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A LAGRANGE MULTIPLIER TEST FOR CROSS-SECTIONAL DEPENDENCE IN A FIXED EFFECTS PANEL DATA MODEL

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Abstract

It is well known that the standard Breusch and Pagan (1980) LM test for cross-equation correlation in a SUR model is not appropriate for testing cross-sectional dependence in panel data models when the number of cross-sectional units (n) is large and the number of time periods (T) is small. In fact, a scaled version of this LM test was proposed by Pesaran (2004) and its finite sample bias was corrected by Pesaran, Ullah and Yamagata (2008). This was done in the context of a heterogeneous panel data model. This paper derives the asymptotic bias of this scaled version of the LM test in the context of a fixed effects homogeneous panel data model. This asymptotic bias is found to be a constant related to n and T, which suggests a simple bias corrected LM test for the null hypothesis. Additionally, the paper carries out some Monte Carlo experiments to compare the finite sample properties of this proposed test with existing tests for cross-sectional dependence.

Keywords: LM Test; Cross-sectional Dependence; Fixed Effects; High Dimensional Inference; John Test; Panel Data

JEL Classification: C13; C33.

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1 Introduction

Cross-sectional dependence, described as the interaction between cross-sectional units (e.g., households, firms and states etc.), has been well discussed in the spatial literature. Intuitively, dependence across "space", can be regarded as the counterpart of serial correlation in time series. It could arise from the behavioral interaction between individuals, e.g., imitation and learning among consumers in a community, or firms in the same industry. This has been widely studied in game theory and industrial organization. It could also be due to unobservable common factors or common shocks popular in macroeconomics.

As is the case under serial correlation in time series, cross-sectional dependence leads to efficiency loss for least squares and invalidates conventional t-tests and F-tests which use standard variance-covariance estimators. In some cases, it could potentially result in inconsistent estimators (Lee, 2002; Andrews, 2005). Several estimators have been proposed to deal with cross-sectional dependence, including the popular spatial methods (Anselin, 1988; Anselin and Bera, 1998; Kelejian and Prucha, 1999; Kapoor, Kelejian and Prucha, 2007; Lee, 2007; Lee and Yu, 2010), and factor models in panel data (Pesaran, 2006, Kapetanios, Pesaran and Yamagata, 2011; Bai, 2009). However, before imposing any structure on the disturbances of our model, it may be wise to test the existence of cross-sectional dependence.

There has been a lot of work on testing for cross-sectional dependence in the spatial econometrics literature, see Anselin and Bera (1998) for cross-sectional data and Baltagi, Song and Koh (2003) for panel data, to mention a few. The latter derives a joint Lagrange Multiplier (LM) test for the existence of spatial error correlation as well as random region effects in a panel data regression model. Panel data provide richer information on the covariance matrix of the errors than cross-sectional data. This is especially relevant for the off-diagonal elements which are of particular importance in determining cross-sectional dependence. With panel data one can test for cross-sectional dependence without imposing *ad hoc* specifications on the error structure generating the covariance matrix, e.g., the spatial autoregressive model in the spatial literature, or the single or multiple factor structures imposed on the errors in the macro literature. Ng (2006) and Pesaran (2004) propose two test procedures based on the sample covariance matrix in panel data. Ng (2006) develops a test tool using spacing method in a panel model. Pesaran (2004) proposes a cross-sectional dependence (CD) test using the pairwise average of the off-diagonal sample correlation coefficients in a seemingly unrelated regressions model. The CD test is closely related to the R_{AVE}

test statistic advanced by Frees (1995). Unlike the traditional Breusch-Pagan (1980) LM test, the CD test is applicable for a large number of cross-sectional units (n) observed over T time periods. Recently, Sarafidis, Yamagata and Robertson (2009) develop a test for cross-sectional dependence based on Sargan's difference test in a linear dynamic panel data model, in which the error cross-sectional dependence is modelled by a multifactor structure. Hsiao, Pesaran and Pick (2009) propose a LM type test for nonlinear panel data models. Baltagi, Feng and Kao (2011) propose a test for sphericity following John (1972) and Ledoit and Wolf (2002) in the statistics literature. Sphericity means that the variance-covariance matrix is proportional to the identity matrix. However, rejection of the null could be due to cross-sectional dependence or heteroskedasticity or both. For a recent survey of some cross-sectional dependence tests in panels, see Moscone and Tosetti (2009).

In the fixed n case and as $T \to \infty$, the Breusch and Pagan's (1980) LM test can be applied to test for the cross-sectional dependence in panels. Under the null hypothesis, the test statistic is asymptotically Chi-square distributed with $n(n-1)/2$ degrees of freedom. However, this test is not applicable when $n \to \infty$. Therefore, Pesaran (2004) proposed a scaled version of this LM test, denoted by CD_{lm} which has a $N(0,1)$ distribution as $T \to \infty$ first, followed by $n \to \infty$. As pointed out by Pesaran (2004), the CD_{lm} test is not correctly centered at zero for finite T and is likely to exhibit large size distortions as n increases. To solve this problem, Pesaran (2004) proposes a diagnostic test based on the average of the sample correlations, which he denotes by the CD test, and this is valid for large n. Additionally, Pesaran, Ullah and Yamagata (2008) develop a bias-adjusted LM test using finite sample approximations in the context of a heterogeneous panel model.

This paper derives the asymptotic bias of this scaled version of the LM test in the context of a fixed effects homogeneous panel data model. Because it is based on the fixed effects residuals, we denote it by LM_P to distinguish it from CD_{lm}. The asymptotic bias of LM_P is found to be a constant related to n and T which suggests a simple bias corrected LM test for the null hypothesis. This paper differs from the bias-adjusted LM test of Pesaran, Ullah and Yamagata (2008) in that the latter assumes a *heterogeneous* panel data model, whereas this paper assumes a fixed effects *homogeneous* panel data model. Also, the bias correction derived in this paper is based on asymptotic results as $(n,T) \to \infty$, while the bias adjustment in Pesaran, Ullah and Yamagata (2008) is obtained using finite sample approximation. Phillips and Moon (1999) provide regression limit theory for panels with $(n,T) \to \infty$. Here, we adopt the asymptotics used in the

statistics literature for high dimensional inference, see Ledoit and Wolf (2002) and Schott (2005), to mention a few. This literature usually deals with multivariate normal distributed variables where the number of variables (in our case n) is comparably as large as the sample size (T). Our paper finds that under this joint asymptotics framework with $(n, T) \to \infty$ simultaneously, the limiting distribution of the LM_P statistic is not standard normal under the assumption of a fixed effects model. Consequently, it can suffer from large size distortions.

The organization of the paper is as follows. The next section discusses the LM tests for cross-sectional dependence in the context of a fixed effects panel data model. Section 3 derives the limiting distribution of the LM_P test in the raw data case. Section 4 derives the corresponding limiting distribution of the LM_P test in the context of a fixed effects model. Section 5 compares the size and power of the proposed test as well as other tests for cross-sectional dependence using Monte Carlo experiments. In section 6, we show that the proposed bias-corrected LM test can be extended to the dynamic panel data model. Section 7 concludes. The appendix contains all the proofs and the technical details.

2 LM Tests for Cross-sectional Dependence

Consider the heterogeneous panel data model:

$$y_{it} = x_{it}'\beta_i + u_{it}, \text{ for } i = 1, ..., n; \ t = 1, ..., T, \tag{1}$$

where i indexes the cross-sectional units and t the time series observations. y_{it} is the dependent variable and x_{it} denotes the exogenous regressors of dimension $k \times 1$ with slope parameters β_i that are allowed to vary across i. u_{it} is allowed to be cross-sectionally dependent but is uncorrelated with x_{it}. Let $U_t = (u_{1t}, \cdots, u_{nt})'$. The $n \times 1$ vectors $U_1, U_{2,} \cdots, U_T$ are assumed iid $N(0, \Sigma_u)$ over time. Let σ_{ij} be the (i, j)th element of the $n \times n$ matrix Σ_u. The errors u_{it} $(i = 1, ..., n; t = 1, ..., T)$ are cross-sectionally dependent if Σ_u is non-diagonal, i.e., $\sigma_{ij} \neq 0$ for $i \neq j$. The null hypothesis of cross-sectional independence can be written as:

$$H_0 : \sigma_{ij} = 0 \text{ for } i \neq j,$$

or equivalently as

$$H_0 : \rho_{ij} = 0 \text{ for } i \neq j, \tag{2}$$

where ρ_{ij} is the correlation coefficient of the errors with $\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_i^2 \sigma_j^2}}$. Under the alternative hypothesis, there is at least one non-zero correlation coefficient ρ_{ij}, i.e., $H_a : \rho_{ij} \neq 0$ for some $i \neq j$.

The OLS estimator of y_{it} on x_{it} for each i, denoted by $\hat{\beta}_i$, is consistent. The corresponding OLS residuals $\hat{u}_{it}$ defined by $\hat{u}_{it} = y_{it} - x'_{it}\hat{\beta}_i$ are used to compute the sample correlation $\breve{\rho}_{ij}$ as follows:

$$\breve{\rho}_{ij} = \left(\sum_{t=1}^{T} \hat{u}_{it}^2\right)^{-1/2} \left(\sum_{t=1}^{T} \hat{u}_{jt}^2\right)^{-1/2} \sum_{t=1}^{T} \hat{u}_{it}\hat{u}_{jt}. \tag{3}$$

In the fixed n case and as $T \to \infty$, the Breusch and Pagan's (1980) LM test can be applied to test for the cross-sectional dependence in heterogeneous panels. In this case it is given by:

$$LM_{BP} = T \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \breve{\rho}_{ij}^2.$$

This is asymptotically distributed under the null as a χ^2 with $n(n-1)/2$ degrees of freedom. However, this Breusch-Pagan LM test statistic is not applicable when $n \to \infty$. In this case, Pesaran (2004) proposes a scaled version of the LM_{BP} test given by:

$$CD_{lm} = \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(T\breve{\rho}_{ij}^2 - 1\right). \tag{4}$$

Pesaran (2004) shows that CD_{lm} is asymptotically distributed as $N(0,1)$, under the null, with $T \to \infty$ first, and then $n \to \infty$. However, as pointed out by Pesaran (2004), for finite T, $E[T\breve{\rho}_{ij}^2 - 1]$ is not correctly centered at zero, and with large n, the incorrect centering of the CD_{lm} statistic is likely to be accentuated. Thus, the standard normal distribution may be a bad approximation of the null distribution of the CD_{lm} statistic in finite samples, and using the critical values of a standard normal may lead to big size distortion. Using finite sample approximations, Pesaran, Ullah and Yamagata (2008) rescale and recenter the CD_{lm} test. The new LM test, denoted as PUY's LM test, is given by

$$\text{PUY's } LM = \sqrt{\frac{2}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{(T-k)\breve{\rho}_{ij}^2 - \mu_{Tij}}{\sigma_{Tij}}, \tag{5}$$

where

$$\mu_{Tij} = \frac{1}{T-k} tr[E(M_i M_j)]$$

is the exact mean of $(T-k)\breve{\rho}_{ij}^2$ and

$$\sigma_{Tij}^2 = \{tr[E(M_i M_j)]\}^2 a_{1T} + 2tr\{E[(M_i M_j)^2]\} a_{2T}$$

is its exact variance. Here

$$\begin{aligned} a_{1T} &= a_{2T} - \frac{1}{(T-k)^2}, \\ a_{2T} &= 3\left[\frac{(T-k-8)(T-k+2)+24}{(T-k+2)(T-k-2)(T-k-4)}\right]^2, \end{aligned}$$

$M_i = I - X_i(X_i'X_i)^{-1}X_i'$, where $X_i = (x_{i1}, \cdots, x_{iT})'$ contains T observations on the k regressors for the i-th individual regression. PUY's LM is asymptotically distributed as $N(0,1)$, under the null, with $T \to \infty$ first, and then $n \to \infty$.

This paper considers the fixed effects homogeneous panel data model

$$y_{it} = \alpha + x_{it}'\beta + \mu_i + v_{it}, \text{ for } i = 1, ..., n; \ t = 1, ..., T \tag{6}$$

where μ_i denotes the time-invariant individual effect. The $k \times 1$ regressors x_{it} could be correlated with μ_i, but are uncorrelated with the idiosyncratic error v_{it}. This is a standard model in the applied panel data literature and differs from (1) in that the β_i's are the same, and heterogeneity is introduced through the μ_i's. The intercept α appears explicitly so that the regressor vector x_{it} includes only time-variant variables. Throughout our derivations for the fixed effects model, we require the following assumptions:

Assumption 1 $\frac{n}{T} \to c \in (0, \infty\ )$ *as* $(n, T) \to \infty$.

c is a non-zero bounded constant. This assumption approximates the case where the dimension n is comparably as large as T.

For a static panel data model, we assume:

Assumption 2 *i) The* $n \times 1$ *vectors of idiosyncratic disturbances* $V_t = (v_{1t}, \cdots, v_{nT})'$, $t = 1, ..., T$, *are iid* $N(0, \Sigma_\nu)$ *over time*[1]*; ii)* $E[v_{it}|x_{i1}, ..., x_{iT}] = 0$ *and* $E[v_{it}|x_{j1}, ..., x_{jT}] = 0$, $i = 1, \cdots, n$, $t = 1, \cdots, T$*; iii) For the demeaned regressors* $\tilde{x}_{it} = x_{it} - \frac{1}{T}\sum_{s=1}^{T} x_{is}$, $\frac{1}{T}\sum_{t=1}^{T} \tilde{x}_{it}$, $\frac{1}{T}\sum_{t=1}^{T} \tilde{x}_{it}\tilde{x}_{jt}'$ *are stochastic bounded for all* $i = 1, \cdots, n$ *and* $j = 1, \cdots, n$, *and* $\lim_{(n,T)\to\infty} \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} \tilde{x}_{it}\tilde{x}_{it}'$ *exists and is nonsingular.*

The normality assumption 2.i) above may be strict but it is a standard assumption in the statistical literature and is also assumed by Pesaran, Ullah and Yamagata (2008). Other distributions will be examined for robustness checks in the Monte Carlo experiments. Assumptions 2.ii) is standard. Assumption 2.iii) excludes nonstationary or trending regressors. Under these assumptions, the within estimator $\tilde{\beta}$ is $\sqrt{nT}$-consistent. This estimator is obtained by regressing $\tilde{y}_{it} = y_{it} - \frac{1}{T}\sum_{s=1}^{T} y_{is}$ on $\tilde{x}_{it}$. The corresponding within residuals given by $\widehat{v}_{it} = \tilde{y}_{it} - \tilde{x}_{it}'\tilde{\beta}$ are used

[1] V_t and Σ_v form triangular arrays as both n and T increase. Strictly speaking, V_t (Σ_v) should be written as $V_{t,n}$ ($\Sigma_{v,n}$). To avoid index cluttering, we suppress the subscript n.

to compute the sample correlation $\hat{\rho}_{ij}$ as follows:

$$\hat{\rho}_{ij} = \left(\sum_{t=1}^{T} \hat{v}_{it}^2\right)^{-1/2} \left(\sum_{t=1}^{T} \hat{v}_{jt}^2\right)^{-1/2} \sum_{t=1}^{T} \hat{v}_{it}\hat{v}_{jt}. \tag{7}$$

For a dynamic panel data model with the lagged dependent variable as a regressor, more assumptions are needed. We will discuss this case in Section 6.

The scaled version of the LM_{BP} test suggested by Pesaran (2004) but now applied to the fixed effects model is given by:

$$LM_P = \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(T\hat{\rho}_{ij}^2 - 1\right). \tag{8}$$

This replaces $\breve{\rho}_{ij}$ with $\hat{\rho}_{ij}$ and it now tests the null given in (2) only applied to the remainder disturbance v_{it}. In order to see this, let $u_{it} = \mu_i + v_{it}$ denote the disturbances in (6). The fixed effects estimator wipes out the individual effects, and that is why it does not matter whether the μ_i's are correlated with the regressors or not. The test for no cross-sectional dependence of the disturbances given in (2) becomes a test for no cross-sectional dependence of the v_{it}. This LM_P test, for the fixed effects model (8), suffers from the same problems discussed by Pesaran (2004) for the corresponding CD_{lm} statistic (4) for the heterogeneous panel model. We show that it will exhibit substantial size distortions due to incorrect centering when n is large. Unlike the finite sample adjustment in Pesaran, Ullah and Yamagata (2008), this paper derives the asymptotic distribution of the LM_P statistic under the null as $(n, T) \to \infty$, and proposes a bias corrected LM test. The asymptotics are done using the high dimensional inference in the statistics literature, see Ledoit and Wolf (2002) and Schott (2005), to mention a few. Our derivation begins with the raw data case and then extends it to a fixed effects regression model. We find that in a fixed effects panel data model, after subtracting a constant that is a function of n and T, the LM_P test is asymptotically distributed, under the null, as a standard normal. Therefore, a bias-corrected LM test is proposed.

3 LM_P Test in the Raw Data Case

In the raw data case, the $n \times 1$ vectors $Z_1, Z_2, \cdots, Z_T$ are a random sample drawn from $N(0, \Sigma_z)$. The tth observation Z_t has n components, $Z_t = (z_{1t}, \cdots, z_{nt})'$. The null hypothesis of independence among these n components is the same as (2) but now pertaining to Σ_z rather than Σ_u. For fixed n, and as $T \to \infty$, the traditional LM test statistic is $T \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} r_{ij}^2$, which converges in distribution

to $\chi^2_{n(n-1)/2}$ under the null of independence. The sample correlation r_{ij} is defined as

$$r_{ij} = \left(\sum_{t=1}^{T} z_{it}^2\right)^{-1/2} \left(\sum_{t=1}^{T} z_{jt}^2\right)^{-1/2} \sum_{t=1}^{T} z_{it} z_{jt}. \tag{9}$$

However, as the dimension n becomes as comparably large as T, this traditional LM test becomes invalid. A scaled LM test statistic

$$LM_z = \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(T r_{ij}^2 - 1\right) \tag{10}$$

is thus considered. This LM_z statistic (10) is closely related to the test statistic proposed by Schott (2005)

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} r_{ij}^2 - \frac{n(n-1)}{2T}.$$

For high-dimensional data, as $n/T \to c \in (0, \infty)$, Schott (Theorem 1, 2005) shows that under the null of independence,

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} r_{ij}^2 - \frac{n(n-1)}{2T} \xrightarrow{d} N\left(0, \lim_{(n,T)\to\infty} \frac{n(n-1)(T-1)}{T^2(T+2)}\right)$$

or, equivalently, that

$$\sqrt{\frac{T^2(T+2)}{n(n-1)(T-1)}} \left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} r_{ij}^2 - \frac{n(n-1)}{2T}\right] \xrightarrow{d} N(0,1).$$

Using Schott's (2005) result and the fact that

$$\sqrt{\frac{T^2(T+2)}{n(n-1)(T-1)}} \left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} r_{ij}^2 - \frac{n(n-1)}{2T}\right] = \sqrt{\frac{T+2}{T-1}} LM_z,$$

it is straightforward to infer that the limiting distribution of LM_z is $N(0,1)$ under the null.[2]

4 LM_P Test in a Fixed Effects Panel Data Model

This section derives the limiting distribution of the LM_P test defined in (8). This tests the null of no cross-sectional dependence in the fixed effects regression model given in (6). The null hypothesis of no cross-sectional dependence is the same as (2) but now pertaining to Σ_ν rather than Σ_u.

Theorem 1 *Under Assumptions 1, 2 and the null hypothesis of no cross-section dependence*

$$LM_P - \frac{n}{2(T-1)} \xrightarrow{d} N(0,1).$$

[2] Srivastava (2005, Theorem 5.1) also derives the null limiting distribution of the LM_z statistic given in (10) using $T \to \infty$ and focusing on the case where $T = O(n^\delta)$ where $0 < \delta \leq 1$.

The proof of the theorem is provided in the Appendix. The asymptotics are derived under the joint asymptotics of $(n, T) \to \infty$ with $n/T \to c \in (0, \infty)$.

Based on this result, this paper proposes a bias-corrected LM test statistic given by:

$$LM_{BC} = LM_P - \frac{n}{2(T-1)} = \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(T\hat{\rho}_{ij}^2 - 1\right) - \frac{n}{2(T-1)}. \tag{11}$$

Theorem 1 shows that, under the null, the limiting distribution of the bias-corrected LM test is standard normal.

Comparing LM_P in the fixed effects model versus the corresponding LM_z in the raw data case, it is clear that LM_P exhibits an asymptotic bias, while LM_z does not. This result is similar to the analysis in Baltagi, Feng and Kao (2011) for the John test for sphericity under the panel regression model with and without fixed effects. The asymptotic bias in the fixed effects model results from the incidental parameter problem. Due to the presence of unobserved heterogeneity μ_i, the idiosyncratic error v_{it} can not be estimated accurately by the within residuals $\widehat{v}_{it} = \tilde{y}_{it} - \tilde{x}_{it}'\tilde{\beta} = v_{it} - \frac{1}{T}\sum_{s=1}^{T} v_{is} - \tilde{x}_{it}'(\tilde{\beta} - \beta)$. The second term $\frac{1}{T}\sum_{s=1}^{T} v_{is}$, created by the within transformation to wipe out the unobserved heterogeneity μ_i, is $O_p(\frac{1}{T})$. Hence, the accuracy of the within residuals depends on T. For small T, the within residuals are inaccurate, and so are the sample correlations $\hat{\rho}_{ij}$'s computed using the within residuals. For large T, the terms involved with odd power of $\frac{1}{T}\sum_{s=1}^{T} v_{is}$ vanish due to the law of large numbers. However, the sum of a large number of squared terms of $\frac{1}{T}\sum_{s=1}^{T} v_{is}$ can not be ignored. The inaccuracy due to the within transformation accumulates in the sum of squared terms of the statistic with comparably large n and $n/T \to c \in (0, \infty)$, consequently, resulting in asymptotic bias.

5 Monte Carlo Simulations

This section employs Monte Carlo simulations to examine the empirical size and power of our bias-corrected LM test defined in (11) in a static panel data model. We compare its performance with that of Pesaran's (2004) CD test given by

$$\text{Pesaran's } CD = \sqrt{\frac{2T}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \breve{\rho}_{ij},$$

and PUY's LM test given in (5). The sample correlations $\breve{\rho}_{ij}$ are computed using OLS residuals, see (3). We also include the John test for the null of sphericity discussed by Baltagi, Feng and Kao (2011). Sphericity means that Σ_ν is proportional to the identity matrix. The John test statistic is

given by

$$J = \frac{T(\frac{1}{n}tr\hat{S})^{-2}\frac{1}{n}tr(\hat{S}^2) - T - n}{2} - \frac{1}{2} - \frac{n}{2(T-1)}$$

where $\hat{S} = \frac{1}{T}\sum_{t=1}^{T}\hat{V}_t\hat{V}_t'$ is the $n \times n$ sample covariance matrix computed using the within residuals $\hat{V}_t = (\hat{v}_{1t}, ..., \hat{v}_{nt})'$. $tr\hat{S}$ is the trace of the matrix $\hat{S}$. Under normality and homoskedasticity, the John test can be used to test for cross-sectional dependence. However, John's test is not robust to heteroskedasticity, and rejection of the null hypothesis using the John test could be due to heteroskedasticity or cross-sectional dependence. For this reason we include the John test in our experiments only under the homoskedastic case. The John test is not recommended for testing cross-section dependence when heteroskedasticity is present.[3]

5.1 Experiment Design

The experiments use the following data generating process:

$$y_{it} = \alpha + \beta x_{it} + \mu_i + v_{it},\ i = 1, \cdots, n;\ t = 1, \cdots, T, \tag{12}$$

$$x_{it} = \zeta x_{i,t-1} + \mu_i + \eta_{it}. \tag{13}$$

Following Im, Ahn, Schmidt and Wooldridge (1999) x_{it} in (13) is correlated with the μ_i, but not with v_{it}.

To calculate the power of the tests considered, two different models of the cross-sectional dependence are used: a factor model and a spatial model. In the former, it is assumed that

$$v_{it} = \gamma_i f_t + \varepsilon_{it}, \tag{14}$$

where f_t $(t = 1, \cdots, T)$ are the factors and γ_i $(i = 1, \cdots, n)$ are the loadings. In a spatial model, we consider a first-order spatial autocorrelation (SAR(1) in (15)) and a spatial moving average (SMA(1) in (16)) model as follows:

$$v_{it} = \delta(0.5v_{i-1,t} + 0.5v_{i+1,t}) + \varepsilon_{it}, \tag{15}$$

$$v_{it} = \delta(0.5\varepsilon_{i-1,t} + 0.5\varepsilon_{i+1,t}) + \varepsilon_{it}. \tag{16}$$

[3] It is also important to note that for the raw data case, Kapetanios (2004) suggests transforming the data by dividing by the estimated standard deviations for each individual before applying the John test. In this case, the sample correlation matrix will be the same as the sample covariance matrix after transforming the data. One can show the asymptotic equivalence between the Kapetanios (2004) two-step John test and the LM_z test given in (10). Both test statistics are based on the sum of the squared off-diagonal sample correlations, but with different scale parameters.

Cross-sectional dependence can also be modelled by including a spatially lagged dependent variable, denoted as the mixed regressive, spatial autoregressive (MRSAR) model:

$$y_{it} = \alpha + \delta(0.5y_{i-1,t} + 0.5y_{i+1,t}) + \beta x_{it} + \mu_i + v_{it}, \tag{17}$$

where, similar to the SAR(1) model in (15), the term $\delta(0.5y_{i-1,t} + 0.5y_{i+1,t})$ represents the spatial interaction in the dependent variable. The null can be regarded as a special case of $\gamma_i = 0$ in the factor model (14) and $\delta = 0$ in the spatial models (15), (16) and (17).

v_{it} (under the null) and ε_{it} (under the alternative) are from $iidN(0, \sigma_i^2)$. To model the heteroskedasticity, we follow Baltagi, Song and Kwon (2009) and Roy (2002) and assume that

$$\sigma_i^2 = \sigma^2(1 + \theta\bar{x}_i)^2, \tag{18}$$

where $\bar{x}_i$ is the individual mean of x_{it}. Here θ is assigned values 0, 0.5 with $\theta = 0$ denoting the homoskedastic case. For non-zero θ, we fix the average value of σ_i^2 across i as 0.5 in our experiments. We obtain the value of $\sigma^2 = 0.5/\left[\frac{1}{n}\sum_{i=1}^{n}(1 + \theta\bar{x}_i)^2\right]$ using (18) and subsequently the value of σ_i^2. For the case of $\theta = 0$, $\sigma_i^2 = \sigma^2$ is fixed at 0.5.

The parameters α and β are set arbitrarily to 1 and 2 respectively. μ_i is drawn from $iidN(\phi_\mu, \sigma_\mu^2)$ with $\phi_\mu = 0$ and $\sigma_\mu^2 = 0.25$ for $i = 1, \cdots, n$. For the regressor in (13), $\zeta = 0.7$ and $\eta_{it} \sim iidN(\phi_\eta, \sigma_\eta^2)$ with $\phi_\eta = 0$ and $\sigma_\eta^2 = 1$. For the factor model in (14) $f_t \sim iidN(0, 1)$ and two sets of experiments are conducted for $\gamma_i \sim iidU(-0.5, 0.55)$ and $\gamma_i \sim iidU(0.1, 0.3)$. For the spatial model $\delta = 0.4$ in (15), (16) and (17).

The Monte Carlo experiments are conducted for $n = 5, 10, 20, 30, 50, 100, 200$ and $T = 10, 20, 30, 50$. For each replication, we compute the bias-corrected LM test, Pesaran's CD, the John test and PUY's LM test. 2,000 replications are performed. To obtain the empirical size, the proposed bias-corrected LM test, PUY's LM and the John tests are conducted at the positive one-sided 5% nominal significance level, while Pesaran's CD test is implemented at the two-sided 5% nominal significance level.

5.2 Results

Tables 1 and 2 present the empirical size of these tests under the null of cross-sectional independence with homoskedasticity ($\theta = 0$) and heteroskedasticity ($\theta = 0.5$), respectively. The size of the bias-corrected LM test is close to 5%, even for micro panels with small T and large n. For example, the size of the bias-corrected LM test is 4.1% and 5.1% for $n = 200$ and $T = 10$, under homoskedasticity

and heteroskedasticity, respectively. The simulation results are consistent with the asymptotic theory given in Theorem 1 in Section 4. As discussed in Pesaran, Ullah and Yamagata (2008), for large T there is no bias issue, so PUY's LM test has the correct size for large T. For large n and small T, it is slightly oversized. For example, the size of PUY's LM test is 7.9% and 9.2% for $T = 10$, $n = 200$ under homoskedasticity and heteroskedasticity, respectively. Pesaran's CD test has the correct size for all combinations of n and T.[4] The size of the John test is also reported in Table 1 for comparison purposes. It performs well except for micro panels, in which case the John test is oversized under homoskedasticity.

Table 3 shows the size adjusted power of these tests under the alternative specified by a factor model. The bias-corrected LM test has bigger size adjusted power than PUY's LM test for small T. However, both tests have size adjusted power that is almost 1 when n and T are larger than 20. By contrast, the power of Pesaran's CD test is much smaller than those of the two LM tests. While the power of the LM tests becomes one for large n and T, the power of the CD test reaches a maximum of 36.5% for $n = 200$ and $T = 50$ when γ_i is drawn from $U(-0.5, 0.55)$. This is expected under the current design. As pointed out by Pesaran, Ullah and Yamagata (2008), in the factor model above in (14), $Cov(v_{it}, v_{jt}) = E[\gamma_i]E[\gamma_j]$, implying that the value of Pesaran's CD test statistic is close to zero if the mean of γ_i is zero. This explains the low power of Pesaran's CD test when γ_i is drawn from $U(-0.5, 0.55)$. However, this is not the case for the proposed LM and PUY's LM tests which involve the squared terms of sample correlation coefficients. For the case of γ_i drawn from $U(0.1, 0.3)$, the power of Pesaran's CD test increases to 1 with n or T.

Tables 4, 5 and 6 give the size adjusted power of these tests under the alternative specifications of SAR(1), SMA(1) and MRSAR, respectively. In these cases, the size adjusted power of Pesaran's CD test performs much better than in the case of a factor model, increasing to 1 with T.[5]

[4] Pesaran's CD test is designed for hetergeneous panels and is based on the sample correlation of the residuals of individual heterogeneous OLS regressions. We performed the experiments again using the CD test but computed with fixed effects residuals. Pesaran's CD test always has correct size for all combinations of n and T under the homoskedastic case. However, it is a little oversized under heteroskedasticity for large n and small T.

[5] Tables 4 and 5 show that in the SAR(1) and SMA(1) models, the size-adjusted power of tests is low when n is relatively large and T is small. For example, in the SAR (1) model, when $T = 10, n = 200$, the size-adjusted power is 73.6%, 45.6% and 52.9% for the proposed bias-corrected LM, PUY's LM and Pesaran's CD tests, respectively. However, when T gets large, the size-adjusted power of these tests increases to 1. By contrast, Table 6 shows that in the MRSAR model, the size-adjusted power of these two LM tests is large and increases to 1 with n no matter whether T is small or large.

The power of the tests under the spatial model depends upon the spatial autocorrelation parameter δ. For example, for $\delta = 0.8$, in the SAR (1) model, when $T = 10$, $n = 100$, the size-adjusted power is 100%, 100% and 93.9% for the proposed bias-corrected LM, PUY's LM and Pesaran's CD tests, respectively. Pesaran (2004) discusses the local power of the CD test in factor model and SAR(1) model. Similarly, one can investigate the asymptotic power of the proposed bias-corrected LM under different alternatives. Since our proposed test statistic is based on the sum of

Table 7 provides the results of robustness check on the size of the tests with some non-normal or asymmetric distributions on the errors. We ran experiments with uniform distribution $U[1,2]$, Chi-square distribution with 1 degree of freedom, χ_1^2, and t-distribution with 5 degrees of freedom, $t(5)$, and we compare these results with those of Gaussian case $N(0, 0.5)$. By and large, these experiments show that the size of the bias-corrected LM, PUY's LM and Pesaran's CD tests are not that sensitive to the normality assumption on the errors. The same results obtain although the magnitude are different. PUY's LM test is still oversized around 8% for large $n = 100$, small $T = 10$ no matter what distribution is used. The bias-corrected LM test has size close to 5% for the uniform and t distributions and is a little oversized for $T \geq 10$ when using the χ_1^2 distribution.[6]

6 Dynamic Panel Data Models

In a dynamic panel data model

$$y_{it} = \alpha + \xi y_{i,t-1} + x_{it}'\beta + \mu_i + v_{it}, \text{ for } i = 1, ..., n; t = 1, ..., T \tag{19}$$

$y_{i,t-1}$ is the lagged dependent variable. As documented by Nickell (1981), the within estimator is inconsistent for finite T as $N \to \infty$. Various consistent estimators have been proposed in the literature, including Anderson and Hsiao (1981), Arellano and Bond (1991), Kiviet (1995), Bun and Carree (2005), Phillips and Sul (2007) etc., to name a few. For a detailed discussion, see Baltagi (2008). Recently, Hahn and Kuersteiner (2002) studied the asymptotic properties of the within estimator in a dynamic panel model with fixed effects when n and T grow at the same rate. They show, after a bias-correction, the within estimator is $\sqrt{nT}$-consistent.

For the dynamic panel data model in (19), let us denote $\theta = (\xi, \beta')'$. Based on the bias-corrected estimator $\widehat{\widehat{\theta}}$ proposed by Hahn and Kuersteiner (2002), we can compute the within residuals $\widehat{v}_{it} = \tilde{y}_{it} - (\tilde{y}_{i,t-1}, \tilde{x}_{it}')\widehat{\widehat{\theta}}$ with $\tilde{y}_{i,t-1} = y_{i,t-1} - \frac{1}{T}\sum_{s=1}^{T} y_{i,s-1}$, and the corresponding sample correlations $\hat{\rho}_{ij}$ and the bias-corrected LM test statistic (LM_{BC}).[7] We show that as long as $\widehat{\widehat{\theta}}$ is $\sqrt{nT}$-consistent,

squared sample correlations constructed from within residuals, these derivations will be quite involved and are not pursued in this paper.

[6] We performed Monte Carlo simulations where the true DGP is a heterogeneous panel. When T is large ($T = 50$) and n is small ($n = 10$), the size of the proposed bias-corrected LM, PUY's LM and Pesaran's CD test is $5.6, 5.4$ and 5.5 respectively. When n is relatively larger than T, our simulations show that the proposed bias-corrected LM test is not robust to slope heterogeneity. For example, the size of our proposed bias-corrected LM test is 13.4% for $T = 30$ and $n = 50$. The proposed test used in the heterogeneous panels is actually CD_{lm} minus $\frac{n}{2(T-1)}$. When T is much larger than n, the CD_{lm} test has size close to the nominal level. Since the term $\frac{n}{2(T-1)}$ is small in this case, the estimated size of the proposed test is also close to the nominal level.

[7] Theorem 1 of Hahn and Kuersteiner (2002) shows that the limiting distribution of $\sqrt{nT}(\tilde{\theta} - \theta)$, where $\tilde{\theta}$ denotes the within estimator, is not centered at zero when both n and T are large. Due to this noncentrality, we find in Monte

the proposed LM_{BC} test in the dynamic panel data model still has standard normal limiting distribution under the null. However, stronger assumptions are needed than the static panel data model. In particular,

Assumption 3 *i)* $\sqrt{nT}(\widehat{\widehat{\theta}} - \theta) = O_p(1)$; *ii)* $|\xi| < 1$; *iii)* $\frac{1}{n}\sum_{i=1}^{n} y_{i,0}^2 = O_p(1)$ *and* $\frac{1}{n}\sum_{i=1}^{n} \mu_i^2 = O_p(1)$; *iv)* $\frac{1}{T}\sum_{s=1}^{T}\sum_{\tau=1}^{s-1} \xi^{\tau-1} x_{i,s-\tau} = O_p(1)$ *and* $\frac{1}{T}\sum_{s=1}^{T}\sum_{\tau=1}^{s-1} \xi^{\tau-1} v_{i,s-\tau} = O_p(T^{-1/2})$.

Assumption 3.iii) is the same as condition 4.iv) in Hahn and Kuersteiner (2002). It implies $y_{i,0} = O_p(1)$ and $\mu_i = O_p(1)$. Under Assumptions 3.iii), iv) the dependent variable y_{it} and its time average $\frac{1}{T}\sum_{t=1}^{T} y_{i,t}$ are stochastically bounded.

Theorem 2 *Under Assumptions 1, 2, 3 and the null hypothesis of no cross-section dependence*

$$LM_{BC} \xrightarrow{d} N(0,1).$$

Under Assumption 3.iii), the proof follows along the same lines as that of Theorem 1. See the Appendix.

To examine the finite sample properties of the proposed bias-corrected LM test in a dynamic panel data model, we follow the same design as that of Hahn and Kuersteiner (2002):

$$y_{it} = \alpha + \xi y_{i,t-1} + \mu_i + v_{it},\ i = 1, ..., n;\ t = -50, -49, ..., 0, 1, ..., T,$$

where v_{it} is assumed $N(0,1)$ independent across i and t, $\mu_i \sim N(0,1)$, $y_{i0}|\mu_i \sim N\left(\frac{\mu_i}{1-\xi}, \frac{Var(v_{it})}{1-\xi^2}\right)$ and $\xi = \{0.3, 0.6, 0.9\}$. For this model, Hahn and Kuersteiner (2002) propose a bias-corrected estimator $\widehat{\widehat{\xi}} = \frac{T+1}{T}\widehat{\xi} + \frac{1}{T}$, where $\widehat{\xi}$ is the within estimator of ξ. Hahn and Kuersteiner (2002) show that $\sqrt{nT}(\widehat{\widehat{\xi}} - \xi) \xrightarrow{d} N(0, 1-\xi^2)$. In our Monte Carlo experiments, heteroskedasticity of v_{it} is allowed. In fact, $v_{it} \sim N(0, \sigma_i^2)$ where $\sigma_i^2 \sim \chi^2(2)/2$ as in the dynamic setup of Pesaran, Ullah and Yamagata (2008). The first 50 observations are discarded to lessen the effects of the initial values of y_{i0} on the results.

Table 8 reports the size of the tests for the dynamic panel data model. It shows that the proposed bias-corrected LM test has the correct size, close to the 5% nominal significance level, e.g., 5.1% and 5.4% for $n = 100$, $T = 10$ and $n = 200$, $T = 10$ in the case of $\xi = 0.3$. For the cases of $\xi = 0.3, 0.6$, it is slightly oversized for $n = 200$, $T = 10$. The PUY's LM test tends to over-reject

Carlo experiments that the proposed bias-corrected LM test using the within estimator is oversized in micro panels when n is much larger than T. This is why we use the bias-corrected estimator $\widehat{\widehat{\theta}}$ proposed by Hahn and Kuersteiner (2002).

in micro panels with large n and small T, and this fact is also observed in Table 6 of Pesaran, Ullah and Yamagata (2008). Pesaran's CD has correct size as in Pesaran (2004) and Pesaran, Ullah and Yamagata (2008).[8]

7 Conclusion

This paper derives the limiting distribution of the scaled version of LM test proposed by Pesaran (2004) but applied to a fixed effects model. We find that this LM test exhibits an asymptotic bias which is related to the number of cross-sectional units n and the number of time periods T. Therefore, a bias-corrected LM test is proposed and its finite sample properties are examined using Monte Carlo experiments. The simulation results show that the bias-corrected LM test successfully controls for size distortions as n gets large relative to T. It also maintains reasonable power under the alternative of a factor model or a spatial SAR, SMA, or MRSAR models. However, our proposed LM test is not robust to slope heterogeneity. More importantly, the simulation results show that the bias-corrected LM test can be applied in typical micro panel data case with large n and small T. The asymptotic distribution of our proposed LM test is derived under the normality assumption and no time series dependence. While these are indeed restrictive assumptions, they are needed because they are also assumed in the statistics literature of high dimensional inference for the raw data case. To our knowledge, these asymptotic results have not been extended in the statistics literature to deal with non-normality.

[8] We also tried the dynamic setup in Pesaran, Ullah and Yamagata (2008) except that a homogeneous slope is assumed. We conducted experiments with $\xi = 0.3, 0.6, 0.9$. The results are similar to those in Table 8 and are available as Table 9 upon request from the authors.

Table 1: Size of Tests under Homoskedasticity (θ=0)

Size	$T \setminus n$	5	10	20	30	50	100	200
Bias-corrected *LM*	10	6.0	4.8	4.5	4.8	4.4	5.3	4.1
	20	5.3	4.7	5.4	5.2	5.4	4.7	4.9
	30	5.3	6.4	5.4	5.8	6.0	4.6	5.2
	50	5.8	5.5	5.5	6.1	6.4	5.1	4.8
PUY's *LM*	10	7.1	6.1	6.0	6.5	6.9	8.4	7.9
	20	5.5	5.3	6.4	6.4	6.5	5.4	6.7
	30	5.7	7.0	5.3	5.8	6.3	4.4	4.7
	50	5.7	5.6	5.5	6.8	6.1	5.5	4.6
Pesaran's *CD*	10	6.0	5.3	5.0	5.2	4.8	5.5	6.8
	20	5.4	5.2	5.5	4.7	5.2	4.8	4.9
	30	5.1	4.7	5.7	5.0	4.9	4.7	5.0
	50	4.9	5.7	5.5	4.3	5.1	4.4	4.8
John	10	5.8	6.5	7.2	6.5	6.7	9.0	7.0
	20	5.4	6.1	6.5	6.0	6.6	6.4	5.8
	30	5.1	6.3	6.4	6.6	6.7	5.8	5.9
	50	4.9	5.4	5.7	6.1	6.9	6.5	5.5

Note: This table reports the size of the bias-corrected LM test, Pesaran, Ullah and Yamagata (2008) (PUY) LM test, Pesaran's (2004) CD and the John test, in a fixed effects panel data model specified in Section 5. The bias-corrected LM, PUY's LM and John tests are one-sided, while Pesaran's CD is two-sided. These tests are conducted at the 5% nominal significance level. Homoskedasticity and normality of the idiosyncratic errors are assumed.

Table 2: Size of Tests under Heteroskedasticity (θ=0.5)

Size	$T \setminus n$	5	10	20	30	50	100	200
Bias-corrected *LM*	10	5.4	5.5	5.8	5.4	6.2	5.9	5.1
	20	5.6	6.3	5.0	4.8	6.2	5.5	5.4
	30	6.5	5.5	5.0	6.1	6.0	6.1	5.3
	50	5.8	6.0	5.4	5.9	5.1	5.7	4.3
PUY's *LM*	10	6.7	6.9	5.9	6.1	6.5	7.3	9.2
	20	6.4	6.3	5.6	6.0	7.2	5.2	6.7
	30	7.0	6.0	4.8	6.0	5.5	5.8	5.7
	50	6.7	6.5	5.8	5.5	4.7	5.3	4.5
Pesaran's *CD*	10	4.9	5.9	5.0	4.9	5.9	5.3	5.4
	20	4.9	5.5	5.3	5.8	4.5	4.7	4.9
	30	5.5	5.1	5.0	6.2	5.1	5.3	4.8
	50	5.0	5.3	5.1	4.8	4.4	4.2	5.4

Note: This table reports the size of the bias-corrected LM test, Pesaran, Ullah and Yamagata (2008) (PUY) LM test and Pesaran's (2004) CD test, in a fixed effects panel data model specified in Section 5. The bias-corrected LM and PUY's LM tests are one-sided, while Pesaran's CD is two-sided. These tests are conducted at the 5% nominal significance level. Heteroskedasticity and normality of the idiosyncratic errors are assumed. The form of heteroskedasticity is specified in Section 5.

Table 3: Size Adjusted Power of Tests: Factor Model

	Size Adjusted Power	$T \setminus n$	5	10	20	30	50	100	200
$\gamma_i \sim iid$ U(-0.5, 0.55)	Bias-corrected *LM*	10	23.8	50.4	82.1	92.9	99.2	99.9	100.0
		20	50.4	82.9	98.5	100.0	100.0	100.0	100.0
		30	61.9	93.2	99.7	100.0	100.0	100.0	100.0
		50	79.1	98.1	100.0	100.0	100.0	100.0	100.0
	PUY's *LM*	10	21.6	44.8	77.9	88.9	98.0	99.7	100.0
		20	49.0	81.7	98.2	99.8	100.0	100.0	100.0
		30	60.5	93.0	99.7	100.0	100.0	100.0	100.0
		50	78.2	97.3	100.0	100.0	100.0	100.0	100.0
	Pesaran's *CD*	10	7.6	7.8	8.0	8.7	9.2	10.6	13.8
		20	16.4	14.2	13.7	12.6	13.3	17.7	21.5
		30	18.0	17.8	17.9	18.4	18.9	22.1	27.2
		50	26.4	25.8	27.1	29.1	29.3	32.8	36.5
$\gamma_i \sim iid$ U(0.1, 0.3)	Bias-corrected *LM*	10	15.3	35.5	64.8	83.3	95.0	99.2	100.0
		20	33.6	68.8	95.6	98.9	100.0	100.0	100.0
		30	46.5	83.4	98.9	100.0	100.0	100.0	100.0
		50	66.7	93.2	99.9	100.0	100.0	100.0	100.0
	PUY's *LM*	10	14.7	29.2	59.6	76.2	91.9	98.0	100.0
		20	33.5	68.7	94.1	98.8	99.9	100.0	100.0
		30	46.3	83.6	98.7	100.0	100.0	100.0	100.0
		50	65.3	92.8	99.9	100.0	100.0	100.0	100.0
	Pesaran's *CD*	10	20.8	51.4	86.5	96.6	99.7	100.0	100.0
		20	42.6	83.3	99.1	99.9	100.0	100.0	100.0
		30	52.8	93.2	100.0	100.0	100.0	100.0	100.0
		50	72.3	98.6	100.0	100.0	100.0	100.0	100.0

Note: This table computes the size adjusted power for a factor structure model that allows for cross-sectional dependence in the errors. Heteroskedasticity is assumed, see Section 5.

Table 4: Size Adjusted Power of Tests: SAR (1) Model

Size Adjusted Power	$T \setminus n$	5	10	20	30	50	100	200
Bias-corrected *LM*	10	62.4	66.0	65.8	68.3	68.2	69.9	73.6
	20	96.0	98.1	99.4	99.9	99.8	99.9	100.0
	30	99.5	100.0	100.0	100.0	100.0	100.0	100.0
	50	100.0	100.0	100.0	100.0	100.0	100.0	100.0
PUY's *LM*	10	57.5	54.9	55.8	53.4	54.3	54.6	45.6
	20	95.4	97.5	98.8	99.4	99.1	99.7	100.0
	30	99.3	100.0	100.0	100.0	100.0	100.0	100.0
	50	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Pesaran's *CD*	10	70.5	59.4	55.6	53.7	52.6	53.9	52.9
	20	94.5	88.6	84.2	83.7	84.2	86.0	83.4
	30	98.5	97.0	95.9	94.4	95.6	95.5	96.1
	50	100.0	100.0	99.8	99.6	99.8	99.7	99.8

Note: This table computes the size adjusted power for a SAR(1) structure model that allows for cross-sectional dependence in the errors. Heteroskedasticity is assumed, see Section 5.

Table 5: Size Adjusted Power of Tests: SMA (1) Model

Size Adjusted Power	$T \setminus n$	5	10	20	30	50	100	200
Bias-corrected *LM*	10	50.3	52.3	53.0	53.0	50.8	52.3	57.4
	20	92.3	95.2	97.7	97.8	97.7	99.0	99.0
	30	99.2	99.9	100.0	100.0	100.0	100.0	100.0
	50	100.0	100.0	100.0	100.0	100.0	100.0	100.0
PUY's *LM*	10	45.1	40.1	45.4	41.8	40.9	40.6	33.6
	20	90.0	93.2	96.0	95.9	95.8	97.0	95.9
	30	98.4	99.8	100.0	100.0	100.0	100.0	100.0
	50	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Pesaran's *CD*	10	46.8	40.7	38.2	37.3	35.6	36.9	37.3
	20	80.5	70.9	66.7	63.1	65.9	69.1	66.4
	30	90.8	87.6	84.2	81.8	80.9	78.4	80.2
	50	99.5	98.1	97.3	97.0	96.1	97.3	96.8

Note: This table computes the size adjusted power for a SMA(1) structure model that allows for cross-sectional dependence in the errors. Heteroskedasticity is assumed, see Section 5.

Table 6: Size Adjusted Power of Tests: MRSAR Model

Size Adjusted Power	$T \setminus n$	5	10	20	30	50	100	200
Bias-corrected *LM*	10	64.5	70.1	86.6	96.3	99.7	100.0	100.0
	20	97.6	99.0	100.0	100.0	100.0	100.0	100.0
	30	99.9	100.0	100.0	100.0	100.0	100.0	100.0
	50	100.0	100.0	100.0	100.0	100.0	100.0	100.0
PUY's *LM*	10	47.7	31.5	41.7	49.9	65.0	84.9	96.7
	20	94.7	94.5	99.4	99.8	100.0	100.0	100.0
	30	99.7	100.0	100.0	100.0	100.0	100.0	100.0
	50	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Pesaran's *CD*	10	78.3	61.3	60.1	54.9	52.6	54.7	53.2
	20	98.9	89.9	84.9	80.8	81.5	83.5	82.7
	30	100.0	97.1	94.7	93.0	92.5	92.5	92.6
	50	100.0	99.7	99.3	99.5	99.3	99.5	99.0

Note: This table computes the size adjusted power for a mixed regressive, spatial autoregressive (MRSAR) structure model that allows for cross-sectional dependence in the errors. Heteroskedasticity is assumed, see Section 5.

Table 7 :Size of Tests: Robustness to Non-normal Errors

		N(0,0.5)			U[1,2]			Chi2(1)			t(5)		
Size	*T* \ *n*	20	50	100	20	50	100	20	50	100	20	50	100
Bias-corrected *LM*	10	5.8	6.2	5.9	5.6	6.2	6.0	6.5	7.4	6.8	6.1	5.7	5.8
	30	5.0	6.0	6.1	5.3	5.4	5.6	7.8	7.5	8.7	6.1	6.0	5.6
PUY's *LM*	10	5.9	6.5	7.3	5.9	6.9	8.3	7.1	7.4	7.9	6.4	8.0	7.6
	30	4.8	5.5	5.8	6.2	5.6	5.5	8.3	7.1	8.0	5.9	5.9	6.2
Pesaran's *CD*	10	5.0	5.9	5.3	4.7	5.7	5.5	5.5	5.2	4.8	4.9	4.5	5.7
	30	5.0	5.1	5.3	4.8	4.7	4.4	4.7	4.4	4.6	5.3	5.0	4.0

Note: In order to check the sensitivity of the tests to non-normal disturbances, the uniform distribution U[1,2], Chi-square distribution with 1 degree of freedom, Chi2(1) and t-distribution with 5 degrees of freedom, t(5) are considered. The normal case is also presented for comparison. The form of heteroskedasticity is specified in Section 5.

Table 8: Size of Tests: a Dynamic Panel Data Model

	Size	$T \setminus n$	5	10	20	30	50	100	200
ξ=0.3	Bias-corrected *LM*	10	5.3	5.8	5.5	4.5	5.6	5.1	5.4
		20	6.5	4.9	5.1	5.5	5.5	4.8	5.0
		30	6.2	6.2	5.6	4.8	5.7	4.8	4.5
		50	6.1	6.1	5.0	5.1	5.1	5.6	5.2
	PUY's *LM*	10	7.2	7.6	9.0	9.9	15.9	29.5	65.5
		20	6.4	5.7	7.2	6.9	7.9	11.1	17.8
		30	7.5	6.1	5.9	6.2	7.4	7.9	8.8
		50	6.0	6.3	6.2	5.4	6.3	6.9	7.0
	Pesaran's *CD*	10	6.5	5.9	5.5	6.2	5.0	6.1	4.5
		20	5.1	5.4	4.5	5.1	5.3	5.1	5.7
		30	5.1	4.6	5.7	5.6	5.1	5.5	5.7
		50	5.2	5.0	4.0	5.0	4.5	4.9	5.4
ξ=0.6	Bias-corrected *LM*	10	4.1	5.2	5.1	4.4	5.2	5.5	6.3
		20	4.9	5.3	4.2	4.7	5.7	5.4	4.9
		30	4.9	4.9	4.6	5.1	5.0	5.2	5.1
		50	6.4	5.1	5.3	5.7	4.8	5.3	5.9
	PUY's *LM*	10	7.4	9.1	11.5	12.4	22.0	42.8	84.6
		20	6.0	6.9	6.0	7.9	9.6	17.9	36.3
		30	6.3	5.8	6.7	7.4	8.2	11.0	17.8
		50	6.7	6.0	6.5	6.9	5.7	7.4	7.8
	Pesaran's *CD*	10	6.2	6.0	5.5	4.6	5.1	5.2	5.7
		20	5.9	5.7	7.1	5.5	6.0	6.4	5.0
		30	6.3	5.3	5.2	4.7	4.9	4.5	5.5
		50	4.6	5.7	5.5	5.5	4.5	5.1	4.5
ξ=0.9	Bias-corrected *LM*	10	5.4	5.3	5.1	5.7	5.2	6.4	7.5
		20	6.2	6.2	5.2	5.0	5.9	5.9	6.5
		30	6.0	5.5	4.6	5.5	5.7	5.8	6.2
		50	6.2	5.6	5.5	5.7	5.3	6.1	4.5
	PUY's *LM*	10	6.7	7.3	9.1	12.1	15.9	29.6	57.9
		20	8.0	7.7	9.7	10.6	14.6	29.0	63.7
		30	7.1	7.0	7.4	8.9	11.9	20.1	47.5
		50	6.9	6.7	6.8	6.9	7.8	13.9	22.0
	Pesaran's *CD*	10	5.8	6.3	5.6	6.6	4.6	5.2	4.3
		20	5.3	6.4	5.0	5.1	5.7	5.4	6.0
		30	5.7	5.1	4.9	5.0	5.1	5.5	5.6
		50	5.0	6.0	5.3	5.0	4.5	5.9	5.1

Note: This table reports the size of the bias-corrected LM test, Pesaran, Ullah and Yamagata (PUY) (2008) LM test and Pesaran's (2004) CD test, in a dynamic panel data model with fixed effects specified in Section 6. This setup follows the Monte Carlo design of Hahn and Kruersteiner (2002), except that heteroskedasticity is allowed here. Hahn and Kuersteiner estimator of the autoregressive parameter is used to compute the proposed bias-corrected LM test. The bias-corrected LM and PUY's LM tests are one-sided N(0,1), while Pesaran's CD is two-sided. These tests are conducted at the 5% nominal significance level.

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Appendix

This appendix includes the proofs of the main results in the text.

In the fixed-effects model $y_{it} = \alpha + x_{it}'\beta + \mu_i + v_{it}$, $\tilde{\beta}$ is the within estimator and the within residuals are given by $\widehat{v}_{it} = \tilde{y}_{it} - \tilde{x}_{it}'\tilde{\beta}$, where $\tilde{y}_{it} = y_{it} - \bar{y}_{i\cdot}$ and $\tilde{x}_{it} = x_{it} - \bar{x}_{i\cdot}$, with $\bar{y}_{i\cdot} = \frac{1}{T}\sum_{s=1}^{T} y_{is}$, and $\bar{x}_{i\cdot}$ similar defined. Define $\tilde{v}_{it} = v_{it} - \bar{v}_{i\cdot}$ with $\bar{v}_{i\cdot} = \frac{1}{T}\sum_{s=1}^{T} v_{is}$. The within residuals can be written as $\hat{v}_{it} = \tilde{v}_{it} - \tilde{x}_{it}'(\tilde{\beta} - \beta)$. Let $V_i = (v_{i1}, \cdots, v_{iT})'$, $\hat{V}_i = (\hat{v}_{i1}, \cdots, \hat{v}_{iT})'$, $\bar{V}_i = (\bar{v}_{i\cdot}, \cdots, \bar{v}_{i\cdot})'$, $X_i = (x_{i1}, \cdots, x_{iT})'$, $\tilde{X}_i = (\tilde{x}_{i1}, \cdots, \tilde{x}_{iT})', Y_i = (y_{i1}, \cdots, y_{iT})'$, $\tilde{Y}_i = (\tilde{y}_{i1}, \cdots, \tilde{y}_{iT})'$ for $i = 1, \cdots, n$. In vector form,

$$\hat{V}_i = V_i - \bar{V}_i - \tilde{X}_i(\tilde{\beta} - \beta). \tag{20}$$

Using this notation, the sample correlation r_{ij} in the raw data case can be written as

$$r_{ij} = \frac{V_i'V_j}{(V_i'V_i)^{1/2}(V_j'V_j)^{1/2}} \tag{21}$$

and its sample counterpart using within residuals in the fixed effects model is given by

$$\hat{\rho}_{ij} = \frac{\hat{V}_i'\hat{V}_j}{(\hat{V}_i'\hat{V}_i)^{1/2}(\hat{V}_j'\hat{V}_j)^{1/2}}. \tag{22}$$

Dividing $\hat{v}_{it}$ by σ_i, we obtain

$$\frac{\hat{v}_{it}}{\sigma_i} = \frac{v_{it}}{\sigma_i} - \frac{1}{T}\sum_{s=1}^{T}\frac{v_{is}}{\sigma_i} - (\frac{\tilde{x}_{it}}{\sigma_i})'(\tilde{\beta} - \beta).$$

As shown below, the terms involving $(\frac{\tilde{x}_{it}}{\sigma_i})'(\tilde{\beta} - \beta)$ have no effect on the test statistic asymptotically. Without loss of generality, σ_i is assumed to be 1 in the derivations below. Under Assumption 2, $\frac{1}{T}\tilde{X}_i'\tilde{X}_i = O_p(1)$, $\frac{1}{T}\tilde{X}_i'\tilde{X}_j = O_p(1)$ and $(\tilde{\beta} - \beta) = O_p((nT)^{-1/2})$. In addition, we need the following lemma in the proofs below.

Lemma 1 *Under Assumptions 1, 2 and the null,*

1) $\frac{1}{T}V_i'V_i = 1 + O_p(T^{-1/2})$;

2) $\frac{1}{T}V_i'V_j = O_p(T^{-1/2})$ for $i \neq j$;

3) $\frac{1}{T}\bar{V}_i'\bar{V}_i = \frac{1}{T}V_i'\bar{V}_i = O_p(T^{-1})$;

4) $\frac{1}{T}\bar{v}_{i\cdot}\bar{v}_{j\cdot} = O_p(T^{-2})$;

5) $\frac{1}{T}\tilde{X}_i'V_i = O_p(T^{-1/2})$;

6) $\frac{1}{T}\tilde{X}_i'\bar{V}_i = O_p(T^{-1/2})$;

7) $\frac{1}{T}\tilde{X}_j'V_i = O_p(T^{-1/2})$;

8) $\frac{1}{T}\tilde{X}_j'\bar{V}_i = O_p(T^{-1/2})$.

Proof. To calculate the order of magnitude of a random variable, we can use Lemma 1 in Baltagi, Feng and Kao (2011). Specifically, for a random sequence $\{Z_n\}$, if $EZ_n^2 = O(n^\nu)$ and $EZ_n = 0$, where ν is a constant, then $Z_n = O_p(n^{\nu/2})$. Using this result, we can prove this lemma by calculating the order of magnitude of the second moments of random variables.

1)

$$\frac{1}{T}V_i'V_i - 1 = \frac{1}{T}\sum_{t=1}^{T}(v_{it}^2 - 1) = O_p(T^{-1/2}).$$

2)

$$\frac{1}{T}V_i'V_j = \frac{1}{T}\sum_{t=1}^{T} v_{it}v_{jt} = O_p(T^{-1/2}).$$

3) Since $\frac{1}{T}V_i'\bar{V}_i = \frac{1}{T}\sum_{t=1}^{T} v_{it}\bar{v}_{i\cdot} = (\bar{v}_{i\cdot})^2 = \frac{1}{T}\bar{v}_i'\bar{v}_i$, the order of magnitude of $\frac{1}{T}\bar{V}_i'\bar{V}_i$ can be obtained as follows:

$$\frac{1}{T}\bar{V}_i'\bar{V}_i = \bar{v}_{i\cdot}^2 = \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} v_{it}v_{is} = \frac{1}{T^2}\sum_{t=1}^{T} v_{it}^2 + \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s\neq t}^{T} v_{it}v_{is} = O_p(T^{-1}). \tag{23}$$

4) For $i \neq j$,

$$\bar{v}_{i\cdot}\bar{v}_{j\cdot} = \left(\frac{1}{T}\sum_{t=1}^{T} v_{it}\right)\left(\frac{1}{T}\sum_{s=1}^{T} v_{js}\right) = O_p(T^{-1/2})O_p(T^{-1/2}) = O_p(T^{-1}).$$

5) Suppose $k = 1$,

$$\frac{1}{T}\tilde{X}_i'V_i = \frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{it}v_{it} = O_p(T^{-1/2}).$$

6) Under Assumption 2, $\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{it} = O_p(1)$,

$$\frac{1}{T}\tilde{X}_i'\bar{V}_i = \frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{it}\bar{v}_{i\cdot} = \left(\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{it}\right)\left(\frac{1}{T}\sum_{s=1}^{T} v_{is}\right) = O_p(1)O_p(T^{-1/2}) = O_p(T^{-1/2}).$$

7) Similar to 5), since v_{it} is uncorrelated with $\tilde{x}_{jt}$,

$$\frac{1}{T}\tilde{X}_j'V_i = \frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{jt}v_{it} = O_p(T^{-1/2}).$$

8) Similar to 6),

$$\frac{1}{T}\tilde{X}_j'\bar{V}_i = \frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{jt}\bar{v}_{i\cdot} = (\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{jt})(\frac{1}{T}\sum_{s=1}^{T} v_{is}) = O_p(1)O_p(T^{-1/2}) = O_p(T^{-1/2}).$$

■

Lemma 2 *Under Assumptions 1, 2 and the null,*

1) $\hat{V}_i'\hat{V}_i = V_i'V_i - \bar{V}_i'\bar{V}_i + E_i$, where $E_i = -2(\tilde{\beta}-\beta)'\tilde{X}_i'V_i + 2(\tilde{\beta}-\beta)'\tilde{X}_i'\bar{V}_i + (\tilde{\beta}-\beta)'\tilde{X}_i'\tilde{X}_i(\tilde{\beta}-\beta) = O_p(n^{-1/2})$;

2) $\hat{V}_i'\hat{V}_j = V_i'V_j - \bar{V}_i'\bar{V}_j + F$, where $F = -(\tilde{\beta}-\beta)'\tilde{X}_j'V_i + (\tilde{\beta}-\beta)'\tilde{X}_j'\bar{V}_i - (\tilde{\beta}-\beta)'\tilde{X}_i'V_j + (\tilde{\beta}-\beta)'\tilde{X}_i'\bar{V}_j + (\tilde{\beta}-\beta)'\tilde{X}_i'\tilde{X}_j(\tilde{\beta}-\beta) = O_p(n^{-1/2})$.

Proof. 1) Using (20), we have

$$\begin{aligned}
\frac{1}{T}\hat{V}_i'\hat{V}_i &= \frac{1}{T}[V_i - \bar{V}_i - \tilde{X}_i(\tilde{\beta}-\beta)]'[V_i - \bar{V}_i - \tilde{X}_i(\tilde{\beta}-\beta)] \\
&= \frac{1}{T}[V_i'V_i - 2V_i'\bar{V}_i - 2(\tilde{\beta}-\beta)'\tilde{X}_i'V_i + \bar{V}_i'\bar{V}_i + 2(\tilde{\beta}-\beta)'\tilde{X}_i'\bar{V}_i + (\tilde{\beta}-\beta)'\tilde{X}_i'\tilde{X}_i(\tilde{\beta}-\beta)] \\
&= \frac{1}{T}V_i'V_i - \frac{1}{T}\bar{V}_i'\bar{V}_i - 2(\tilde{\beta}-\beta)'\frac{1}{T}\tilde{X}_i'V_i + 2(\tilde{\beta}-\beta)'\frac{1}{T}\tilde{X}_i'\bar{V}_i + (\tilde{\beta}-\beta)'\frac{1}{T}\tilde{X}_i'\tilde{X}_i(\tilde{\beta}-\beta).
\end{aligned}$$

By Lemma 1,

$$\begin{aligned}
(\tilde{\beta}-\beta)'\frac{1}{T}\tilde{X}_i'V_i &= O_p((nT)^{-1/2})O_p(T^{-1/2}) = O_p(n^{-1/2}T^{-1}); \\
(\tilde{\beta}-\beta)'\frac{1}{T}\tilde{X}_i'\bar{V}_i &= O_p((nT)^{-1/2})O_p(T^{-1/2}) = O_p(n^{-1/2}T^{-1}); \\
(\tilde{\beta}-\beta)'\frac{1}{T}\tilde{X}_i'\tilde{X}_i(\tilde{\beta}-\beta) &= O_p((nT)^{-1/2})O_p(1)O_p((nT)^{-1/2}) = O_p(n^{-1}T^{-1}).
\end{aligned}$$

Thus,

$$\hat{V}_i'\hat{V}_i = V_i'V_i - \bar{V}_i'\bar{V}_i + E_i,$$

where $E_i = -2(\tilde{\beta}-\beta)'\tilde{X}_i'V_i + 2(\tilde{\beta}-\beta)'\tilde{X}_i'\bar{V}_i + (\tilde{\beta}-\beta)'\tilde{X}_i'\tilde{X}_i(\tilde{\beta}-\beta) = O_p(n^{-1/2})$.

2) Similarly, $\hat{V}_i'\hat{V}_j$ can be written as:

$$\begin{aligned}
\hat{V}_i'\hat{V}_j &= [V_i - \bar{V}_i - \tilde{X}_i(\tilde{\beta}-\beta)]'[V_j - \bar{V}_j - \tilde{X}_j(\tilde{\beta}-\beta)] \\
&= V_i'V_j - V_i'\bar{V}_j - V_i'\tilde{X}_j(\tilde{\beta}-\beta) - \bar{V}_i'V_j + \bar{V}_i'\bar{V}_j + \bar{V}_i'\tilde{X}_j(\tilde{\beta}-\beta) \\
&\quad -(\tilde{\beta}-\beta)'\tilde{X}_i'V_j + (\tilde{\beta}-\beta)'\tilde{X}_i'\bar{V}_j + (\tilde{\beta}-\beta)'\tilde{X}_i'\tilde{X}_j(\tilde{\beta}-\beta) \\
&= V_i'V_j - T\bar{v}_{i\cdot}\bar{v}_{j\cdot} - (\tilde{\beta}-\beta)'\tilde{X}_j'V_i + (\tilde{\beta}-\beta)'\tilde{X}_j'\bar{V}_i \\
&\quad -(\tilde{\beta}-\beta)'\tilde{X}_i'V_j + (\tilde{\beta}-\beta)'\tilde{X}_i'\bar{V}_j + (\tilde{\beta}-\beta)'\tilde{X}_i'\tilde{X}_j(\tilde{\beta}-\beta).
\end{aligned}$$

By Lemma 1, $T\bar{v}_{i\cdot}\bar{v}_{j\cdot} = O_p(1)$ and

$$\begin{aligned}
&-(\tilde{\beta}-\beta)'\tilde{X}_j'V_i + (\tilde{\beta}-\beta)'\tilde{X}_j'\bar{V}_i - (\tilde{\beta}-\beta)'\tilde{X}_i'V_j + (\tilde{\beta}-\beta)'\tilde{X}_i'\bar{V}_j + (\tilde{\beta}-\beta)'\tilde{X}_i'\tilde{X}_j(\tilde{\beta}-\beta) \\
&= O_p((nT)^{-1/2})O_p(T^{1/2}) + O_p((nT)^{-1/2})O_p(T)O_p((nT)^{-1/2}) \\
&= O_p(n^{-1/2}),
\end{aligned}$$

thus, we obtain

$$\hat{V}_i'\hat{V}_j = V_i'V_j - \bar{V}_i'\bar{V}_j + F,$$

where $F = -(\tilde{\beta}-\beta)'\tilde{X}_j'V_i + (\tilde{\beta}-\beta)'\tilde{X}_j'\bar{V}_i - (\tilde{\beta}-\beta)'\tilde{X}_i'V_j + (\tilde{\beta}-\beta)'\tilde{X}_i'\bar{V}_j + (\tilde{\beta}-\beta)'\tilde{X}_i'\tilde{X}_j(\tilde{\beta}-\beta) = O_p(n^{-1/2})$. ■

Lemma 3 *Under Assumptions 1, 2 and the null,*

1) $(\hat{V}_i'\hat{V}_j)^2-(\hat{V}_i'\hat{V}_i)(\hat{V}_j'\hat{V}_j)(V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)]=G+H$, where $G=(\bar{V}_i'\bar{V}_j)^2-2(V_i'V_j)(\bar{V}_i'\bar{V}_j)+(\bar{V}_j'\bar{V}_j)(V_i'V_j)^2/(V_j'V_j)+(\bar{V}_i'\bar{V}_i)(V_i'V_j)^2/(V_i'V_i)-(\bar{V}_i'\bar{V}_i)(\bar{V}_j'\bar{V}_j)(V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)]+2(V_i'V_j)F=O_p(1)+O_p(\sqrt{\frac{T}{n}})$, and $H=F^2-2(\bar{V}_i'\bar{V}_j)F-[(V_i'V_i)E_j-(\bar{V}_i'\bar{V}_i)E_j+(V_j'V_j)E_i-(\bar{V}_j'\bar{V}_j)E_i+E_iE_j](V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)]=O_p(n^{-1/2})$;

2) $(\frac{\hat{V}_i'\hat{V}_i}{T})(\frac{\hat{V}_j'\hat{V}_j}{T})=(1-\frac{1}{T})^2+O_p(T^{-1/2})$.

Proof. 1) Using Lemma 2, we obtain,

$$
\begin{aligned}
& (\hat{V}_i'\hat{V}_j)^2-(\hat{V}_i'\hat{V}_i)(\hat{V}_j'\hat{V}_j)(V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)] && (24)\\
= \;& (V_i'V_j-\bar{V}_i'\bar{V}_j+F)^2-[V_i'V_i-\bar{V}_i'\bar{V}_i+E_i][V_j'V_j-\bar{V}_j'\bar{V}_j+E_j](V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)]\\
= \;& (V_i'V_j)^2+(\bar{V}_i'\bar{V}_j)^2+F^2-2(V_i'V_j)(\bar{V}_i'\bar{V}_j)+2(V_i'V_j)F-2(\bar{V}_i'\bar{V}_j)F\\
& -[(V_i'V_i)(V_j'V_j)-(V_i'V_i)(\bar{V}_j'\bar{V}_j)+(V_i'V_i)E_j-(\bar{V}_i'\bar{V}_i)(V_j'V_j)+(\bar{V}_i'\bar{V}_i)(\bar{V}_j'\bar{V}_j)\\
& -(\bar{V}_i'\bar{V}_i)E_j+(V_j'V_j)E_i-(\bar{V}_j'\bar{V}_j)E_i+E_iE_j](V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)]\\
= \;& (\bar{V}_i'\bar{V}_j)^2-2(V_i'V_j)(\bar{V}_i'\bar{V}_j)+(\bar{V}_j'\bar{V}_j)(V_i'V_j)^2/(V_j'V_j)+(\bar{V}_i'\bar{V}_i)(V_i'V_j)^2/(V_i'V_i) && (25)\\
& -(\bar{V}_i'\bar{V}_i)(\bar{V}_j'\bar{V}_j)(V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)]+2(V_i'V_j)F && (26)\\
& +F^2-2(\bar{V}_i'\bar{V}_j)F && (27)\\
& -[(V_i'V_i)E_j-(\bar{V}_i'\bar{V}_i)E_j+(V_j'V_j)E_i-(\bar{V}_j'\bar{V}_j)E_i+E_iE_j](V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)]. && (28)
\end{aligned}
$$

By denoting G the sum of expressions (25) and (26), and denoting H the sum of expressions (27) and (28), we obtain $(\hat{V}_i'\hat{V}_j)^2-(\hat{V}_i'\hat{V}_i)(\hat{V}_j'\hat{V}_j)(V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)]=G+H$.

By Lemma 1,

$$
\begin{aligned}
G &= (\bar{V}_i'\bar{V}_j)^2-2(V_i'V_j)(\bar{V}_i'\bar{V}_j)+(\bar{V}_j'\bar{V}_j)(V_i'V_j)^2/(V_j'V_j)+(\bar{V}_i'\bar{V}_i)(V_i'V_j)^2/(V_i'V_i)\\
&\quad -(\bar{V}_i'\bar{V}_i)(\bar{V}_j'\bar{V}_j)(V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)]+2(V_i'V_j)F\\
&= O_p(T^{-2})+O_p(T^{1/2})O_p(T^{-1})+O_p(T^{-1})O_p(T)+O_p(T^{-1})O_p(T)\\
&\quad +O_p(T^{-1})O_p(T^{-1})O_p(T)+O_p(T^{1/2})O_p(n^{-1/2})\\
&= O_p(1)+O_p(\sqrt{\frac{T}{n}})
\end{aligned}
$$

and

$$
\begin{aligned}
H &= F^2-2(\bar{V}_i'\bar{V}_j)F-[(V_i'V_i)E_j-(\bar{V}_i'\bar{V}_i)E_j+(V_j'V_j)E_i-(\bar{V}_j'\bar{V}_j)E_i+E_iE_j](V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)]\\
&= O_p(n^{-1})+O_p(T^{-1})O_p(n^{-1/2})+[T(1+O_p(T^{-1/2}))O_p(n^{-1/2})+O_p(1)O_p(n^{-1/2})\\
&\quad +T(1+O_p(T^{-1/2}))O_p(n^{-1/2})+O_p(1)O_p(n^{-1/2})+O_p(n^{-1})]\frac{O_p(T^{-1})}{1+O_p(T^{-1/2})}\\
&= O_p(n^{-1/2}). && (29)
\end{aligned}
$$

Note that the term H contains terms involved with F, E_i and E_j. We will show that this term vanishes asymptotically.

2) As in the proof of Lemma 1,

$$\begin{aligned}
& \frac{V_i'V_i}{T} - \frac{\bar{V}_i'\bar{V}_i}{T} \\
= & \frac{1}{T}\sum_{t=1}^{T} v_{it}^2 - \frac{1}{T^2}\sum_{t=1}^{T} v_{it}^2 + \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s\neq t}^{T} v_{it}v_{is} \\
= & (1-\frac{1}{T}) + O_p(T^{-1/2}).
\end{aligned}$$

By Lemma 2, it follows that

$$\begin{aligned}
& (\frac{\hat{V}_i'\hat{V}_i}{T})(\frac{\hat{V}_j'\hat{V}_j}{T}) \\
= & (\frac{V_i'V_i}{T} - \frac{\bar{V}_i'\bar{V}_i}{T} + \frac{E_i}{T})(\frac{V_j'V_j}{T} - \frac{\bar{V}_j'\bar{V}_j}{T} + \frac{E_j}{T}) \\
= & \left[(1-\frac{1}{T}) + O_p(T^{-1/2}) + O_p(n^{-1/2}T^{-1})\right]\left[(1-\frac{1}{T}) + O_p(T^{-1/2}) + O_p(n^{-1/2}T^{-1})\right] \\
= & (1-\frac{1}{T})^2 + O_p(T^{-1/2}). \qquad (30)
\end{aligned}$$

■

Lemma 4 *Under the Assumptions 1, 2 and the null,*

a) $\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T} v_{it}^2 v_{jt}^2 = \frac{n(n-1)}{2T} + O_p(\frac{n\sqrt{n}}{T\sqrt{T}})$;

b) $\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{\tau\neq t}^{T} v_{it}^2 v_{jt}v_{j\tau} = O_p(\frac{n\sqrt{n}}{T})$;

c) $\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T} v_{it}v_{is}v_{jt}^2 = O_p(\frac{n\sqrt{n}}{T})$;

d) $\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T} v_{it}v_{jt}v_{is}v_{js} = O_p(\frac{n}{T})$;

e) $\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\sum_{\tau\neq t\neq s}^{T} v_{it}v_{jt}v_{is}v_{j\tau} = O_p(\frac{n}{\sqrt{T}})$.

Proof. a)

$$
\begin{aligned}
& \frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T} v_{it}^2 v_{jt}^2 \\
= & \frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\left[(v_{it}^2-1)(v_{jt}^2-1)+(v_{it}^2-1)+(v_{jt}^2-1)+1\right] \\
= & \frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}(v_{it}^2-1)(v_{jt}^2-1)+\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}(v_{it}^2-1) \\
& +\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}(v_{jt}^2-1)+\frac{n(n-1)}{2T} \\
= & \frac{n(n-1)}{2T}+O_p(\frac{n}{T\sqrt{T}})+O_p(\frac{n\sqrt{n}}{T\sqrt{T}}) \\
= & \frac{n(n-1)}{2T}+O_p(\frac{n\sqrt{n}}{T\sqrt{T}}).
\end{aligned}
$$

b) Since

$$
\begin{aligned}
& E\left[\left(\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{\tau\neq t}^{T} v_{it}^2 v_{jt} v_{j\tau}\right)^2\right] \\
= & \frac{1}{T^4}E\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{\tau\neq t}^{T}\sum_{i1=1}^{n-1}\sum_{j1=i1+1}^{n}\sum_{t1=1}^{T}\sum_{\tau1\neq t1}^{T} v_{it}^2 v_{jt} v_{j\tau} v_{i1t1}^2 v_{1jt1} v_{j1\tau1}\right] \\
= & \frac{1}{T^4}E\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{\tau\neq t}^{T}\sum_{i1=1}^{n-1} v_{it}^2 v_{jt}^2 v_{j\tau}^2 v_{i1t1}^2\right] \\
= & \frac{1}{T^4}O(n^3T^2)=O(\frac{n^3}{T^2}),
\end{aligned}
$$

we obtain

$$
\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{\tau\neq t}^{T} v_{it}^2 v_{jt} v_{j\tau}=O_p(\frac{n\sqrt{n}}{T}).
$$

c) Similar to the proof of b),

$$
\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T} v_{it} v_{is} v_{jt}^2=O_p(\frac{n\sqrt{n}}{T}).
$$

d) Since

$$
\begin{aligned}
& E\left[\left(\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T} v_{it} v_{jt} v_{is} v_{js}\right)^2\right] \\
= & \frac{1}{T^4}E\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\sum_{i1=1}^{n-1}\sum_{j1=i1+1}^{n}\sum_{t1=1}^{T}\sum_{s1\neq t1}^{T} v_{it} v_{jt} v_{is} v_{js} v_{i1t1} v_{j1t1} v_{i1s1} v_{j1s1}\right] \\
= & \frac{1}{T^4}E\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T} v_{it}^2 v_{jt}^2 v_{is}^2 v_{js}^2\right] \\
= & O(\frac{1}{T^4}n^2T^2)=O(\frac{n^2}{T^2}),
\end{aligned}
$$

we have

$$\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T} v_{it}v_{jt}v_{is}v_{js} = O_p(\frac{n}{T}).$$

e) Similarly, since

$$\begin{aligned}
& E\left[\left(\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\sum_{\tau\neq t\neq s}^{T} v_{it}v_{jt}v_{is}v_{j\tau}\right)^2\right] \\
= & \frac{1}{T^4}E\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\sum_{\tau\neq t\neq s}^{T}\sum_{i1=1}^{n-1}\sum_{j1=i1+1}^{n}\sum_{t1=1}^{T}\sum_{s1\neq t1}^{T}\sum_{\tau 1\neq t1\neq s1}^{T} v_{it}v_{jt}v_{is}v_{j\tau}v_{i1t1}v_{j1t1}v_{i1s1}v_{j1\tau 1}\right] \\
= & \frac{1}{T^4}E\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\sum_{\tau\neq t\neq s}^{T} v_{it}^2v_{jt}^2v_{is}^2v_{j\tau}^2\right] \\
= & \frac{1}{T^4}O(n^2T^3) = O(\frac{n^2}{T}),
\end{aligned}$$

we have

$$\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\sum_{\tau\neq t\neq s}^{T} v_{it}v_{jt}v_{is}v_{j\tau} = O_p(\frac{n}{\sqrt{T}}).$$

■

Lemma 5 *Under Assumptions 1, 2 and the null,*

1). $\sqrt{\frac{1}{n(n-1)}}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{T}(V_i'V_j)(\bar{V}_i'\bar{V}_j) = \sqrt{\frac{1}{n(n-1)}}\left[\frac{n(n-1)}{2T} + O_p(\frac{n\sqrt{n}}{T}) + O_p(\frac{n}{\sqrt{T}})\right]$;

2). $\sqrt{\frac{1}{n(n-1)}}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{T}(\bar{V}_i'\bar{V}_j)^2 = \sqrt{\frac{1}{n(n-1)}}\left[\frac{n(n-1)}{2T} + O_p(\frac{n\sqrt{n}}{T})\right]$;

3). $\sqrt{\frac{1}{n(n-1)}}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{T}(\bar{V}_j'\bar{V}_j)(V_i'V_j)^2/(V_j'V_j) = \sqrt{\frac{1}{n(n-1)}}\left[\frac{n(n-1)(T+2)}{2T^2} + O_p(\frac{n\sqrt{n}}{T})\right]$;

4). $\sqrt{\frac{1}{n(n-1)}}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{T}(\bar{V}_i'\bar{V}_i)(V_i'V_j)^2/(V_i'V_i) = \sqrt{\frac{1}{n(n-1)}}\left[\frac{n(n-1)(T+2)}{2T^2} + O_p(\frac{n\sqrt{n}}{T})\right]$;

5). $\sqrt{\frac{1}{n(n-1)}}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{T}(\bar{V}_i'\bar{V}_i)(\bar{V}_j'\bar{V}_j)(V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)] = \sqrt{\frac{1}{n(n-1)}}\left[\frac{n(n-1)(T^2+20T+60)}{2T^4} + O_p(\frac{n\sqrt{n}}{T^2\sqrt{T}})\right]$;

6). $\sqrt{\frac{1}{n(n-1)}}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{T}(V_i'V_j)F = \sqrt{\frac{1}{n(n-1)}}\left[O_p(\frac{n}{T}) + O_p(\sqrt{\frac{n}{T}})\right]$.

The proof of part 1) is given below. Part 2) through part 6) can be shown in the same way. The proofs are included in the Supplementary Appendix which is available upon request from the authors.

Proof of 1).

$$
\begin{aligned}
& \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{T}(V_i'V_j)(\bar{V}_i'\bar{V}_j) \\
= & \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}v_{it}v_{jt}\right)T\left(\frac{1}{T}\sum_{s=1}^{T}v_{is}\right)\left(\frac{1}{T}\sum_{\tau=1}^{T}v_{j\tau}\right) \\
= & \frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{\tau=1}^{T}v_{it}v_{jt}v_{is}v_{j\tau}. \qquad (31)
\end{aligned}
$$

There are 5 cases of (t,s,τ): (1) $t=s=\tau$; (2) $(t=s)\neq\tau$; (3) $(t=\tau)\neq s$; (4) $t\neq(s=\tau)$; (5) $t\neq s\neq\tau$. We can write

$$
\begin{aligned}
& \frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{\tau=1}^{T}v_{it}v_{jt}v_{is}v_{j\tau} \\
= & \frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}v_{it}^2v_{jt}^2+\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{\tau\neq t}^{T}v_{it}^2v_{jt}v_{j\tau}+\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T}v_{it}v_{is}v_{jt}^2 \\
& +\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T}v_{it}v_{jt}v_{is}v_{js}+\frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\sum_{\tau\neq t\neq s}^{T}v_{it}v_{jt}v_{is}v_{j\tau}.
\end{aligned}
$$

Using Lemma 4, we get

$$
\begin{aligned}
& \frac{1}{T^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{\tau=1}^{T}v_{it}v_{jt}v_{is}v_{j\tau} \\
= & \frac{n(n-1)}{2T}+O_p(\frac{n}{T\sqrt{T}})+O_p(\frac{n\sqrt{n}}{T})+O_p(\frac{n\sqrt{n}}{T})+O_p(\frac{n}{T})+O_p(\frac{n}{\sqrt{T}}) \\
= & \frac{n(n-1)}{2T}+O_p(\frac{n\sqrt{n}}{T})+O_p(\frac{n}{\sqrt{T}}).
\end{aligned}
$$

■

Now we are in good position to prove Theorem 1.

Proof of Theorem 1. It is equivalent to show that for large n and T,

$$
LM(\hat{\rho}_{it})-LM(r_{it})-\frac{n}{2(T-1)}=o_p(1).
$$

By (21), (22) and Lemma 3,

$$
\begin{aligned}
& LM(\hat{\rho}_{ij}) - LM(r_{it}) \\
= & \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (T\hat{\rho}_{ij}^2 - 1) - \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (Tr_{ij}^2 - 1) \\
= & \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} T \frac{(\hat{V}_i'\hat{V}_j)^2 - (\hat{V}_i'\hat{V}_i)(\hat{V}_j'\hat{V}_j)(V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)]}{(\hat{V}_i'\hat{V}_i)(\hat{V}_j'\hat{V}_j)} \\
= & \frac{1}{(1-\frac{1}{T})^2} \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\frac{1}{T}(G+H)}{(\hat{V}_i'\hat{V}_i/T)(\hat{V}_j'\hat{V}_j/T)/(1-\frac{1}{T})^2} \\
= & \frac{1}{(1-\frac{1}{T})^2} \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{G}{T} + \frac{1}{(1-\frac{1}{T})^2} \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{H}{T} \\
& + \frac{1}{(1-\frac{1}{T})^2} \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left[\frac{1}{(\hat{V}_i'\hat{V}_i/T)(\hat{V}_j'\hat{V}_j/T)/(1-\frac{1}{T})^2} - 1 \right] \frac{1}{T}(G+H).
\end{aligned}
$$

Using $H = O_p(n^{-1/2})$ from (29), the second term above can be written as follows:

$$
\frac{1}{(1-\frac{1}{T})^2} \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{H}{T} = \frac{1}{(1-\frac{1}{T})^2} \frac{1}{T} \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} O_p(n^{-1/2}) = O_p(\frac{\sqrt{n}}{T}).
$$

By Lemma 3, $(\frac{1}{T}\hat{V}_i'\hat{V}_i)(\frac{1}{T}\hat{V}_j'\hat{V}_j) = (1-\frac{1}{T})^2 + O_p(T^{-1/2})$, it follows that $\frac{1}{(\hat{V}_i'\hat{V}_i/T)(\hat{V}_j'\hat{V}_j/T)/(1-\frac{1}{T})^2} - 1 = O_p(T^{-1/2})$. Thus, it is straightforward to calculate the order of magnitude of the third term,

$$
\begin{aligned}
& \frac{1}{(1-\frac{1}{T})^2} \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left[\frac{1}{(\hat{V}_i'\hat{V}_i/T)(\hat{V}_j'\hat{V}_j/T)/(1-\frac{1}{T})^2} - 1 \right] \frac{G+H}{T} \\
= & \frac{1}{(1-\frac{1}{T})^2} \frac{1}{T} \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} O_p(T^{-1/2})[O_p(1) + O_p(\sqrt{\frac{T}{n}}) + O_p(n^{-1/2})] \\
= & O_p(\frac{n}{T\sqrt{T}}) + O_p(\frac{\sqrt{n}}{T}).
\end{aligned}
$$

Now we consider the first term,

$$
\begin{aligned}
& \frac{1}{(1-\frac{1}{T})^2} \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{G}{T} \\
= & \frac{1}{(1-\frac{1}{T})^2} \sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{T}[(\bar{V}_i'\bar{V}_j)^2 - 2(V_i'V_j)(\bar{V}_i'\bar{V}_j) + (\bar{V}_j'\bar{V}_j)(V_i'V_j)^2/(V_j'V_j) \\
& + (\bar{V}_i'\bar{V}_i)(V_i'V_j)^2/(V_i'V_i) - (\bar{V}_i'\bar{V}_i)(\bar{V}_j'\bar{V}_j)(V_i'V_j)^2/[(V_i'V_i)(V_j'V_j)] + 2(V_i'V_j)F].
\end{aligned}
$$

By Lemma 5,

$$
\begin{aligned}
& \frac{1}{(1-\frac{1}{T})^2}\sqrt{\frac{1}{n(n-1)}}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{G}{T} \\
= & \frac{1}{(1-\frac{1}{T})^2}\sqrt{\frac{1}{n(n-1)}}[-2\frac{n(n-1)}{2T}+O_p(\frac{n\sqrt{n}}{T})+O_p(\frac{n}{\sqrt{T}}) \\
& +\frac{n(n-1)}{2T}+O_p(\frac{n\sqrt{n}}{T}) \\
& +\frac{n(n-1)(T+2)}{2T^2}+O_p(\frac{n\sqrt{n}}{T}) \\
& +\frac{n(n-1)(T+2)}{2T^2}+O_p(\frac{n\sqrt{n}}{T}) \\
& -\frac{n(n-1)(T^2+20T+60)}{2T^4}+O_p(\frac{n\sqrt{n}}{T^2\sqrt{T}}) \\
& +O_p(\frac{n}{T})+O_p(\sqrt{\frac{n}{T}})]. \qquad (32)
\end{aligned}
$$

For large n and T, the expression above (32) can be approximated by

$$
\begin{aligned}
& \frac{1}{(1-\frac{1}{T})^2}(-2\frac{n}{2T}+\frac{n}{2T}+\frac{n}{2T}+\frac{n}{2T}-\frac{n}{2T^2})+O_p(\frac{n}{T^2})+O_p(\frac{\sqrt{n}}{T})+O_p(\frac{1}{\sqrt{T}}) \\
= & \frac{n}{2(T-1)}+O_p(\frac{n}{T^2})+O_p(\frac{\sqrt{n}}{T})+O_p(\frac{1}{\sqrt{T}}).
\end{aligned}
$$

Combining these 3 terms, we obtain

$$
\begin{aligned}
& LM(\hat{\rho}_{it})-LM(r_{it}) \\
= & [\frac{n}{2(T-1)}+O_p(\frac{n}{T^2})+O_p(\frac{\sqrt{n}}{T})+O_p(\frac{1}{\sqrt{T}})]+O_p(\frac{\sqrt{n}}{T})+[O_p(\frac{n}{T\sqrt{T}})+O_p(\frac{\sqrt{n}}{T})] \\
= & \frac{n}{2(T-1)}+O_p(\frac{n}{T^2})+O_p(\frac{\sqrt{n}}{T})+O_p(\frac{1}{\sqrt{T}}).
\end{aligned}
$$

Therefore, as $(n,T)\to\infty$ with $n/T\to c\in(0,\infty)$,

$$
LM(\hat{\rho}_{ij})-LM(r_{it})-\frac{n}{2(T-1)}\xrightarrow{p}0.
$$

■

Proof of Theorem 2

For the dynamic panel data model (19), the sample correlations $\hat{\rho}_{ij}$ used in the bias-corrected LM test statistic LM_{BC} are computed using the within residuals $\widehat{v}_{it} = \tilde{y}_{it} - (\tilde{y}_{i,t-1},\, \tilde{x}'_{it})\widehat{\widehat{\theta}}$ where $\widehat{\widehat{\theta}}$ is the bias-corrected estimator proposed by Hahn and Kuersteiner (2002). Denote the regressors in (19) by $w_{it} = (y_{i,t-1},\, x'_{it})'$ and the demeaned regressors by $\tilde{w}_{it} = (\tilde{y}_{i,t-1},\, \tilde{x}'_{it})'$. Let $\tilde{W}_i = (\tilde{w}_{i1}, \cdots, \tilde{w}_{iT})'$. Under Assumption 3.i) $\sqrt{nT}(\widehat{\widehat{\theta}} - \theta) = O_p(1)$. Replacing $\tilde{\beta}$ and $\tilde{X}_i$ with $\widehat{\widehat{\theta}}$ and $\tilde{W}_i$, the proof of Theorem 2 follows along the same lines as above. We need to verify that Lemmas 1.5, 1.6, 1.7, 1.8 and Lemma 3.6 still hold for the dynamic panel data model.

Lemma 5. Under Assumptions 1, 2, 3 and the null,

1). $\frac{1}{T}\tilde{W}'_i V_i = O_p(T^{-1/2})$;

2). $\frac{1}{T}\tilde{W}'_i \bar{V}_i = O_p(T^{-1/2})$;

3). $\frac{1}{T}\tilde{W}'_j V_i = O_p(T^{-1/2})$;

4). $\frac{1}{T}\tilde{W}'_j \bar{V}_i = O_p(T^{-1/2})$;

5). $\sqrt{\frac{1}{n(n-1)}} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{T}(V'_i V_j)F = \sqrt{\frac{1}{n(n-1)}}\left[O_p(\frac{n}{T}) + O_p(\sqrt{\frac{n}{T}})\right]$, where $F = -(\widehat{\widehat{\theta}} - \theta)'\tilde{W}'_j V_i + (\widehat{\widehat{\theta}} - \theta)'\tilde{W}'_j \bar{V}_i - (\widehat{\widehat{\theta}} - \theta)'\tilde{W}'_i V_j + (\widehat{\widehat{\theta}} - \theta)'\tilde{W}'_i \bar{V}_j + (\widehat{\widehat{\theta}} - \theta)'\tilde{W}'_i \tilde{W}_j (\widehat{\widehat{\theta}} - \theta)$.

Proof. In (19), the within residual is given by $\widehat{v}_{it} = \tilde{y}_{it} - \tilde{w}'_{it}\widehat{\widehat{\theta}}$, $\tilde{w}_{it} = (\tilde{y}_{i,t-1},\, \tilde{x}'_{it})'$ with $\tilde{y}_{i,t-1} = y_{i,t-1} - \frac{1}{T}\sum_{s=1}^{T} y_{i,s-1}$. From (19),

$$
\begin{aligned}
y_{i,s-1} &= \alpha + \xi y_{i,s-2} + x'_{i,s-1}\beta + \mu_i + v_{i,s-1} \\
&= \alpha + \xi(\alpha + \xi y_{i,s-3} + x'_{i,s-2}\beta + \mu_i + v_{i,s-2}) + x'_{i,s-1}\beta + \mu_i + v_{i,s-1} \\
&= \xi^2 y_{i,s-3} + (\alpha + \mu_i)(1+\xi) + (x'_{i,s-1}\beta + x'_{i,s-2}\beta\xi) + (v_{i,s-1} + \xi v_{i,s-2}) \\
&= \xi^2(\alpha + \xi y_{i,s-4} + x'_{i,s-3}\beta + \mu_i + v_{i,s-3}) + (\alpha + \mu_i)(1+\xi) + (x'_{i,s-1}\beta + x'_{i,s-2}\beta\xi) \\
&\quad + (v_{i,s-1} + \xi v_{i,s-2}) \\
&= \cdots \\
&= (\alpha + \mu_i)(1 + \xi + \cdots + \xi^{s-2}) + (x'_{i,s-1}\beta + x'_{i,s-2}\beta\xi + \cdots + x'_{i,1}\beta\xi^{s-2}) \\
&\quad + (v_{i,s-1} + \xi v_{i,s-2} + \cdots + \xi^{s-2} v_{i,1}) + \xi^{s-1} y_{i,0}
\end{aligned}
$$

so,

$$
\begin{aligned}
&\frac{1}{T}\sum_{s=1}^{T} y_{i,s-1} \\
= \;& \frac{(\alpha + \mu_i)}{T}\sum_{s=1}^{T}\sum_{\tau=0}^{s-2}\xi^{\tau} + \frac{1}{T}\beta'\left(\sum_{s=1}^{T}\sum_{\tau=0}^{s-2}\xi^{\tau} x_{i,s-\tau-1}\right) + \frac{1}{T}\sum_{s=1}^{T}\sum_{\tau=0}^{s-2}\xi^{\tau} v_{i,s-\tau-1} + \frac{y_{i,0}}{T}\frac{1-\xi^{T-1}}{1-\xi}.
\end{aligned}
$$

Under Assumption 3, $\frac{1}{T}\sum_{s=1}^{T} y_{i,s-1} = O_p(1)$.

Proof of 1): Without loss of generality, assume $k = 1$,

$$\frac{1}{T}\tilde{W}_i'V_i = \frac{1}{T}\sum_{t=1}^{T}(\tilde{y}_{i,t-1},\ \tilde{x}_{it})v_{it} = \left(\frac{1}{T}\sum_{t=1}^{T}\tilde{y}_{i,t-1}v_{it},\ \frac{1}{T}\sum_{t=1}^{T}\ \tilde{x}_{it}v_{it}\right).$$

Under Assumption 2, as in the proof of Lemma 1.5 in the static model above, $\frac{1}{T}\sum_{t=1}^{T}\ \tilde{x}_{it}v_{it} = O_p(T^{-1/2})$. Consider

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\tilde{y}_{i,t-1}v_{it} &= \frac{1}{T}\sum_{t=1}^{T}(y_{i,t-1} - \frac{1}{T}\sum_{s=1}^{T}y_{i,s-1})v_{it} \\ &= \frac{1}{T}\sum_{t=1}^{T}y_{i,t-1}v_{it} - \frac{1}{T}\sum_{t=1}^{T}(\frac{1}{T}\sum_{s=1}^{T}y_{i,s-1})v_{it}\end{aligned}$$

Since v_{it} is uncorrelated with $y_{i,s-1}$ for $s < t$ and v_{is} is uncorrelated with $y_{i,t-1}$ for $s > t$,

$$E\left[(\frac{1}{T}\sum_{t=1}^{T}y_{i,t-1}v_{it})^2\right] = \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}E[y_{i,t-1}v_{it}y_{i,s-1}v_{is}] = \frac{1}{T^2}\sum_{t=1}^{T}E[y_{i,t-1}^2v_{it}^2] = O(\frac{1}{T}),$$

the first term is $\frac{1}{T}\sum_{t=1}^{T}y_{i,t-1}v_{it} = O_p(T^{-1/2})$. Consider the second term, we obtain

$$\frac{1}{T}\sum_{t=1}^{T}(\frac{1}{T}\sum_{s=1}^{T}y_{i,s-1})v_{it} = (\frac{1}{T}\sum_{s=1}^{T}y_{i,s-1})(\frac{1}{T}\sum_{t=1}^{T}v_{it}) = O_p(1)O_p(T^{-1/2}) = O_p(T^{-1/2}).$$

The proof of 2): Since $\bar{v}_{i\cdot} = (\frac{1}{T}\sum_{s=1}^{T}v_{is}) = O_p(T^{-1/2})$, $\frac{1}{T}\sum_{t=1}^{T}\tilde{y}_{i,t-1} = \frac{1}{T}\sum_{t=1}^{T}(y_{i,t-1} - \frac{1}{T}\sum_{s=1}^{T}y_{i,s-1}) = O_p(1)$ and $\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{it} = O_p(1)$, we obtain

$$\frac{1}{T}\tilde{W}_i'\bar{V}_i = \frac{1}{T}\sum_{t=1}^{T}(\tilde{y}_{i,t-1},\ \tilde{x}_{it})\bar{v}_{i\cdot} = \left((\bar{v}_{i\cdot})\frac{1}{T}\sum_{t=1}^{T}\tilde{y}_{i,t-1},\ (\bar{v}_{i\cdot})\frac{1}{T}\sum_{t=1}^{T}\tilde{x}_{it}\right) = O_p(T^{-1/2}).$$

The proofs of 3) and 4) are similar. The proof of 5) can be found in the supplementary appendix which is available upon request from the authors. ■