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Alan Middleton
Syracuse University

Creighton K. Thomas
Texas A & M University - College Station

David A. Huse
Princeton University

Zero and low temperature behavior of the two-dimensional $\pm J$ Ising spin glass

Creighton K. Thomas,[1] David A. Huse,[2] and A. Alan Middleton[3]

[1]*Department of Physics and Astronomy, Texas A&M University, College Station, Texas 77843-4242, USA*
[2]*Department of Physics, Princeton University, Princeton, NJ 08544, USA*
[3]*Department of Physics, Syracuse University, Syracuse, NY 13244, USA*

Scaling arguments and precise simulations are used to study the square lattice $\pm J$ Ising spin glass, a prototypical model for glassy systems. Droplet theory explains, and our numerical results show, entropically-stabilized long range spin glass order at zero temperature, which resembles the energetic stabilization of long range order in higher dimensional models at finite temperature. At low temperature, a temperature-dependent crossover length scale is used to predict the power-law dependence on temperature of the heat capacity and clarify the importance of disorder distributions.

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Glassy systems, characterized by extremely slow relaxation and resultant complex hysteresis and memory effects, are difficult to study because their dynamics encompass a great range of time scales [1]. Glassy materials include those without intrinsic disorder, such as silica glass, and those where quenched disorder influences the active degrees of freedom. An example of the latter is the Edwards-Anderson spin glass model [2], which includes the disorder and frustration necessary to capture many of the complex behaviors seen in disordered magnetic materials. Many aspects of this prototypical spin glass model remain poorly understood. The droplet and replica-symmetry-breaking pictures provide distinct views of spin glasses [3]. Analytical results are few, so numerical approaches are invaluable for both testing and motivating new ideas. But numerics can also be exceedingly difficult in the general case: computing spin glass ground states is believed to require exponential time to solve exactly, as it is an NP-hard problem [4, 5].

A fortunate special case that avoids this computational intractability is the two-dimensional Ising spin glass (2DISG). The Hamiltonian is $\mathcal{H} = \sum_{\langle ij \rangle} J_{ij} s_i s_j$. The nearest-neighbor couplings $\mathcal{J} = \{J_{ij}\}$ are independent random variables coupling Ising spins $s_i = \pm 1$ at sites i on a square lattice with L^2 sites. The random signs of the J_{ij} lead to competing interactions that can not be simultaneously satisfied. Here we study the model with the $\pm J$ distribution, where each bond value is $J_{ij} = \pm 1$ with equal probability. Highly developed numerical algorithms [4, 6–9] can efficiently circumvent the complexity due to disorder and frustration: ground states and finite-temperature partition functions of the 2DISG may be computed in time polynomial in L. These algorithms have given us more insight into model glassy systems.

In this Letter, we develop the droplet scaling theory as applied to the zero and low temperature properties of the $\pm J$ model. We obtain reliable numerical results for very low temperatures T and large L (well beyond any Monte Carlo simulation results), including the probability distributions of spin correlations. We show that the $\pm J$ model at $T = 0$ in 2D has equilibrium correlations similar to those of a higher-dimensional spin glass at $T > 0$ in its spin glass ordered phase. Thus the 2DISG provides a computationally-tractable model with spin-glass order that closely mimics spin glass order in higher dimensions, where large samples can not be studied numerically.

The ground state of a $\pm J$ model is highly degenerate with an extensive entropy [6, 10, 11]. For many years it was assumed that the ground state is critical, with power-law spin-spin correlations as a function of distance [11–14]. Jorg, *et al.* [15] then presented evidence that instead the ground state has true long-range spin glass order. Recent results [16] about rigid clusters of spins with fixed relative orientations are also suggestive of long range correlations. Here we add to that evidence, and present the corresponding droplet theory, which differs in some important respects [17] from the scenarios proposed in Refs. [15, 16, 18]. We show how the $T = 0$ properties predict a new temperature-dependent crossover length scale $\ell_x(T)$ where the probability distribution of the droplet free energies crosses over from discrete to continuous. We use this to predict the low-T scaling of the specific heat. This prediction is shown to agree well with our numerical results, resolving the long-debated question of the specific heat behavior in this model. These results provide new insight into the role of entropy and the importance of the choice of disorder distribution in spin glasses.

Computational technique We start by computing the partition function $Z_{\mathrm{P}}(\mathcal{J})$ in an L-by-L sample with periodic boundary conditions, adopting previously published techniques [6, 7], but using arbitrary precision arithmetic [9]. The efficient computation of Z relies on a hierarchical decomposition of the sample [7]: starting from the smallest pieces of the spatial decomposition, single plaquettes, neighboring pieces are joined together to recursively compute the partition function. The same four Pfaffians [9] used to find Z_{P} also give Z_{AP}, the partition function for antiperiodic boundary conditions, where the horizontal bonds along a vertical column have negated J_{ij}. Note that these computations are exact to within numerical precision: there is no question of convergence as there is with Markov chain Monte

Carlo methods and we have verified the numerical stability of our procedure [9]. The two partition functions differ due to a domain wall induced by the change in boundary conditions. We use finite differences over T of $F = -kT\ln(Z)$ to compute equilibrium averages for energies E and entropies S. The sample-dependent differences $\delta X(\mathcal{J}) \equiv X_{\rm P} - X_{\rm AP}$, with $X = F$, E, and S, give relative domain wall (free) energies and entropies. Error bars in our plots indicate 1σ statistical errors.

We have extended this technique to compute arbitrary spin-spin correlation functions [19]. These correlations can be computed at sufficiently low T, $T = 0.02$ for $L \leq 256$, and with sufficient precision, 4 096 bits, such that the contribution of excited states can be clearly separated from the ground state contribution to correlations. If spins have fixed relative orientation with probability greater than $1 - e^{-2/T}$, we take the spins to be rigidly correlated at $T = 0$. Such spins have fluctuations that are clearly smaller than non-rigid spins by a factor of at least $\approx 10^{21}$, exceeding multiplicity (entropic) factors for excited states. We have confirmed that the assignment of rigid correlations is unchanged if T is lowered or the precision increased, using over 200 samples for $L = 128$. An example of $T = 0$ rigid and non-rigid nearest-neighbor spin correlations $\langle s_i s_j \rangle_0$, and the domain wall due to a change in boundary conditions are shown in Fig. 1.

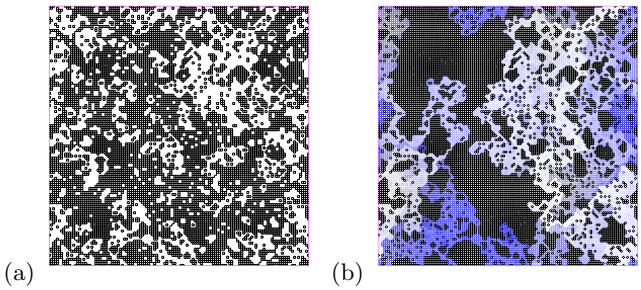

FIG. 1: [color online] An $L = 128$ sample. (a) Nearest neighbor correlations: Black lines are bonds that are rigidly correlated at $T = 0$. (b) Relative domain wall. Black lines are bonds whose correlations are not affected by the change P $\rightarrow$ AP in the horizontal boundary conditions. The shaded bonds indicate the change in the correlations on a logarithmic scale, with light colors for bonds with $|\delta\langle s_i s_j \rangle_0| \sim 1$, and the darkest colors indicating $|\delta\langle s_i s_j \rangle_0| \sim e^{-20}$. This "diffuse" domain wall appears to have a multifractal geometry [20].

Domain walls, droplets, and rigid clusters Consider a given ground state configuration of a particular realization $\mathcal{J}$ of the $\pm J$ model. A relative domain wall loop is the boundary of a simply connected region where all the spins are flipped relative to the given ground state configuration. The droplet excitation [21] at site i and length scale ℓ is the set of all lowest-energy domain wall loops of linear size of order ℓ enclosing site i and of order ℓ^2 other sites. We can ask about the probability distributions of the energy and of the entropy of this droplet excitation. Here we do not directly measure these droplets, but computations for the very similar system-size relative domain walls due to changing between P and AP boundary conditions give insight into the effect of these droplets on the spin-spin correlations.

We first note previous work (e.g., [22]) which shows that system-size domain walls have a probability distribution for δE that is independent of L, as $L \rightarrow \infty$ (given a parity for L). For large even L, we find that $77.5 \pm 1.0\%$ of the samples do have zero energy domain walls. Thus, assuming the droplets have a similar probability distribution, the event that there is no zero energy droplet of scale ℓ surrounding a site i has probability strictly between zero and one. This is also true for smaller scales $\ell/2$, $\ell/4$, etc., with the correlations between the events at each scale presumably falling off rapidly across scales. As a result [14], a given site i has no zero energy droplets at any scale less than ℓ with a probability $p(\ell)$ that decreases with increasing ℓ as a power of ℓ.

In a finite sample at $T = 0$, we choose to identify the "backbone" of the ground states as the largest set of rigidly correlated spins [16]. Spins on the backbone can not be flipped at zero energy by any droplets without flipping the entire backbone. By the above argument, the number of such spins scales with a power of L less than two: the backbone is a fractal. Hartmann's results imply a backbone fractal dimension $d_b = 1.78(2)$ [14]; our results for rigidly correlated clusters are consistent with this [20]. As seen in previous work [16], we sometimes see backbone spins that are not connected by a path of rigid bonds; the backbone can have "gaps". Thus the backbone of the sample in Fig. 1(a) is all spins on the largest cluster connected by rigid bonds plus possibly some other spins or clusters whose rigidity is not detected by nearest-neighbor correlations. The backbone on this sample percolates; the fraction of large samples where the backbone percolates is a number between zero and one [20].

Entropic enhancement of correlations So far, we have discussed the rigid correlations that arise from locations where there are no zero energy droplets. What about zero energy droplets, where spin clusters flip in the ground state? Saul and Kardar [6] found that zero-energy relative domains walls have a typical entropy $\delta S \sim L^{\theta_S}$, with $\theta_S \cong 0.50$. Our data from $L = 32$ to $L = 256$ is very well fit by a power law, with $\theta_S = 0.50 \pm 0.01$, (in contrast with Ref. 23). It is intriguing that this exponent is consistent with the simple rational number 1/2. The probability distribution of δS is continuous through zero, so $|\delta S|$ is less than or of order one with a probability $\sim L^{-\theta_S}$. We take the probability distribution for δS for the zero-energy droplets to have the same scaling.

This leads to an essential new aspect of the droplet picture for the two-dimensional $\pm J$ Ising spin glass at $T = 0$: A droplet at a large ℓ typically either has a nonzero energy so never flips or has a large entropy so it is flipped away from its usual orientation only with exponentially

small probability $\sim \exp(-\ell^{\theta_S})$. It is only the *entropically active* droplets with $|\delta S|$ less than or of order one that flip at $T = 0$ with a substantial probability. These active droplets are power-law rare in the length scale ℓ: only a fraction $\sim \ell^{-\theta_S}$ of the sample's area is active at scale ℓ. This scenario is very similar to the expected behavior within the droplet theory [21] for an Ising spin glass in 3 or more dimensions at a nonzero temperature within the spin glass ordered phase, where only a fraction $\sim \ell^{-\theta}$ of the droplets at scale ℓ have δF less than order of order $k_B T$ so are *thermally* active, and the typical free energy of a droplet scales as $\delta F \sim \ell^{\theta}$, with $\theta > 0$.

This droplet picture allows us to apply many of the predictions of the droplet theory [21], with the usual free energy exponent θ replaced by the entropy exponent $\theta_S \cong 0.50$. In particular, it predicts long range spin glass order, with the $T = 0$ spin-glass correlation function for spins at spacing $\vec{r}$, $G_0(\vec{r}) = [\langle s_{\vec{0}} s_{\vec{r}} \rangle_0^2]$, where the square brackets denote an average over samples, behaving at large r as

$$G_0(\vec{r}) - G_0(\infty) \sim r^{-\theta_S} \ . \tag{1}$$

There is long range order because the total probability, found by summing over all scales, of flipping a given spin is bounded away from unity, since the probability of the droplet being entropically active decays as $\ell^{-\theta_S}$. The active droplets with scale $\ell > r$ are unlikely to flip only one of a pair of a spins separated by distance r, so such droplets do not substantially reduce $G_0(\vec{r})$. Thus $G_0(\vec{r})$ is larger than $G_0(\infty)$ by an amount proportional to the probability $\sim r^{-\theta_S}$ of droplets of scale r being active.

The results of numerical calculations for $G_T(\vec{r})$ for $T \ll L^{-\theta_S}$, shown in Fig. 2(a), are consistent with Eq. (1). To verify this and to more precisely determine the long range order $G_0(\infty)$, we measure $G_T(\vec{r})$ for higher T. At nonzero temperature and finite L, the difference between $G_T(\vec{r})$ and $G_0(\infty)$ should scale as $L^{-\theta_S} g(\vec{r}/L, TL^{\theta_S})$, for a scaling function g. The scaling parameter TL^{θ_S} naturally arises from the competition between energy and entropy. From such scaling of the correlations for several values of $\vec{r}/L$, we infer $G_0(\infty) = 0.395 \pm 0.010$. A specific example of a scaling collapse, with fixed $\theta_S = 0.50$ and variable $G_0(\infty)$, is shown in Fig. 2(b) for $\vec{r} = (L/4, L/4)$. In fact this long range order with $G_0(\infty) \cong 0.4$ is apparent in Poulter and Blackman [13] (their Fig. 6), although that is not how they interpreted those data. A similar entropically-stabilized long range order has recently also been seen numerically in a diluted $\pm J$ spin glass in 3D where the ground state stiffness is zero [24].

Crossover to effectively continuous disorder Using this droplet picture, we can also understand the crossover to the $T > 0$ behavior. The competition between entropy and energy [17, 21, 25, 26] in spin glasses causes extreme sensitivity of the long-distance spin correlations to changes in T ("chaos"). We use a related argument at very low T to derive the low-T scaling of the specific heat in the $\pm J$ model.

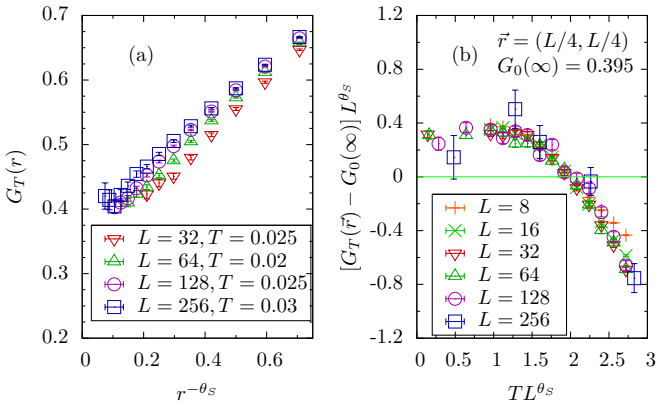

FIG. 2: (a) Correlation functions at low temperatures for $L = 32, 64, 128, 256$ plotted vs. $r^{-\theta_S}$, using $\theta_S = 0.50$. Droplet theory based on the entropy of the droplets predicts a linear curve at small values of $r^{-\theta_S}$ and large L, as is seen. (b) Sample scaling collapse for correlation function $G_T(\vec{r})$. Using nonzero-temperature scaling, we expect $L^{\theta_S}[G_T(\vec{r}) - G_0(\infty)] \approx g(\vec{r}/L, TL^{\theta_S})$. Adjusting $G_0(\infty)$ for the best collapse, we find $G_0(\infty) = 0.395 \pm 0.010$.

At $T = 0$, the droplet energies δE are integer multiples of 4, so the distribution of droplet energies is discrete. However, the droplet entropies $\delta S(T = 0) \sim \ell^{\theta_S}$ are continuously distributed for large ℓ. We define a crossover length $\ell_x(T)$ to be the scale at which the typical $T\delta S(\ell_x)$ becomes $\mathcal{O}(1)$, causing $\delta F = \delta E - T\delta S$ to become continuously distributed. Thus at low T

$$\ell_x(T) \sim T^{-1/\theta_S} \ . \tag{2}$$

For $\ell \ll \ell_x(T)$ and very low T only zero-energy droplets can be active. But at scales at and above $\ell_x(T)$ the droplets with $\delta E = 4$ or more can, due to their large entropy, have δF near zero and thus also be thermally active (Ref. 15 gave a discussion of this crossover from discrete to continuous behavior, but not of $\ell_x(T)$.)

Specific heat The low-T specific heat C in a 2DISG is governed by the thermally active droplets (those with excitation free energies $\delta F \leq \mathcal{O}(T)$) that have nonzero energy [21], with the *smallest* droplets dominating, since they have the highest density. If the disorder distribution is continuous, these smallest droplets have size $\mathcal{O}(1)$, energy of order T and density proportional to T, so give $C \sim T$ at low T. For $\pm J$ disorder, on the other hand, the droplets with size $\ell < \ell_x(T)$ have either zero energy δE or $\delta E = \mathcal{O}(1)$ so do not contribute to the low-T specific heat. The smallest active droplets with $\delta E \neq 0$ are of size of order $\ell_x(T)$. Due to their $\mathcal{O}(1)$ energy gap, these droplets each contribute $\sim 1/T^2$ to C. They occur with density $\sim T/\ell_x^2(T)$ so the specific heat $C \sim T^{(2/\theta_S)-1}$ at low T in an infinite sample (alternatively, the entropy density is $\ell_x^{\theta_S - 2}$, which scales as C). For finite L, this power-law specific heat is cut off at the lowest temperatures when the size $\ell_x(T)$ of these active

droplets exceeds L. This produces the low-T finite-size scaling form $C \approx T^{(2/\theta_S)-1} c(TL^{\theta_S})$, where $c(x)$ is a scaling function that goes to a constant for large argument (where $1 \ll \ell_x(T) \ll L$). Our specific heat data are shown in this scaling form in Fig. 3. Our data for intermediate temperatures, $T \cong 0.35$, are consistent with Monte Carlo results [27], which saw an effective exponent $\alpha \cong -4.2$, with $C \sim T^{-\alpha}$. However, the effective exponent crosses over to a lower magnitude consistent with $\alpha = 1 - \frac{2}{\theta_S} \cong -3.0$ at low T and the largest L. Contributions from the *largest* active droplets give a subleading correction, which from standard hyperscaling is $\alpha' = -2\nu$ (though see Ref. 17), that is too weak to detect yet. Our result, $\alpha = 1 - \frac{2}{\theta_S}$, gives the leading contribution to C at low T for the 2D $\pm J$ model in the infinite volume limit.

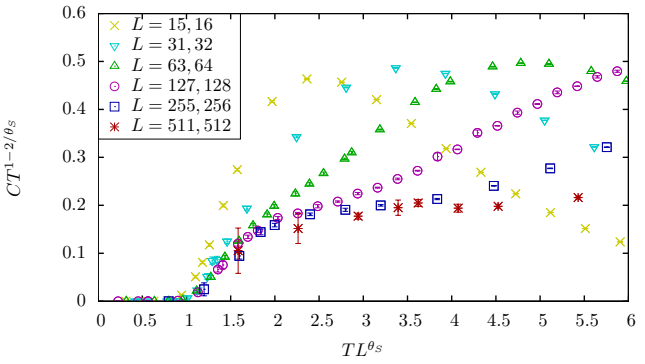

FIG. 3: [color online] Scaling plot for the low-T specific heat C. For $T < L^{\theta_S}$, there are no active droplets, so C is exponentially suppressed. The low-T scaling behavior in large-enough systems is seen when CT^{1-2/θ_S} is flat at $TL^{\theta_S} > 1$. The value [28] of $\theta_S = 0.50$ is independently set by the scaling of the entropy of domain walls.

Discussion The $\pm J$ spin glass model in 2D can be used to efficiently study the thermodynamics of a glassy model. Large scale simulations verify conceptually novel scaling arguments for this model: these concepts have broad implications. Strictly correlated spins at $T = 0$ form fractal clusters that can be explained using a droplet theory [14], leading to backbone correlations that decay as a power law. Yet the spin-spin correlations at $T = 0$ have long range order. Though large zero-energy excitations are abundant, the scale-dependent entropy of these excitations prevents them from destabilizing the long range order. This entropic "stiffness" at $T = 0$ is very similar to the (free-)energetic stiffness in $d \geq 3$ models that gives a stable spin glass with long range order at nonzero T, making the 2D $\pm J$ model a good candidate for exploring the spin glass phase in general. The difference between discrete and continuous disorder is a general problem in statistical mechanics; here we show how this difference can have striking effects on the scaling through a temperature dependent crossover length. When the temperature is nonzero, application of this crossover length $\ell_x(T)$ leads to a new prediction for the low temperature specific heat of the $\pm J$ model in 2D, which we have verified numerically.

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