\frac{\kappa_5^2}{\ell}\frac{\dot\varphi_0}{6a^2H^2}\delta\dot\varphi_k\,.\tag{20}$$

The gravitational potential is thus enhanced by the non-local curvature corrections encoded in δE^μ_ν. *A priori* this could imply significant changes in the initial conditions for ζ, which will then propagate to late times and freeze out. However, at short wavelengths, $|\nu|^2 \approx 4k\tau$, so that using the relation (11) in the limit $k\tau \gg 1$, $r \approx -\ell H/2\sqrt{k\tau}$ (Cf. appendix C).

Due to this overall factor, the contribution from Ψ is thus still negligible compared to that of the scalar field in the definition of ζ: $\Psi_k \propto \delta\dot\varphi_k/\sqrt{k\tau} \ll \delta\dot\varphi_k$, so that the initial conditions for ζ will still be dominated by $\zeta_k \approx \frac{aH}{V'}\delta\dot\varphi_k$, as was the case in the standard chaotic inflation scenario.

We can therefore conclude that despite the fact that the gravitational potential Ψ is modified at short wavelengths in this scenario, its overall contribution to the initial conditions for ζ remains negligible. The initial conditions for ζ and $\delta\varphi$ will thus remain unaffected by the presence of the extra-dimension, and since their evolution is governed by a set of equations which is identical to that in four dimensions, we must conclude that no signature of the extra-dimension will be present in the power spectrum and the spectral index to leading order in the slow-roll parameters, except for the possibility of "background effects".

Remarks on Anisotropy

In order to focus on the key point of the argument, anisotropy has been neglected in the previous development. It is however important to notice that the previous result is independent of this assumption and is completely generalizable to the case where $\Phi \neq \Psi$. At long wavelengths, the anisotropic part of the Weyl tensor vanishes, so that one should recover $\Phi = \Psi$. At short

wavelengths, however, the relation between the two gravitational potentials can be derived from the last constraint G^i_j (with $i \neq j$) to give

$$\Phi = \Psi - r \frac{\kappa_5^2}{\ell} \frac{\dot{\varphi}_0}{3a^2 H^2} \delta\dot{\varphi} \,. \tag{21}$$

Despite this difference, the relation between Ψ and $\delta\varphi$ remains unchanged and is still given by (20), (Cf. appendix C). The contribution from the gravitational potential in the initial conditions for ζ will therefore remain negligible.

Furthermore, as seen in the appendix C, at short wavelengths the equation for the scalar field is similarly unaffected by the anisotropy (C11). Since the anisotropy disappears at long wavelengths, and the transition between the short-wavelength oscillating modes and the long wavelength frozen modes occurs almost instantaneously, the equation of motion for the scalar field will be unaffected by the extra-dimension (through the Weyl tensor) during its entire evolution. The argument given previously will therefore remain valid even in the presence of anisotropy and no signature to the power spectrum will be present to leading order in the slow-roll parameters, besides that of the background effects.

IV. CONCLUSION

In this paper, we have developed a formalism capable of tracking non-local effects of the KK continuum at the perturbed level on a single de Sitter brane while preserving general covariance on the brane. As a consistency check, this formalism was shown to reproduce the correct Minkowski limit developed in [7]. We argue that to leading order in the slow-roll parameters, this effective theory provides a good framework to study models of braneworld inflation for which the inflaton scalar field is confined to the brane.

In particular, we use this analysis to study the power spectrum of curvature perturbations. Although non-local terms typically present in braneworld scenarios can be very important at short scales, a close analysis to their effects during a standard model of inflation shows that the corrections to the power spectrum are completely negligible in the slow-approximation besides any possible background effect.

This is in complete agreement with the results of [16], which appeared during the latest stages of this work. Both studies use similar approximations (slow-roll) but different methods to derive very analogous results. We may point out that the present analysis does not rely on any numerical computations and is thus very easily reproducible. Another advantage of our analysis resides in

its covariant derivation. It can thus very easily be extended to the study of the tensor modes or to non-inflationary scenarios which left for a further study.

Acknowledgments

APPENDIX A: EFFECTIVE THEORY ON A MINKOWSKI AND DE SITTER BRANE

1. Four-dimensional decomposition

We consider the five-dimensional theory (2), and work in the gauge (1) where the brane is fixed at $y = 0$.

The electric part of the five-dimensional Weyl tensor $E_{\mu\nu}$ is defined as:

$$E_{\mu\nu}(y,x) \equiv {}^{(5)}C^y{}_{ayb}\, q^a_\mu q^b_\nu. \tag{A1}$$

From the properties of the five-dimensional Weyl tensor ${}^{(5)}C^a{}_{bcd}$, we can easily check that $E_{\mu\nu}$ is traceless with respect to $q_{\mu\nu}$. In the frame (1), using the five-dimensional Einstein equation $G_{ab} = -\kappa_5^2 \Lambda g_{ab}$, the expression for $E_{\mu\nu}$ is

$$E^\mu_\nu = {}^{(5)}R^y{}_{\alpha y \nu}q^{\alpha\mu} + \frac{1}{\ell^2}\,\delta^\mu_\nu = -\partial_y K^\mu_\nu - K^\mu_\alpha K^\alpha_\nu + \frac{1}{\ell^2}\delta^\mu_\nu. \tag{A2}$$

It has been shown in the literature, that from a four-dimensional brane perspective, the projected Weyl tensor on the brane is the only quantity mediating between the brane and the bulk. Indeed, the projected Ricci tensor on the brane can be expressed as

$$R_{\mu\nu}(x) = \frac{\kappa_5^2}{\ell}\left(T_{\mu\nu} - \frac{1}{2}T q_{\mu\nu}\right) - \frac{\kappa_5^4}{4}\left(T^\alpha_\mu T_{\alpha\nu} - \frac{1}{3}T T_{\mu\nu}\right) - E_{\mu\nu}(0,x). \tag{A3}$$

The evolution of the Weyl tensor through the bulk is governed by the equation:

$$\begin{aligned}
\partial_y E^\mu_\nu ={}& 2K^\alpha_\nu E^\mu_\alpha - \frac{3}{2}K E^\mu_\nu - \frac{1}{2}K^\alpha_\beta E^\beta_\alpha \delta^\mu_\nu + C^\mu{}_{\alpha\nu\beta}K^{\alpha\beta} + (K^3)^\mu_\nu \\
&- \frac{1}{2}D^\alpha\left[D^\mu K_{\alpha\nu} + D_\nu K^\mu_\alpha - 2D_\alpha K^\mu_\nu\right],
\end{aligned} \tag{A4}$$

where $(K^3)^\mu_\nu$ are some cubic terms in the traceless part of the extrinsic curvature whose exact form will not be relevant for the purpose of this study (it is enough to note that this term vanishes if $K^\mu_\nu \sim \delta^\mu_\nu$). The previous results have first been developed in ref. [8].

2. Linearized gravity on a Minkowski brane

We now use this formalism to recover the well-know result of Garriga and Tanaka in ref. [7] that first studied linearized gravity around a Minkowski brane, expressing explicitly the KK tower and their effect on the brane.

For an empty brane, the background bulk geometry is exactly AdS: $\bar{q}_{\mu\nu}(y,x) = e^{-2y/\ell}\bar{\gamma}_{\mu\nu}(x)$, where $\bar{\gamma}_{\mu\nu}$ is the metric on the brane, here $\bar{\gamma}_{\mu\nu} = \eta_{\mu\nu}$. Since the background in conformally invariant, the Weyl tensor vanishes at that order. From the evolution equation (A2), together with the boundary condition (4), the extrinsic curvature is thus simply $\bar{K}^\mu_\nu(y,x) = -\frac{1}{\ell}\delta^\mu_\nu$.

We now consider small fluctuations induced by matter on the brane $T^\mu_\nu = 0 + \delta T^\mu_\nu$, so that the extrinsic curvature and the Weyl tensor become:

$$E^\mu_\nu = 0 + \delta E^\mu_\nu$$

$$K^\mu_\nu = \bar{K}^\mu_\nu + \delta K^\mu_\nu$$

$$\delta K^\mu_\nu(y \equiv 0) = -\frac{\kappa_5^2}{2}(\delta T^\mu_\nu - \frac{1}{3}\delta T \delta^\mu_\nu).$$

At the linearized level, the evolution equations (A2,A4) for the extrinsic curvature and the Weyl tensor thus simplify to

$$\partial_y \delta K^\mu_\nu = -\delta E^\mu_\nu + \frac{2}{\ell}\delta K^\mu_\nu \tag{A5}$$

$$\delta_y \delta E^\mu_\nu = \frac{4}{\ell}\delta E^\mu_\nu + e^{2y/\ell}\left(\Box \delta K^\mu_\nu - \bar{\gamma}^{\mu\alpha}\delta K_{;\alpha\nu}\right), \tag{A6}$$

where $\Box$ is the four-dimensional d'Alembertien in Minkowski: $\Box = \gamma^{\alpha\beta}D_\alpha D_\beta$. Combining these two equations we get the second order differential evolution equation for the Weyl tensor:

$$\partial_y^2 \delta E^\mu_\nu = \frac{8}{\ell}\partial_y \delta E^\mu_\nu - \frac{16}{\ell^2}\delta E^\mu_\nu - e^{2y/l}\Box \delta E^\mu_\nu, \tag{A7}$$

which has the simple solution:

$$\delta E^\mu_\nu(y,x) = e^{4y/\ell}\left(I_0(e^{y/\ell}\ell\sqrt{-\Box})A^\mu_\nu(x) + K_0(e^{y/\ell}\ell\sqrt{-\Box})B^\mu_\nu(x)\right), \tag{A8}$$

where K_0, I_0 are the two independent Bessel functions, and A^μ_ν, B^μ_ν are two "integration constants" to be determined. One of them is imposed by the requirement that the five-dimensional metric

(and thus the Weyl tensor) remains finite at spatial infinity ($y \to \infty$). This requirement sets A^μ_ν to zero. The second tensor B^μ_ν is fixed by the Israël matching condition on the brane (4). Using the relation (A6), one has:

$$\Box \, \delta K^\mu_\nu(0) - \bar{\gamma}^{\mu\alpha} \, \delta K_{;\alpha\nu}(0) = -\frac{\kappa_5^2 \Box}{2} \delta \Sigma^\mu_\nu \equiv -\frac{\kappa_5^2 \Box}{2} \left(\delta T^\mu_\nu - \frac{1}{3} \delta T + \frac{1}{3\Box} \bar{\gamma}^{\mu\alpha} \delta T_{;\alpha\nu} \right)$$
$$= -\sqrt{-\Box} \, K_1(\ell\sqrt{-\Box}) \, B^\mu_\nu \,, \tag{A9}$$

so that the Weyl tensor on the brane is

$$\delta E^\mu_\nu(0, x) = -\frac{\kappa_5^2}{2} \sqrt{-\Box} \, \frac{K_0(\ell\sqrt{-\Box})}{K_1(\ell\sqrt{-\Box})} \, \delta \Sigma^\mu_\nu \,. \tag{A10}$$

The modified Einstein equation on the brane is therefore:

$$\delta R^\mu_\nu = \frac{\kappa_5^2}{\ell} \left(\delta T^\mu_\nu - \frac{1}{2} \delta T \delta^\mu_\nu \right) + \frac{\kappa_5^2}{2} \sqrt{-\Box} \, \frac{K_0(\ell\sqrt{-\Box})}{K_1(\ell\sqrt{-\Box})} \, \delta \Sigma^\mu_\nu, \tag{A11}$$

which is precisely the result obtained by Garriga and Tanaka in ref. [7]. Motivated by this result, we can now use the same formalism to compute the KK tower for a dS brane, which will be useful if we are to study inflation on the brane.

3. Linearized gravity on a de Sitter brane

Still working in the frame (1), where the brane location is fixed at $y = 0$, we now consider a positive cosmological constant on the brane (or equivalently, assume that the brane tension is larger than its conical value λ), so that for the background

$$T^\mu_\nu = -V \delta^\mu_\nu. \tag{A12}$$

The modified Friedmann equation is then given by

$$H^2 = \frac{\kappa_5^2}{3\ell} V \left(1 + \frac{\kappa_5^2 \ell}{12} V \right), \tag{A13}$$

and at the perturbed level, the Einstein equation on the brane will be:

$$\delta R^\mu_\nu = \frac{\kappa_5^2}{\ell} \left(1 + \frac{\kappa_5^2 \ell}{6} V \right) \left(\delta T^\mu_\nu - \frac{1}{2} T \delta^\mu_\nu \right) - \delta E^\mu_\nu. \tag{A14}$$

Solving the evolution equation (A2) with the boundary condition imposed by the Israël matching condition (4), we thus have:

$$\bar{K}^\mu_\nu(y) = -\frac{1}{\ell} k(y) \delta^\mu_\nu = -\frac{1}{\ell} \coth(\bar{v} - \frac{y}{\ell}) \delta^\mu_\nu, \tag{A15}$$

where $\bar{v}$ is an integration constant fixed such that the right boundary condition is recovered on the brane i.e. $\bar{K}^\mu_\nu(0) = -\frac{1}{\ell}(1+\frac{\kappa_5^2 \ell}{6}V)$, imposing $\bar{v} = \mathrm{arccoth}(1+\frac{\kappa_5^2 \ell}{6}V)$. The metric profile is therefore:

$$\partial_y \bar{q}_{\mu\nu} = 2\bar{K}_{\mu\nu} \quad \Rightarrow \quad \bar{q}_{\mu\nu}(y,x) = \frac{\sinh^2(\bar{v} - y/\ell)}{\sinh^2 \bar{v}} \bar{\gamma}_{\mu\nu}(x), \tag{A16}$$

where here again $\bar{\gamma}_{\mu\nu}(x)$ is the induced metric on the brane. We choose here to work in conformal time, so that $\bar{\gamma}_{\mu\nu} = a(\tau)^2 \eta_{\mu\nu}$, with $H = \dot{a}/a^2$. In the limit $V \to 0$, $\bar{v} \to +\infty$, we recover the result for a Minkowski brane: $k(y) = 1$ and $\bar{q}_{\mu\nu}(y,x) = e^{-2y/\ell} \bar{\gamma}_{\mu\nu}(x)$.

We may notice at this point, that in this gauge, spatial infinity is reached at finite value of y, i.e. at $y = \bar{y}_\infty = \ell \bar{v} > 0$. The perturbations perceived on the brane will depend on the boundary conditions imposed at spatial infinity. The mode functions should be normalized over the region $0 < y < \bar{y}_\infty$, which will require them to fall as $y \to \bar{y}_\infty$ or behave as plane waves. Note that at this point, the extrinsic curvature is infinite $k(y) \to \infty$ as $y \to \bar{y}_\infty$.

<center>*Weyl tensor*</center>

As seen earlier, the key point in obtaining the KK tower at the perturbed level, is to find the expression for the Weyl tensor. We proceed as for the Minkowski brane, and the evolution equations (A2, A4) for the extrinsic curvature and the Weyl tensor at the perturbed level are:

$$\partial_y \delta K^\mu_\nu = -\delta E^\mu_\nu + \frac{2}{\ell} k(y) \delta K^\mu_\nu \tag{A17}$$

$$\partial_y \delta E^\mu_\nu = \frac{4}{\ell} k(y) \delta E^\mu_\nu + \frac{1}{\ell^2 H^2 \sinh^2(x - \frac{y}{l})} \left[\hat{O} \delta K^\mu_\nu - \hat{O}^\mu_\nu \delta K \right], \tag{A18}$$

with the operator $\hat{O}^\mu_\nu = \bar{\gamma}^{\mu\alpha} D_\alpha D_\nu - H^2 \delta^\mu_\nu$, and $\hat{O} = \hat{O}^\alpha_\alpha = \Box - 4H^2$. Combining these two equations, and using the variable $z = k(y) > 1$, we get the evolution equation for the Weyl tensor

$$\partial_y^2 \delta E^\mu_\nu = \frac{8}{\ell} k(y) \partial_y \delta E^\mu_\nu + \frac{4}{\ell} \left(k'(y) - \frac{4}{\ell} k^2(y) \right) \delta E^\mu_\nu - \frac{1}{\ell^2 H^2} \hat{O} \delta E^\mu_\nu$$

$$(z^2 - 1)^2 \partial_z^2 \delta E^\mu_\nu = 6z(z^2 - 1) \partial_z \delta E^\mu_\nu - 4(3z^2 + 1) \delta E^\mu_\nu - (z^2 - 1) \frac{1}{H^2} \hat{O} \delta E^\mu_\nu. \tag{A19}$$

The general profile of the Weyl tensor through the bulk can be expressed of the form

$$\delta E^\mu_\nu(z) = (z^2 - 1)^2 \left[P_\nu(z) C^\mu_\nu + Q_\nu(z) D^\mu_\nu \right], \tag{A20}$$

where P_ν and Q_ν are the Legendre functions, and $\nu = \frac{1}{2}\left(-1 + \sqrt{1 - \frac{4}{H^2}\hat{O}} \right)$. The profile of the Weyl tensor is thus drastically modified compared to the situation in Minkowski. The integration tensors C^μ_ν and D^μ_ν are fixed by the requirement that the Weyl tensor falls at spatial infinity $z \to \infty$ or behaves as plane waves, and by the Israël matching condition.

The constraint at spatial infinity, together with the requirement that the Weyl tensor is real, sets its general profile to be

$$\delta E^\mu_\nu(z) = (z^2 - 1)^2 \left[\mathcal{Q}_\nu(z) + \mathcal{Q}_{\nu^\star}(z) \right] C^\mu_\nu, \tag{A21}$$

where $\mathcal{Q}_\nu$ is the modified Legendre function:

$$\mathcal{Q}_\nu(z) = z^{-\nu-1} {}_2F_1 \left(\frac{1+\nu}{2}, \frac{2+\nu}{2}, \frac{3+2\nu}{2}, z^{-2} \right) + \text{cc},$$

and ${}_2F_1$ is the Hypergeometric function of second kind.

Finally, the second integration constant C^μ_ν is fixed by the Israël matching condition on the brane (A18) which reduces to

$$\mathcal{Q}'_\nu(\bar z) \, C^\mu_\nu = \frac{1}{\ell^6 H^6} \frac{\ell \kappa_5^2}{2} \Box \delta \Sigma^\mu_\nu, \tag{A22}$$

where $\Box$ is the now d'Alembertian in de Sitter space, $\Box = \bar\gamma^{\mu\nu} D_\mu D_\nu$, and $\bar z$ is the value of z on the brane: $\bar z = \left(1 + \frac{\kappa_5^2 \ell}{6} V \right) = \sqrt{1 + \ell^2 H^2}$. The source tensor $\delta \Sigma^\mu_\nu$ is given by

$$
\begin{aligned}
\Box \delta \Sigma^\mu_\nu &= \frac{2}{\kappa_5^2} \left[\hat O \delta K^\mu_\nu(0) - \hat O^\mu_\nu \, \delta K(0) \right] \\
&= \left(\Box \delta T^\mu_\nu - \frac{1}{3} \Box \delta T \, \delta^\mu_\nu + \frac{1}{3} \gamma^{\mu\alpha} \delta T_{; \alpha\nu} \right) - 4H^2 \left(\delta T^\mu_\nu - \frac{1}{4} \delta T \delta^\mu_\nu \right).
\end{aligned} \tag{A23}
$$

One can check that $\delta \Sigma^\mu_\nu$ is indeed transverse and traceless with respect to the background de Sitter metric $\bar\gamma_{\mu\nu}$. For any arbitrary function $f(\Box)$, the tensor $f(\Box) \delta \Sigma^\mu_\nu$, will thus also remain transverse and traceless.

Using this boundary condition, the Weyl tensor on the brane is thus of the form:

$$\delta E^\mu_\nu(\bar z) = \frac{1}{\ell^2 H^2} \frac{\mathcal{Q}_\nu(\bar z)}{\mathcal{Q}'_\nu(\bar z)} \frac{\ell \kappa_5^2}{2} \Box \, \delta \Sigma^\mu_\nu, \tag{A24}$$

with the perturbed Einstein equation

$$\delta G^\mu_\nu = \frac{\kappa_5^2}{\ell} \bar z \, \delta T^\mu_\nu - \delta E^\mu_\nu. \tag{A25}$$

APPENDIX B: MINKOWSKI VERSUS DE SITTER BRANE

The aim of this appendix is to show that the dS effective theory reduces to that of Minkowski in the limit $H \to 0$. The main difficulty in taking this limit resides in the fact that ν is then

ill-defined: $\nu \sim \sqrt{-\frac{\square}{H^2}}$. We thus first consider the limit $\bar{z} = \sqrt{1 + \ell^2 H^2} \to 1$, with fixed ν.

$$
\begin{aligned}
\frac{1}{\ell^2 H^2} \frac{\mathcal{Q}_\nu(\bar{z})}{\mathcal{Q}'_\nu(\bar{z})} &= \left[\frac{1}{\nu} + \left(\bar{\Gamma} + \bar{\Gamma}(\nu) + \log\left(\frac{\ell H}{2} \right) \right) \right] \\
&\quad - \frac{\ell^2 H^2}{8\nu} \Bigg[2\Big(\nu \left(2\nu(1+\nu)\bar{\Gamma}^2 + \nu(\nu-1)(1+\log 4) - 2(1+\nu)(\log 4\nu + \nu - 2)\bar{\Gamma} - 4\log 2 + 1 \right) \\
&\quad + 2\nu(1+\nu)\log(H\ell)(\nu \log(H\ell) + \nu(-1 + 2\bar{\Gamma} - \log 4) + 2) + 2\Big) \\
&\quad + \nu(1+\nu)\left(\nu \left(\log 4 - 2\bar{\Gamma}(\nu) \right)^2 + 4(2\nu \log(H\ell) + \nu(2\bar{\Gamma} - 1) + 2)\bar{\Gamma}(\nu) \right) \Bigg] \\
&\quad + \cdots ,
\end{aligned}
$$

with $\bar{\Gamma}$ the Euler number ($\bar{\Gamma} \sim 0.57$), and $\bar{\Gamma}(\nu)$ the digamma function.

Using this expansion, we may now set $\nu = \sqrt{-\frac{\square}{H^2}}$ and take the limit $H \to 0$ to obtain:

$$
\frac{1}{\ell^2 H^2} \frac{\mathcal{Q}_\nu(\bar{z})}{\mathcal{Q}'_\nu(\bar{z})} = \left[\bar{\Gamma} + \log\left(\frac{\ell k}{2} \right) \right] - \frac{\ell^2 k^2}{4} \left[1 + 2\left(-1 + \bar{\Gamma} + \log\left(\frac{\ell k}{2} \right) \right) \left(\bar{\Gamma} + \log\left(\frac{\ell k}{2} \right) \right) \right] + \cdots , \quad \text{(B1)}
$$

where for simplicity, we used the notation $k = \sqrt{-\square}$.

We can see that this result corresponds to a derivative expansion in the limit where $H = 0$. We can therefore check that this expansion is the same as the one obtained in the Minkowski case for which (cf. eq (A10))

$$
\delta E_{\text{Min}}{}^\mu{}_\nu = -\frac{\kappa_5^2}{2} k \frac{K_0(\ell k)}{K_1(\ell k)} \delta \Sigma^\mu{}_\nu , \quad \text{(B2)}
$$

where both source terms $\delta \Sigma^\mu{}_\nu$ clearly coincide when $H = 0$. One can easily check that expanding $-\frac{1}{\ell k} \frac{K_0(\ell k)}{K_1(\ell k)}$ to second order in k reduces precisely to the previous result (B1), confirming that we recover the Minkowski limit as $H \to 0$, at least to leading orders in k. The comparison can be continued order by order in k to check the validity of the theory to all order in derivatives in the Minkowski limit.

APPENDIX C: SHORT-WAVELENGTH REGIME

We compute in this section the source term (10) using the perturbed contribution from the stress-energy tensor (5). At short wavelengths and to leading order in the slow-roll parameters, its expression is simply of the form

$$
S^\mu{}_\nu \equiv \square \delta \Sigma^\mu{}_\nu = \frac{\dot{\varphi}_0}{3a^2} \begin{pmatrix} -5\nabla^2 \delta\dot{\varphi} + 3\delta\dddot{\varphi} & \delta\ddot{\varphi}^{,i} - 3\nabla^2 \delta\varphi^{,i} \\ -\delta\ddot{\varphi}_{,j} + 3\nabla^2 \delta\varphi_{,j} & \left(\nabla^2 \delta\dot{\varphi} - \delta\dddot{\varphi} \right) \delta^i_j + 2\delta\dot{\varphi}^{,i}_{,j} \end{pmatrix} , \quad \text{(C1)}
$$

with ∇^2 the three-dimensional Laplacian $\nabla^2 = \partial^i \partial_i$.

The operator of $\Box$ on this tensor is highly non-trivial, however in the short-wavelength regime, the time-dependance of $\delta\varphi$ is given by (17), so expanding $\delta\varphi$ in Fourier modes, on has for each mode, $\delta\varphi_k \sim \frac{e^{-ik\tau}}{a}$. Using this behaviour, we get at short wavelength,

$$\Box S^\mu_\nu \simeq 4\frac{H}{a}\partial_\tau S^\mu_\nu \simeq -4i\frac{kH}{a}S^\mu_\nu \,. \tag{C2}$$

This relation can be used to evaluate the action of the operator $r \equiv \frac{\mathcal{Q}_\nu(\bar{z})}{\mathcal{Q}'_\nu(\bar{z})}$ on this source term. Recalling $\nu = \frac{1}{2}(-1 + \sqrt{1 + 4(4 - \frac{\Box}{H^2})})$. At short wavelengths, the action of ν on the source term S^μ_ν will therefore be $\nu^2 \simeq -\frac{\Box}{H^2} \simeq 4i\frac{k}{aH} \simeq -4ik\tau$ as $k\tau \gg 1$, where we used $a = -1/H\tau$ in the slow-roll regime.

Using the asymptotic behaviour (11), the action of r on the source term S^μ_ν will therefore be

$$r \to -\frac{\ell H}{|\nu|} \simeq -\frac{\ell H}{2\sqrt{k\tau}} \quad \text{for} \quad k\tau \gg 1 \,. \tag{C3}$$

We may now study the modified Einstein equation

$$\delta G^\mu_\nu = \frac{\kappa_5^2 \bar{z}}{\ell}\delta T^\mu_\nu - \frac{\ell\kappa_5^2}{2\ell^2 H^2}r\,S^\mu_\nu \,, \tag{C4}$$

at the perturbed level. The relation between the two gravitational potentials Ψ and Φ can be derived from the (ij) (with $i \neq j$) component of this equation which reads

$$\partial^i\partial_j \left[3a^2 H^2 (\Phi - \Psi) + r\,\dot\varphi_0 \frac{\kappa_5^2}{\ell}\delta\dot\varphi\right] = 0 \,. \tag{C5}$$

This expression can be used in the two other constraints. In particular the momentum constraints is of the form

$$\partial_i \left[12a^3 H^2\Phi + 12a^2 H^2\dot\Psi + 6\frac{\kappa_5^2 \bar{z}}{\ell}\dot\varphi_0 a^2 H^2\delta\varphi + r\frac{\kappa_5^2}{\ell}\dot\varphi_0 \left(3k^2\delta\varphi + \delta\ddot\varphi\right)\right] = 0 \,, \tag{C6}$$

and the Hamiltonian constraint is

$$-12a^2 H^2 \left(k^2\Psi + 3a^2 H^2\Phi + 3aH\dot\Psi\right) + 6\frac{\kappa_5^2 \bar{z}}{\ell}\dot\varphi_0 a^2 H^2 \left(\delta\dot\varphi - 3aH\delta\varphi\right) \tag{C7}$$

$$+ r\frac{\kappa_5^2}{\ell}\dot\varphi_0 \left(5k^2\delta\dot\varphi - 9aHk^2\delta\varphi - 9aH\delta\ddot\varphi + 3\delta\dddot\varphi\right) = 0 \,.$$

Using the anisotropy relation (C5) and the two previous constraints, we may express Ψ in terms of $\delta\varphi$ as

$$\Psi = \frac{\kappa_5^2}{12\ell}\frac{\dot\varphi_0}{a^2 H^2} \left(6\bar{z}a^2 H^2\delta\dot\varphi + r\left(6k^2\delta\dot\varphi - 6aH\delta\ddot\varphi + 3\delta\dddot\varphi\right)\right) \,. \tag{C8}$$

This expression for Ψ has been derived while taking into account the anisotropy (C5), but we may point out that the same expression exactly would have hold have we made the hypothesis $\Phi = \Psi$ in (C6) and (C7).

In the absence of the Weyl term ($r = 0$), we recover the usual four-dimensional result. At short wavelengths, the expression for Ψ is dominated by

$$\Psi \simeq r \, \frac{\kappa_5^2}{4\ell} \frac{\dot{\varphi}_0}{a^2 H^2} \left(2k^2 \delta\dot{\varphi} + \delta\dddot{\varphi} \right) . \tag{C9}$$

Furtheremore, the conservation of energy for the scalar field takes the standard form:

$$\delta\ddot{\varphi} + k^2 \delta\varphi + 2aH\delta\dot{\varphi} + \dot{\varphi}_0 \left(\dot{\Phi} + 3\dot{\Psi} \right) + 4aH\dot{\varphi}_0 \Phi = 0 . \tag{C10}$$

The presence of anisotropy (C5) will thus modify slightly this relation by adding an additional term proportional to r:

$$\delta\ddot{\varphi} + k^2 \delta\varphi + 2aH\delta\dot{\varphi} + 4\dot{\varphi}_0 \, \dot{\Psi} + 4aH\dot{\varphi}_0 \Psi = r \, \frac{\kappa_5^2}{\ell} \frac{\dot{\varphi}_0}{3a^2 H^2} \left[\delta\ddot{\varphi} + 2aH\delta\dot{\varphi} \right] . \tag{C11}$$

However, as seen previously, at short wavelengths $r \sim -\ell H/\sqrt{k\tau}$, and this additional contribution on the right hand side will therefore be negligible. The equation of motion of the scalar field will thus be unaffected by the anisotropy at short wavelengths. Since the at long wavelengths the anisotropy vanishes, we therefore conclude that the equation of motion for the scalar field will be unaffected by the presence of the Weyl tensor, apart in the very short transition regime between the short and the long wavelength regime. But since this transition occurs very rapidly, one can to a very good approximation match both regimes directly.

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