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On the Local Spectral Properties of Weighted Shift Operators

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Abstract. In this paper, we study the local spectral properties for both unilateral and bilateral weighted shift operators.

1. Introduction

The weighted shift operators have been proven to be a very interesting collection of operators, providing examples and counter-examples to illustrate many properties of operators. The basic facts concerning their spectral theory have been fully investigated, and can be found in A. Shields' excellent survey article [15]. The description of their local spectra, and the characterization of their local spectral properties is natural, and there has been some progress in the literature. In K. Laursen, and M. Neumann's remarkable monograph [10], it is shown that each local spectrum of a unilateral weighted shift operator S contains a closed disc centered at the origin. Sufficient conditions are given for a unilateral weighted shift operator S to satisfy Dunford's condition (C) and for S not to possess Bishop's property (β). It is also shown that a unilateral weighted shift operator is decomposable if and only if it is quasi-nilpotent. But, it is unknown which unilateral or bilateral weighted shift operators satisfy Dunford's condition (C) or possess Bishop's property (β). It is also unknown which bilateral weighted shift operators are decomposable. The main goal of this paper is to study the local spectral properties of both unilateral and bilateral weighted shift operators. We study the question of determining the local spectra of a weighted shift S in terms of the weight sequence defining S. Several necessary and sufficient conditions for S to satisfy Dunford's condition (C) or Bishop's property (β) are given.

A part of the material of this paper is contained in the Ph.D thesis of the author, [6], that was defended on April 2001. After the first version of this paper, [8], has been submitted for publication and circulated among some specialists in the domain, the author learned from Professor M. M. Neumann that some results of [8] had also appeared in [13]. However, the scope of

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the present paper is wider and the arguments of the proofs are simpler that the ones given in [6] and [13]. In [13], the authors established some results abscent in the present paper about inner and outer radii of arbitrary Banach space operator and gave growth condition for a Banach space operator to possess Bishop's property (β).

Throughout this paper, $\mathcal{H}$ shall denote a complex Hilbert space, and $\mathcal{L}(\mathcal{H})$ the algebra of all linear bounded operators on $\mathcal{H}$. For an operator $T \in \mathcal{L}(\mathcal{H})$, let T^*, $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_p(T)$, $W(T)$, $r(T)$, and $w(T)$ denote the adjoint, the spectrum, the approximate point spectrum, the point spectrum, the numerical range, the spectral radius, and the numerical radius, respectively, of T. Let $m(T) := \inf\{\|Tx\| : \|x\| = 1\}$ denote the lower bound of T, and let $r_1(T)$ denote $\sup\limits_{n\geq 1}[m(T^n)]^{\frac{1}{n}}$ which equals $\lim\limits_{n\to+\infty}[m(T^n)]^{\frac{1}{n}}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *single-valued extension property* provided that for every non-empty open set $U \subset \mathbb{C}$, the equation $(T-\lambda)f(\lambda) = 0$ admits the zero function $f \equiv 0$ as a unique analytic solution on U. The *local resolvent set* $\rho_{_T}(x)$ of T at a point $x \in \mathcal{H}$ is the union of all open subsets $U \subset \mathbb{C}$ for which there exists an analytic function $f : U \to \mathcal{H}$ such that $(T-\lambda)f(\lambda) = x$ on U. The complement in $\mathbb{C}$ of $\rho_{_T}(x)$ is called the *local spectrum* of T at x and will be denoted by $\sigma_{_T}(x)$. Recall that the *local spectral radius* of T at x is given by $r_{_T}(x) := \limsup\limits_{n\to+\infty}\|T^n x\|^{\frac{1}{n}}$ and equals $\max\{|\lambda| : \lambda \in \sigma_{_T}(x)\}$ whenever T has the single-valued extension property. Moreover, if T enjoys this property then for every $x \in \mathcal{H}$, there exists a unique maximal analytic solution $\widetilde{x}(.)$ on $\rho_{_T}(x)$ for which $(T-\lambda)\widetilde{x}(\lambda) = x$ for all $\lambda \in \rho_{_T}(x)$, and satisfies $\widetilde{x}(\lambda) = -\sum\limits_{n\geq 0}\frac{T^n x}{\lambda^{n+1}}$ on $\{\lambda \in \mathbb{C} : |\lambda| > r_{_T}(x)\}$. If in addition T is invertible, then $\widetilde{x}(\lambda) = \sum\limits_{n\geq 1}\lambda^{n-1}T^{-n}x$ on the open disc $\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{r_{_{T^{-1}}}(x)}\}$. For a closed subset F of $\mathbb{C}$, let $\mathcal{H}_{_T}(F) := \{x \in \mathcal{H} : \sigma_{_T}(x) \subset F\}$ be the corresponding analytic spectral subspace; it is a $T-$hyperinvariant subspace, generally non-closed in $\mathcal{H}$. The operator T is said to satisfy *Dunford's condition* (C) if for every closed subset F of $\mathbb{C}$, the linear subspace $\mathcal{H}_{_T}(F)$ is closed. Let U be an open subset of $\mathbb{C}$ and let $\mathcal{O}(U,\mathcal{H})$ be the space of analytic $\mathcal{H}-$valued functions on U. Equipped with the topology of uniform convergence on compact subsets of U, the space $\mathcal{O}(U,\mathcal{H})$ is a Fréchet space. Note that every operator $T \in \mathcal{L}(\mathcal{H})$ induces a continuous mapping $T_{_U}$ on $\mathcal{O}(U,\mathcal{H})$ defined by $T_{_U}f(\lambda) = (T-\lambda)f(\lambda)$ for every $f \in \mathcal{O}(U,\mathcal{H})$ and $\lambda \in U$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to possess *Bishop's property* (β) provided that, for every open subset U of $\mathbb{C}$, the mapping $T_{_U}$ is injective and has a closed range. Equivalently, if for every open subset U of $\mathbb{C}$ and for every sequence $(f_n)_n$ of $\mathcal{O}(U,H)$, the convergence of $(T_{_U}f_n)_n$ to 0 in $\mathcal{O}(U,\mathcal{H})$ should always entail the convergence to 0 of the sequence $(f_n)_n$ in $\mathcal{O}(U,\mathcal{H})$. It is known that hyponormal operators, $M-$hyponormal operators and more generally subscalar operators possess

Bishop's property (β). It is also known that Bishop's property (β) implies Dunford's condition (C) and it turns out that the single-valued extension property follows from Dunford's condition (C). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable* if, for every open cover of $\mathbb{C}$ by two open subsets U_1 and U_2, there exist $T-$invariant closed linear subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ and $\sigma(T_{|\mathcal{H}_i}) \subset U_i$ for $i = 1,\ 2$. Immediate examples of decomposable operators are provided by the normal operators and more generally by the spectral operators in the sense of Dunford. It is known that an operator $T \in \mathcal{L}(\mathcal{H})$ is decomposable if and only if both T and T^* possess Bishop's property (β). For thorough presentations of the local spectral theory, we refer to the monographs [9], and [10].

Throughout this paper, let S be a weighted shift operator on $\mathcal{H}$ with a positive bounded weight sequence $(\omega_n)_n$, that is

$$Se_n = \omega_n e_{n+1}\ \forall n,$$

where $(e_n)_n$ is an orthonormal basis of $\mathcal{H}$. If the index n runs over the non-negative integers (resp. all integers), then S is called a *unilateral* (resp. *bilateral*) *weighted shift* .

Before proceeding, we would like to recall some simple but useful remarks which will be repeatedly used in the sequel.

- (R_1) For every $\alpha \in \mathbb{C}$, $|\alpha| = 1$, we have $(\alpha S)U_\alpha = U_\alpha S$, where U_α is the unitary operator defined on $\mathcal{H}$ by $U_\alpha e_n = \alpha^n e_n,\ \forall n$.
- (R_2) Let K be a non-empty compact subset of $\mathbb{C}$. If K is connected and invariant by circular symmetry about the origin, then there are two real numbers a, and b with $0 \leq a \leq b$ such that $K = \{\lambda \in \mathbb{C} : a \leq |\lambda| \leq b\}$.
- (R_3) Let T and R in $\mathcal{L}(\mathcal{H})$ such that R is invertible. Then for every $x \in \mathcal{H}$, we have $\sigma_T(x) = \sigma_{RTR^{-1}}(Rx)$.

Finally, we need to introduce the local functional calculus, developed by L. R. Williams in [18], of operators having the single-valued extension property; this local functional calculus extends the Riesz functional calculus for operators. P. McGuire has also studied this local functional calculus, but only for operators with no eigenvalues (see [11]). Let U be an open subset of $\mathbb{C}$ and K be a compact subset of U. Let $(\gamma_i)_{1 \leq i \leq n}$ be a finite family of disjoint closed rectifiable Jordan curves in $U \backslash K$. The formal sum, $\gamma := \gamma_1 + ... + \gamma_n$, of the curves is called *an oriented envelope of* K in U if its winding number equals 1 on K and 0 on $\mathbb{C}\backslash U$, i.e.,

$$\eta_\gamma(\lambda) := \frac{1}{2\pi i}\oint_\gamma \frac{1}{\alpha - \lambda} d\alpha = \begin{cases} 1 & \text{for every } \lambda \in K \\ 0 & \text{for every } \lambda \in \mathbb{C}\backslash U \end{cases}$$

Let $T \in \mathcal{L}(\mathcal{H})$, and let $v \in \mathcal{H}$. Assume that T has the single-valued extension property. For an analytic complex valued function f on a neighborhood U

of $\sigma_T(v)$, we define a vector $f(T,v)$ by the equation

$$f(T,v) := \frac{-1}{2\pi i}\oint_\gamma f(\lambda)\widetilde{v}(\lambda)d\lambda,$$

where γ is an oriented envelope of $\sigma_T(v)$ in U. Note that $f(T,v)$ is independent of the choice of γ. We end this introduction by quoting, without proof, the following theorem from [18] which will be used in the sequel.

Theorem 1.1. *Let $T \in \mathcal{L}(\mathcal{H})$, and let $v \in \mathcal{H}$. Assume that T has the single-valued extension property. If f is an analytic function on a neighborhood O of $\sigma_T(v)$ and is not identically zero on each connected component of O, then the following statements hold*

(a) $Z_T(f,v) := \{\lambda \in \sigma_T(v) : f(\lambda) = 0\}$ *is a finite set.*
(b) $\sigma_T(v) = \sigma_T(f(T,v)) \cup Z_T(f,v)$.

Moreover, if $\sigma_p(T) = \emptyset$, then $\sigma_T(v) = \sigma_T(f(T,v))$.

2. Preliminaries

In this section, we assemble some simple but useful results which will be needed in the sequel. These results remain valid in the general setting of Banach space operators.

Let $\lambda_0 \in \mathbb{C}$; recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* (resp. possess *Bishop's property* (β)) at λ_0 if there is an open neighbourhood V of λ_0 such that for every open subset U of V, the mapping T_U is injective (resp. is injective and has a closed range). Obviously, T has the single-valued extension property at λ_0 whenever T possesses Bishop's property (β) at λ_0. Note that T possesses Bishop's property (β) at λ_0 if and only if there is an open neighbourhood V of λ_0 such that for every open subset U of V and for every sequence $(f_n)_n$ of $\mathcal{O}(U,H)$, the convergence of $(T_U f_n)_n$ to 0 in $\mathcal{O}(U,\mathcal{H})$ should always entail the convergence to 0 of the sequence $(f_n)_n$ in $\mathcal{O}(U,\mathcal{H})$. Moreover, if T possesses Bishop's property (β) at any point $\lambda \in \mathbb{C}$ then T possesses Bishop's classical property (β).

For an arbitrary $T \in \mathcal{L}(\mathcal{H})$, we shall denote

$$\sigma_\beta(T) := \big\{\lambda \in \mathbb{C} : T \text{ fails to possess Bishop's property } (\beta) \text{ at } \lambda\big\}.$$

It is clear that $\sigma_\beta(T)$ is a closed subset of $\mathbb{C}$ contained in $\sigma(T)$. In fact more can be established in the following result.

Proposition 2.1. *For any operator $T \in \mathcal{L}(\mathcal{H})$, we have $\sigma_\beta(T) \subset \sigma_{ap}(T)$.*

Proof. Let $\lambda_0 \notin \sigma_{ap}(T)$, we shall show that T possesses Bishop's property (β) at λ_0. By replacing T with $T - \lambda_0$, we may assume without loss of generality that $\lambda_0 = 0$. In this case, there is $R \in \mathcal{L}(\mathcal{H})$ such that $RT = 1$.

Therefore, for every $\lambda \in \mathbb{C}$ and for every $x \in \mathcal{H}$, we have

$$\begin{aligned} \|(T-\lambda)x\| &= \|(1-\lambda R)Tx\| \\ &\geq m(1-\lambda R)m(T)\|x\| \\ &= \frac{1}{\|R\|}m(1-\lambda R)\|x\|. \end{aligned}$$

If we set $V := \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{\|R\|}\}$ then for every $\lambda \in V$, $1-\lambda R$ is invertible. Hence, for every $\lambda \in V$ and for every $x \in \mathcal{H}$, we have

$$\|x\| \leq \|R\|\|(1-\lambda R)^{-1}\|\|(T-\lambda)x\|. \tag{2.1}$$

Now, let U be an open subset of the open disc V and let $(f_n)_{n\geq 0}$ be a sequence of $\mathcal{O}(U,\mathcal{H})$ such that $(T_U f_n)_n$ converges to 0 in $\mathcal{O}(U,\mathcal{H})$. Let K be a compact subset of U, it follows from (2.1) that

$$\|f_n(\lambda)\| \leq \|R\| \max\left\{\|(1-\mu R)^{-1}\| : \mu \in K\right\}\|(T-\lambda)f_n(\lambda)\|, \ \forall \lambda \in K.$$

This shows that the sequence $(f_n)_n$ converges to 0 uniformly on K; and so, S possesses Bishop's property (β) any point of V. In particular, S possesses Bishop's property (β) at $\lambda_0 = 0$. This finishes the proof. $\square$

For every operator $T \in \mathcal{L}(\mathcal{H})$, we shall denote

$$\Re(T) := \left\{\lambda \in \mathbb{C} : T \text{ does not have the single-valued extension property at } \lambda\right\}.$$

It is an open subset of $\mathbb{C}$ contained in $\sigma(T)$. The description of $\Re(T)$ for several operators T can be found in [2] and [7]. Note that

$$\overline{\sigma(T)\backslash\sigma_{ap}(T)} \subset \Re(T^*)$$

for every operator $T \in \mathcal{L}(\mathcal{H})$. This containment fails to be equality in general as counter-example can be given by a unilateral weighted shift operator. In [7], it is shown that if T is a cyclic operator possessing Bishop's property (β) then

$$\Re(T^*) = \overline{\sigma(T)\backslash\sigma_{ap}(T)}.$$

In fact, this result remain valid for every rationally cyclic injective operator possessing Bishop's property (β). Recall that an operator (resp. injective operator) $T \in \mathcal{L}(\mathcal{H})$ is said to be *cyclic* (resp. *rationally cyclic*) if there is a vector $x \in \mathcal{H}$ (resp. $x \in T^\infty\mathcal{H} := \bigcap\limits_{n\geq 0} T^n\mathcal{H}$) such that the closed linear subspace generated by $\{T^n x : n \geq 0\}$ (resp. $\{T^n x : n \in \mathbb{Z}\}$) is dense in $\mathcal{H}$.

The proof of the following result is an adaptation of the one of theorem 3.1 of [7] and will be omitted.

Proposition 2.2. *Assume that $T \in \mathcal{L}(\mathcal{H})$ is a rationally cyclic injective operator. If T possesses Bishop's property (β), then $\Re(T^*) = \overline{\sigma(T)\backslash\sigma_{ap}(T)}$.*

The following two lemmas are easy to verify, we omit the proof of the next one.

Lemma 2.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces, and let*

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

be a 2×2 upper triangular operator matrix on $\mathcal{H}_1 \oplus \mathcal{H}_2$. If both A and B possess Bishop's property (β) at a point $\lambda_0 \in \mathbb{C}$, then M_C possesses Bishop's property (β) at λ_0.

Lemma 2.4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces, and let*

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

be a 2×2 upper triangular operator matrix on $\mathcal{H}_1 \oplus \mathcal{H}_2$. If B have the single-valued extension property, then for every $x \in \mathcal{H}_1$, we have

$$\sigma_{M_C}(x \oplus 0) = \sigma_A(x).$$

Proof. Let $x \in \mathcal{H}_1$.

Let U be an open subset of $\mathbb{C}$ and $f : U \to \mathcal{H}_1$ be an analytic function such that

$$(A - \lambda)f(\lambda) = x, \text{ for every } \lambda \in U.$$

If for every $\lambda \in U$, we set $F(\lambda) = f(\lambda) \oplus 0$, then

$$(M_C - \lambda)F(\lambda) = x \oplus 0, \text{ for every } \lambda \in U.$$

This shows that

$$\sigma_{M_C}(x \oplus 0) \subset \sigma_A(x).$$

Conversely, let U be an open subset of $\mathbb{C}$ and $F : U \to \mathcal{H}$ be an analytic function such that

$$(M_C - \lambda)F(\lambda) = x \oplus 0, \text{ for every } \lambda \in U.$$

Write $F = f_1 \oplus f_2$, where $f_1 : U \to \mathcal{H}_1$ and $f_2 : U \to \mathcal{H}_2$ are analytic functions. For every $\lambda \in U$, we have

$$(A - \lambda)f_1(\lambda) + Cf_2(\lambda) = x \text{ and } (B - \lambda)f_2(\lambda) = 0.$$

Since B has the single-valued extension property, we have $f_2 \equiv 0$. Hence,

$$(A - \lambda)f_1(\lambda) = x \text{ for every } \lambda \in U.$$

This shows that

$$\sigma_A(x) \subset \sigma_{M_C}(x \oplus 0).$$

And the proof is complete. $\square$

Obviously, it is seen from (R_1) that $\sigma(S)$, $\sigma_{ap}(S)$ and $\sigma_p(S)$ have circular symmetry about the origin. However, the fact that $\sigma_\beta(S)$ is circularly symmetric deserves a proof.

Lemma 2.5. *If S is a unilateral or bilateral weighted shift on $\mathcal{H}$, then $\sigma_\beta(S)$ is invariant under circular symmetry about the origin.*

Proof. Assume that S possesses Bishop's property (β) at a point $\lambda_0 \in \mathbb{C}$. There is $\delta > 0$ such that for every open subset U of the disc $B := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \delta\}$, and for every sequence $(f_n)_n$ of $\mathcal{O}(U, \mathcal{H})$, the convergence of $(S_U f_n)_n$ to 0 in $\mathcal{O}(U, \mathcal{H})$ implies the convergence of $(f_n)_n$ to 0 in $\mathcal{O}(U, \mathcal{H})$. Let $\alpha \in \mathbb{C}$, $|\alpha| = 1$; it should be seen that S possesses Bishop's property (β) at $\alpha\lambda_0$. Indeed, let V be an open subset of αB, and let $(g_n)_n$ be a sequence of $\mathcal{O}(V, \mathcal{H})$ for which $(S_V g_n)_n$ converges to 0 in $\mathcal{O}(V, \mathcal{H})$. Let K be a compact subset of V; we shall show that $\lim_{n\to+\infty} \big[\sup_{\lambda\in K} \|g_n(\lambda)\|\big] = 0$. Note that in view of (R_1), we have

$$\|(S - \alpha\lambda)x\| = \|(S - \lambda)U_\alpha x\|, \ \forall \lambda \in \mathbb{C}, \ \forall x \in \mathcal{H}. \tag{2.2}$$

And so, if we set $f_n(\lambda) := g_n(\alpha\lambda)$, $\lambda \in \alpha^{-1}V$, then it follows from (2.2) that

$$\begin{aligned}
\sup_{\lambda\in\alpha^{-1}K} \|(S-\lambda)U_\alpha f_n(\lambda)\| &= \sup_{\lambda\in\alpha^{-1}K} \|(S-\alpha\lambda)f_n(\lambda)\| \\
&= \sup_{\lambda\in\alpha^{-1}K} \|(S-\alpha\lambda)g_n(\alpha\lambda)\| \\
&= \sup_{\lambda\in K} \|(S-\lambda)g_n(\lambda)\|.
\end{aligned}$$

As $\lim_{n\to+\infty} \big[\sup_{\lambda\in K} \|(S-\lambda)g_n(\lambda)\|\big] = 0$, and S possesses Bishop's property (β) at λ_0, we have

$$\lim_{n\to+\infty} \Big[\sup_{\lambda\in\alpha^{-1}K} \|U_\alpha f_n(\lambda)\|\Big] = 0.$$

Since U_α is an unitary operator, we have

$$\lim_{n\to+\infty} \big[\sup_{\lambda\in K} \|g_n(\lambda)\|\big] = \lim_{n\to+\infty} \Big[\sup_{\lambda\in\alpha^{-1}K} \|f_n(\lambda)\|\Big] = 0.$$

And the proof is complete. □

3. Local spectral properties of unilateral weighted shift operators

In dealing with unilateral weighted shift operators, let $(\beta_n)_{n\geq 0}$ be the sequence given by

$$\beta_n = \begin{cases} \omega_0 \ldots \omega_{n-1} & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}$$

and let us associate to S the following quantities,

$$r_2(S) := \liminf_{n\to\infty} \big[\beta_n\big]^{\frac{1}{n}}, \text{ and } r_3(S) := \limsup_{n\to\infty} \big[\beta_n\big]^{\frac{1}{n}}.$$

Note that,

$$r(S) = \lim_{n\to\infty} \left[\sup_{k\geq 0} \frac{\beta_{n+k}}{\beta_k}\right]^{\frac{1}{n}}, \ r_1(S) = \lim_{n\to\infty} \left[\inf_{k\geq 0} \frac{\beta_{n+k}}{\beta_k}\right]^{\frac{1}{n}},$$

and

$$m(S) \leq r_1(S) \leq r_2(S) \leq r_3(S) \leq r(S) \leq w(S) \leq \|S\|.$$

3.1. **On the local spectra of unilateral weighted shift operators.** The following result can be deduced from proposition 1.6.9 of [10]; we give here a direct proof.

Proposition 3.1. *For every non-zero $x \in \mathcal{H}$, we have*

$$\{\lambda \in \mathbb{C} : |\lambda| \leq r_2(S)\} \subset \sigma_S(x).$$

Proof. As $\bigcap_{n\geq 0} S^n\mathcal{H} = \{0\}$, we get that $0 \in \sigma_S(x)$ for every $x \in \mathcal{H}\backslash\{0\}$. Thus, we may assume that $r_2(S) > 0$. Let $O := \{\lambda \in \mathbb{C} : |\lambda| < r_2(S)\}$, and consider the following analytic $\mathcal{H}$–valued function on O,

$$k(\lambda) := \sum_{n=0}^{+\infty} \frac{\lambda^n}{\beta_n} e_n.$$

It is easy to see that $(S-\lambda)^* k(\overline{\lambda}) = 0$ for every $\lambda \in O$. Now, let $x = \sum_{n\geq 0} a_n e_n \in \mathcal{H}$ such that $O \cap \rho_S(x) \neq \emptyset$. So, for every $\lambda \in O \cap \rho_S(x)$, we have

$$\begin{aligned} \widehat{x}(\lambda) &:= \langle x; k(\overline{\lambda})\rangle \\ &= \langle (S-\lambda)\widetilde{x}(\lambda); k(\overline{\lambda})\rangle \\ &= \langle \widetilde{x}(\lambda); (S-\lambda)^* k(\overline{\lambda})\rangle \\ &= 0. \end{aligned}$$

Since $\widehat{x}(\lambda) = \sum_{n=0}^{+\infty} \frac{a_n}{\beta_n}\lambda^n e_n$ for every $\lambda \in O$, we see that $a_n = 0$ for every $n \geq 0$, and therefore, $x = 0$. Thus, the proof is complete. □

Corollary 3.2. *For every $x \in \mathcal{H}$, $\sigma_S(x)$ is a connected set.*

Proof. Suppose for the sake of contradiction that there is a non-zero element x of $\mathcal{H}$ such that $\sigma_S(x)$ is disconnected. So, there is two non-empty disjoint compact subsets σ_1 and σ_2 of $\mathbb{C}$ such that $\sigma_S(x) = \sigma_1 \cup \sigma_2$. It follows from proposition 1.2.16 (g) of [10] that $\mathcal{H}_S\big(\sigma_S(x)\big) = \mathcal{H}_S(\sigma_1) \oplus \mathcal{H}_S(\sigma_2)$. Therefore, there are two non-zero elements x_1 and x_2 of $\mathcal{H}$ such that $x = x_1 + x_2$ and $\sigma_S(x_i) \subset \sigma_i$, $i = 1, 2$. In particular, $\sigma_S(x_1) \cap \sigma_S(x_2) = \emptyset$, which is a contradiction to proposition 3.1. □

We give here a simple proof of theorem 4 of [15] from the point of view of the local spectral theory.

Corollary 3.3. *The spectrum of S is the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq r(S)\}$.*

Proof. By proposition 1.3.2 of [10], we have $\sigma(S) = \bigcup_{x\in\mathcal{H}\backslash\{0\}} \sigma_S(x)$. As each $\sigma_S(x)$ is a connected set, and $\{\lambda \in \mathbb{C} : |\lambda| \leq r_2(S)\} \subset \bigcap_{x\in\mathcal{H}\backslash\{0\}} \sigma_S(x) \neq \emptyset$ (see proposition 3.1), we deduce that $\sigma(S)$ is also a connected set containing the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq r_2(S)\}$. On the other hand, from (R_1) we see that $\sigma(S)$ has circular symmetry about the origin. So, by (R_2), $\sigma(S)$ must be the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq r(S)\}$. □

Remark 3.4. Let $x = \sum_{n\geq 0} a_n e_n$ be a non-zero element of $\mathcal{H}$. It follows from proposition 3.1, that $r_2(S) \leq r_S(x)$. In fact, more can be shown:

$$r_3(S) \leq r_S(x) \leq r(S). \tag{3.3}$$

Indeed, for every integer $n \geq 0$, we have

$$\begin{aligned} S^n x &= \sum_{k\geq 0} a_k S^n e_k \\ &= \sum_{k\geq 0} a_k \frac{\beta_{n+k}}{\beta_k} e_{n+k}. \end{aligned}$$

Since there is an integer $k_0 \geq 0$ such that $a_{k_0} \neq 0$, it then follows that

$$|a_{k_0}| \frac{\beta_{n+k_0}}{\beta_{k_0}} \leq \left[\sum_{k\geq 0} |a_k|^2 \left| \frac{\beta_{n+k}}{\beta_k} \right|^2 \right]^{\frac{1}{2}} = \|S^n x\|, \ \forall n \geq 0.$$

Now, by taking the nth root and then lim sup as $n \to +\infty$, we get $r_3(S) \leq r_S(x)$.

Let x be a non-zero element of $\mathcal{H}$; from corollary 3.2, we see that every circle of radius r, $0 \leq r \leq r_S(x)$, intersects $\sigma_S(x)$. So, in view of remark 3.4, $\sigma_S(x)$ may contain points which are not in the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq r_2(S)\}$. The next result gives a complete description of local spectrum of S at most of the points in $\mathcal{H}$, and refines the local spectral inclusion in proposition 3.1. For every non-zero $x = \sum_{n\geq 0} a_n e_n \in \mathcal{H}$, we set

$$R_\omega(x) := \liminf_{n\to+\infty} \left| \frac{\beta_n}{a_n} \right|^{\frac{1}{n}}.$$

Note that, $r_2(S) \leq R_\omega(x) \leq +\infty$ for every non-zero $x \in \mathcal{H}$.

Theorem 3.5. *For every non-zero $x = \sum_{n\geq 0} a_n e_n \in \mathcal{H}$, we have:*

(*a*) *If $r_3(S) < R_\omega(x)$, then $\sigma_S(x) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_3(S)\}$.*
(*b*) *If $r_3(S) \geq R_\omega(x)$, then $\{\lambda \in \mathbb{C} : |\lambda| \leq R_\omega(x)\} \subset \sigma_S(x)$.*

Proof. Let us first show that

$$\sigma_S(e_0) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_3(S)\}. \tag{3.4}$$

Let $\alpha \in \mathbb{C}$, $|\alpha| = 1$; by (R_1), we have $\alpha S = U_\alpha S U_\alpha^*$. And so, by (R_3), we have $\sigma_S(e_0) = \sigma_{\alpha S}(U_\alpha e_0)$; hence, $\sigma_S(e_0) = \alpha\sigma_S(e_0)$. This shows that $\sigma_S(e_0)$ has circular symmetry about the origin. As $\sigma_S(e_0)$ is a connected set containing the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq r_2(S)\}$, it follows from (R_2) that $\sigma_S(e_0)$ must be the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq r_S(e_0)\}$. Therefore, (3.4) holds, since $r_S(e_0) = r_3(S)$.

Next, keep in mind that for every $\lambda \in \rho_S(e_0) = \{\lambda \in \mathbb{C} : |\lambda| > r_3(S)\}$, we have

$$\begin{aligned}\widetilde{e_0}(\lambda) &= -\sum_{n\geq 0} \frac{S^n e_0}{\lambda^{n+1}} \\ &= -\sum_{n\geq 0} \frac{\beta_n}{\lambda^{n+1}} e_n.\end{aligned}$$

Now, we are able to prove the first statement.

(a) Suppose that $r_3(S) < R_\omega(x)$. Note that the following function $f(\lambda) := \sum\limits_{n\geq 0} \frac{a_n}{\beta_n}\lambda^n$ is analytic on the neighborhood, $\{\lambda \in \mathbb{C} : |\lambda| < R_\omega(x)\}$, of $\sigma_S(e_0)$. Let r be a real number such that $r_3(S) < r < R_\omega(x)$, we have

$$\begin{aligned}f(S, e_0) &= \frac{-1}{2\pi i}\oint_{|\lambda|=r} f(\lambda)\widetilde{e_0}(\lambda)d\lambda \\ &= \frac{-1}{2\pi i}\oint_{|\lambda|=r} f(\lambda)\Big[-\sum_{n\geq 0}\frac{\beta_n}{\lambda^{n+1}}e_n\Big]d\lambda \\ &= \sum_{n\geq 0}\Big[\frac{1}{2\pi i}\oint_{|\lambda|=r}\frac{f(\lambda)}{\lambda^{n+1}}d\lambda\Big]\beta_n e_n \\ &= x.\end{aligned}$$

And so, by theorem 1.1, we have

$$\sigma_S(x) = \sigma_S(e_0) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_3(S)\}.$$

(b) Suppose that $r_3(S) \geq R_\omega(x)$. If $r_2(S) = R_\omega(x)$, then according to proposition 3.1 there is nothing to prove; thus we may suppose that $r_2(S) < R_\omega(x)$. For each $n \geq 0$, let

$$A_n(\lambda) = -\frac{a_0}{\lambda^{n+1}}\beta_n - \frac{a_1}{\lambda^n}\frac{\beta_n}{\beta_1} - \frac{a_2}{\lambda^{n-1}}\frac{\beta_n}{\beta_2} - ... - \frac{a_n}{\lambda},\ \lambda \in \mathbb{C}\backslash\{0\},$$

and

$$B_n(\lambda) = a_0 + \frac{a_1}{\beta_1}\lambda + \frac{a_2}{\beta_2}\lambda^2 + ... + \frac{a_n}{\beta_n}\lambda^n,\ \lambda \in \mathbb{C}.$$

We have,

$$A_n(\lambda) = -\frac{\beta_n}{\lambda^{n+1}}B_n(\lambda),\ \lambda \in \mathbb{C}\backslash\{0\}. \tag{3.5}$$

By writing $\widetilde{x}(\lambda) := \sum\limits_{n\geq 0}\widetilde{A}_n(\lambda)e_n,\ \lambda \in \rho_S(x)$, we get from the following equation,

$$(S-\lambda)\widetilde{x}(\lambda) = x,\ \lambda \in \rho_S(x),$$

that for every $\lambda \in \rho_S(x)$, we have

$$\begin{cases} -\lambda\widetilde{A}_0(\lambda) = a_0 \\ \\ \widetilde{A}_n(\lambda)\omega_n - \lambda\widetilde{A}_{n+1}(\lambda) = a_{n+1} \quad \text{for every } n \geq 0.\end{cases}$$

Therefore, for every $n \geq 0$, we have

$$\widetilde{A}_n(\lambda) = A_n(\lambda) = -\frac{a_0}{\lambda^{n+1}}\beta_n - \frac{a_1}{\lambda^n}\frac{\beta_n}{\beta_1} - \frac{a_2}{\lambda^{n-1}}\frac{\beta_n}{\beta_2} - \ldots - \frac{a_n}{\lambda},\ \lambda \in \rho_S(x).$$

Since $\|\widetilde{x}(\lambda)\|^2 = \sum\limits_{n\geq 0} |\widetilde{A}_n(\lambda)|^2 = \sum\limits_{n\geq 0} |A_n(\lambda)|^2 < +\infty$ for every $\lambda \in \rho_S(x)$, it then follows that

$$\lim_{n\to+\infty} A_n(\lambda) = 0 \text{ for every } \lambda \in \rho_S(x). \tag{3.6}$$

We shall show that (3.6) is not satisfied for most of the points in the open disc $V(x) := \{\lambda \in \mathbb{C} : |\lambda| < R_\omega(x)\}$. It is clear that the sequence $(B_n)_{n\geq 0}$ converges uniformly on compact subsets of $V(x)$ to the non-zero power series $B(\lambda) = \sum\limits_{n\geq 0} \frac{a_n}{\beta_n}\lambda^n$. Now, let $\lambda_0 \in V(x)\backslash\{0\}$ such that $B(\lambda_0) \neq 0$; there is $\epsilon > 0$ and an integer n_0 such that $\epsilon < |B_n(\lambda_0)|$ for every $n \geq n_0$. On the other hand, $|\lambda_0| < r_3(S)$, then there is a subsequence $(n_k)_{k\geq 0}$ of integers greater than n_0 such that $|\lambda_0|^{n_k} < \beta_{n_k}$. Thus, (3.5) gives

$$|A_{n_k}(\lambda_0)| = |-\frac{\beta_{n_k}}{\lambda_0^{n_k+1}}B_{n_k}(\lambda_0)| \geq \frac{\epsilon}{|\lambda_0|},\ \text{for every } k \geq 0.$$

And so, by (3.6), $\lambda_0 \notin \rho_S(x)$. Since the set of zeros of B is at most countable, we have $\{\lambda \in \mathbb{C} : |\lambda| \leq R_\omega(x)\} \subset \sigma_S(x)$. □

Remark 3.6. Let $\mathcal{H}_0$ denote all the finite linear combinations of the vectors e_n, $(n \geq 0)$; it is clearly a dense subspace of $\mathcal{H}$. It follows from theorem 3.5(a) that for every non-zero $x \in \mathcal{H}_0$, we have

$$\sigma_S(x) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_3(S)\}.$$

This can also be deduced from theorem 1.1, and (3.4), since for every non-zero $x \in \mathcal{H}_0$, there is a non-zero polynomial p such that $x = p(S)e_0$.

Question 1. Does $\sigma_S(x) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_S(x)\}$ for every non-zero $x \in \mathcal{H}$?

In view of proposition 3.1, and (3.3), an interesting special case of this question is suggested.

Question 2. Is $\{\lambda \in \mathbb{C} : |\lambda| \leq r_3(S)\} \subset \sigma_S(x)$ for every non-zero $x \in \mathcal{H}$?

3.2. **Which unilateral weighted shift operators satisfy Dunford's condition** (C)**?** The following result gives a necessary condition for the unilateral weighted shift operator S to enjoy Dunford's condition (C).

Theorem 3.7. *If S satisfies Dunford's condition (C), then $r(S) = r_3(S)$.*

Proof. Suppose that S satisfies Dunford's condition (C), and let $F := \{\lambda \in \mathbb{C} : |\lambda| \leq r_3(S)\}$. It follows from the statement (a) of theorem 3.5 that $\mathcal{H}_S(F)$ contains a dense subset of $\mathcal{H}$. As the subspace $\mathcal{H}_S(F)$ is closed, we get $\mathcal{H}_S(F) = \mathcal{H}$; this means that, $\sigma_S(x) \subset F$ for every $x \in \mathcal{H}$. And so, $\sigma(S) = \bigcup\limits_{x\in\mathcal{H}} \sigma_S(x) \subset F$ (see proposition 1.3.2 of [10]). As $r_3(S) \leq r(S)$, and $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq r(S)\}$, we get $r(S) = r_3(S)$. □

An interesting special case occurs when the sequence $([\beta_n]^{\frac{1}{n}})_{n\geq 1}$ converges. Then by combining proposition 3.1, and theorem 3.7, we see that the following statements are equivalent.

(a) $\sigma_S(x) = \sigma(S)$ for every non-zero $x \in \mathcal{H}$.
(b) S satisfies Dunford's condition (C).
(c) $r(S) = \lim\limits_{n\to+\infty} [\beta_n]^{\frac{1}{n}}$.

In considering the general case, we note that if $r(S) = r_3(S)$, then $\sigma_S(x) = \sigma(S)$ for most of the points x in $\mathcal{H}$. Thus, we conjecture that the following statements are equivalent.

(a) $\sigma_S(x) = \sigma(S)$ for every non-zero $x \in \mathcal{H}$.
(b) S satisfies Dunford's condition (C).
(c) $r(S) = r_3(S)$.

Note that a positive answer to question 2 will prove this conjecture.

3.3. **Which unilateral weighted shift operators possess Bishop's property** (β)**?** In [12], T. L. Miller and V. G. Miller have shown that if $r_1(S) < r_2(S)$, then S does not possess Bishop's property (β). In fact, this result could be obtained by combining theorem 3.1 of [19] and theorem 10(ii) of [15]. But the proof given in [12] is simple, and elegant. Here, we refine this result as follows.

Theorem 3.8. *If S possesses Bishop's property (β), then $r_1(S) = r(S)$. Conversely, if $r_1(S) = r(S)$, then either S possesses Bishop's property (β), or $\sigma_\beta(S) = \{\lambda \in \mathbb{C} : |\lambda| = r(S)\}$.*

Proof. Suppose that S possesses Bishop's property (β). According to the above discussion, we have

$$r_1(S) = r_2(S). \tag{3.7}$$

On the other hand, it follows from [3] that S is power-regular, that is the sequence $(\|S^n x\|^{\frac{1}{n}})_{n\geq 0}$ converges for all $x \in \mathcal{H}$. In particular, $(\|S^n e_0\|^{\frac{1}{n}})_{n\geq 0}$ converges; so,

$$r_2(S) = r_3(S). \tag{3.8}$$

As S satisfies Dunford's condition (C), we get from theorem 3.7 that

$$r_3(S) = r(S). \tag{3.9}$$

Therefore, the result follows from (3.7), (3.8), and (3.9).

Conversely, suppose that $r_1(S) = r(S)$. Since $\sigma_{ap}(S) = \{\lambda \in \mathbb{C} : |\lambda| = r(S)\}$ (see theorem 1 of [14] or theorem 6 of [15]), the desired result holds by combining proposition 2.1 and lemma 2.5. $\square$

In view of theorem 3.8, the following question is suggested.

Question 3. Suppose that $r_1(S) = r(S)$. Does S possess Bishop's property (β)?

The next two propositions give a condition on the weight $(\omega_n)_{n\geq 0}$ for which the corresponding weighted shift possesses Bishop's property (β).

Proposition 3.9. *If $m(S) = w(S)$, then S possesses Bishop's property (β).*

Proof. It is well known that for every operator $T \in \mathcal{L}(\mathcal{H})$, and for every λ not in the closure of $W(T)$, the norm of the resolvent $(T-\lambda)^{-1}$ admits the estimate

$$\|(T-\lambda)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, W(T))}. \tag{3.10}$$

If $m(S) = w(S) = 0$, then S is quasi-nilpotent; and so, by proposition 1.6.14 of [10], S is decomposable. Therefore, S possesses Bishop's property (β). Now, without loss of generality suppose that $m(S) = w(S) = 1$. It follows from proposition 16 of [15] that

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset W(S) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\};$$

and so, by (3.10) we have

$$\|(S-\lambda)^{-1}\| \leq \frac{1}{|1-|\lambda||}, \text{ for every } \lambda \in \mathbb{C},\ |\lambda| > 1.$$

This shows that, for every $x \in \mathcal{H}$, we have

$$|1-|\lambda||\|x\| \leq \|(S-\lambda)x\|, \text{ for all } \lambda \in \mathbb{C},\ |\lambda| > 1.$$

On the other hand, for every $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, $|\lambda| < 1$, we have

$$\begin{aligned} \|(S-\lambda)x\| &\geq \|Sx\| - |\lambda|\|x\| \\ &\geq m(S)\|x\| - |\lambda|\|x\| \\ &\geq (1-|\lambda|)\|x\|. \end{aligned}$$

Therefore, for every $\lambda \in \mathbb{C}\backslash\{\mu \in \mathbb{C} : |\mu| = 1\}$ and for every $x \in \mathcal{H}$, we have

$$|1-|\lambda||\|x\| \leq \|(S-\lambda)x\|.$$

And so, it follows from theorem 1.7.1 of [10] that S possesses Bishop's property (β). □

The next example gives a collection of unilateral weighted shift operators possessing Bishop's property (β) which show that the assumption $m(S) = w(S)$ in proposition 3.9 is not a necessary condition.

Example 3.10. Let a be a positive real number, and let S_a be the unilateral weighted shift with the corresponding weight $(\omega_n(a))_{n\geq 0}$ given by

$$\omega_n(a) = \begin{cases} a & \text{if } n = 0 \\ 1 & \text{if } n > 0 \end{cases}$$

It is clear that $m(S_a) = 1$, and S_a possesses Bishop's property (β) since S_a and the unweighted shift operator are unitarily equivalent. On the other hand, it was shown in [5] (see also proposition 2 of [16]) that

(a) If $a \leq \sqrt{2}$, then $w(S_a) = m(S_a) = 1$.

(*b*) If $\sqrt{2} < a$, then $w(S_a) = \frac{a^2}{2\sqrt{a^2-1}} > m(S_a) = 1$.

Proposition 3.11. *If $(\omega_n)_{n\geq 0}$ is a periodic sequence, then S possesses Bishop's property (β).*

Proof. Suppose that $(\omega_n)_{n\geq 0}$ is a periodic sequence of period k. So, for every $x \in \mathcal{H}$, we have $\|S^k x\| = \omega_0 ... \omega_{k-1}\|x\|$. Hence, $\frac{1}{\omega_0...\omega_{k-1}} S^k$ is an isometry; so, by proposition 1.6.7 of [10], S^k possesses Bishop's property (β). Therefore, it follows from theorem 3.3.9 of [10] that S possesses Bishop's property (β). □

Recall that the weight $(\omega_n)_{n\geq 0}$ is said to be *almost periodic* if there is a periodic positive sequence $(p_n)_{n\geq 0}$ such that $\lim\limits_{n\to+\infty}(\omega_n - p_n) = 0$. Note that, if k is the period of $(p_n)_{n\geq 0}$, then

$$r_1(S) = r(S) = (p_0...p_{k-1})^{\frac{1}{k}}.$$

This suggests a weaker version of question 3.

Question 4. Suppose that the weight $(\omega_n)_{n\geq 0}$ is almost periodic. Does S possess Bishop's property (β)?

4. Local spectral properties of bilateral weighted shift operators

In dealing with bilateral weighted shift operators, let $(\beta_n)_{n\in\mathbb{Z}}$ be the sequence given by

$$\beta_n = \begin{cases} \omega_0...\omega_{n-1} & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ \frac{1}{\omega_n...\omega_{-1}} & \text{if } n < 0 \end{cases}$$

and set

$$\begin{aligned} r^-(S) &= \lim_{n\to+\infty}\Big[\sup_{k>0}\frac{\beta_{-k}}{\beta_{-n-k}}\Big]^{\frac{1}{n}}, & r^+(S) &= \lim_{n\to+\infty}\Big[\sup_{k\geq 0}\frac{\beta_{n+k}}{\beta_k}\Big]^{\frac{1}{n}}, \\ r_1^-(S) &= \lim_{n\to+\infty}\Big[\inf_{k>0}\frac{\beta_{-k}}{\beta_{-n-k}}\Big]^{\frac{1}{n}}, & r_1^+(S) &= \lim_{n\to+\infty}\Big[\inf_{k\geq 0}\frac{\beta_{n+k}}{\beta_k}\Big]^{\frac{1}{n}}, \\ r_2^-(S) &= \liminf_{n\to+\infty}\Big[\frac{1}{\beta_{-n}}\Big]^{\frac{1}{n}}, & r_2^+(S) &= \liminf_{n\to+\infty}\Big[\beta_n\Big]^{\frac{1}{n}}, \\ r_3^-(S) &= \limsup_{n\to+\infty}\Big[\frac{1}{\beta_{-n}}\Big]^{\frac{1}{n}}, \text{ and} & r_3^+(S) &= \limsup_{n\to+\infty}\Big[\beta_n\Big]^{\frac{1}{n}}. \end{aligned}$$

Note that,

$$r_1(S) = \min(r_1^-(S), r_1^+(S)), \qquad r(S) = \max(r^-(S), r^+(S)),$$

$$r_1^-(S) \leq r_2^-(S) \leq r_3^-(S) \leq r^-(S), \text{ and } \quad r_1^+(S) \leq r_2^+(S) \leq r_3^+(S) \leq r^+(S).$$

4.1. **On the local spectra of bilateral weighted shift operators.** We begin this section by pointing out that there are bilateral weighted shift operators which do not have the single-valued extension property. The following result, which can easily be proved, gives a necessary and sufficient condition for a bilateral weighted shift operator S to enjoy this property.

Proposition 4.1. *S has the single-valued extension property if and only if $r_2^-(S) \leq r_3^+(S)$.*

Proof. If $r_2^-(S) \leq r_3^+(S)$, then according theorem 9 of [15], $\sigma_p(S)$ has empty interior. And so, S has the single-valued extension property.

Conversely, suppose that $r_3^+(S) < r_2^-(S)$. Let

$$O := \{\lambda \in \mathbb{C} : r_3^+(S) < |\lambda| < r_2^-(S)\}.$$

It is easy to see that the following analytic $\mathcal{H}-$valued function on O,

$$f(\lambda) := \sum_{n \in \mathbb{Z}} \frac{\beta_n}{\lambda^n} e_n,$$

satisfies the equation

$$(S - \lambda)f(\lambda) = 0, \ \ \lambda \in O.$$

This shows that S does not have the single-valued extension property. □

Corollary 4.2. *S^* has the single-valued extension property if and only if $r_2^+(S) \leq r_3^-(S)$.*

Proof. As S^* is also a bilateral weighted shift operator with $r_2^-(S^*) = r_2^+(S)$, and $r_3^+(S^*) = r_3^-(S)$, the result is a direct consequence of proposition 4.1. □

An analogue of proposition 3.1 is given.

Proposition 4.3. *For every non-zero $x \in \mathcal{H}$, we have*

$$\{\lambda \in \mathbb{C} : r_3^-(S) < |\lambda| < r_2^+(S)\} \subset \sigma_S(x).$$

Proof. If $r_2^+(S) \leq r_3^-(S)$, then there is nothing to prove. Assume that $r_3^-(S) < r_2^+(S)$, and set

$$O := \{\lambda \in \mathbb{C} : r_3^-(S) < |\lambda| < r_2^+(S)\}.$$

Now, consider the following analytic $\mathcal{H}$-valued function k defined on O by

$$k(\lambda) = \sum_{n \in \mathbb{Z}} \frac{\lambda^n}{\beta_n} e_n.$$

We have $(S - \lambda)^* k(\overline{\lambda}) = 0$ for every $\lambda \in O$. And so, the rest of the proof goes as in the proof of proposition 3.1. □

Corollary 4.4. *If $r_3^-(S) < r_2^+(S)$, then each $\sigma_S(x)$ is connected.*

Proof. The proof is similar to the proof of corollary 3.2. □

Remark 4.5. A similar proof to that of (3.3) yields

$$r_3^+(S) \le r_S(x) \le r(S), \quad \text{for every non-zero } x \in \mathcal{H}. \tag{4.11}$$

If S is invertible, then S^{-1} is also a bilateral weighted shift with the corresponding weight $(\frac{1}{\omega_n})_{n\in\mathbb{Z}}$. So, by applying (4.11) to S^{-1}, we get

$$\frac{1}{r_2^-(S)} \le r_{S^{-1}}(x) \le r\big(S^{-1}\big), \quad \text{for every non-zero } x \in \mathcal{H}. \tag{4.12}$$

Let $\mathcal{H}_0$ denote all finite linear combinations of the vectors e_n, $(n \in \mathbb{Z})$. The following proposition gives a complete description of the local spectrum of S at a point $x \in \mathcal{H}_0$.

Proposition 4.6. *For every non-zero $x \in \mathcal{H}_0$, we have*

$$\sigma_S(x) = \{\lambda \in \mathbb{C} : r_2^-(S) \le |\lambda| \le r_3^+(S)\}.$$

Proof. First, let us prove that

$$\sigma_S(e_n) = \{\lambda \in \mathbb{C} : r_2^-(S) \le |\lambda| \le r_3^+(S)\}, \ \forall n \in \mathbb{Z}. \tag{4.13}$$

To prove (4.13), we shall limit ourselves to considering the case $n = 0$; the proof for arbitrary $n \in \mathbb{Z}$ is similar.

Let $O_1 := \{\lambda \in \mathbb{C} : |\lambda| > r_3^+(S)\}$, and consider the following $\mathcal{H}$–valued function, defined on O_1 by

$$f_1(\lambda) := \sum_{n=0}^{+\infty} \frac{\beta_n}{\lambda^{n+1}} e_n.$$

As $(S - \lambda)f_1(\lambda) = e_0$ for every $\lambda \in O_1$, we note that

$$\sigma_S(e_0) \subset \{\lambda \in \mathbb{C} : |\lambda| \le r_3^+(S)\}. \tag{4.14}$$

We also have

$$\sigma_S(e_0) \subset \{\lambda \in \mathbb{C} : |\lambda| \ge r_2^-(S)\}. \tag{4.15}$$

Indeed, we may assume $r_2^-(S) > 0$. The following $\mathcal{H}$–valued function, defined on $O_2 := \{\lambda \in \mathbb{C} : |\lambda| < r_2^-(S)\}$ by

$$f_2(\lambda) := \sum_{n=1}^{+\infty} \lambda^{n-1} \beta_{-n} e_{-n},$$

satisfies $(S - \lambda)f_2(\lambda) = e_0$ for every $\lambda \in O_2$. This proves (4.15). And so, it follows from (4.14), and (4.15) that

$$\sigma_S(e_0) \subset \{\lambda \in \mathbb{C} : r_2^-(S) \le |\lambda| \le r_3^+(S)\}. \tag{4.16}$$

If $r_2^-(S) > r_3^+(S)$, then

$$\sigma_S(e_0) = \{\lambda \in \mathbb{C} : r_2^-(S) \le |\lambda| \le r_3^+(S)\} = \emptyset.$$

Thus, we may assume now that $r_2^-(S) \le r_3^+(S)$. Recall that the condition $r_2^-(S) \le r_3^+(S)$ means that S has the single-valued extension property (see proposition 4.1). As in the beginning of the proof of theorem 3.5, one can show that $\sigma_S(e_0)$ has circular symmetry about the origin. Since $r_S(e_0) = r_3^+(S)$, we have

$$\{\lambda \in \mathbb{C} : |\lambda| = r_3^+(S)\} \subset \sigma_S(e_0). \tag{4.17}$$

Now, let us show the following

$$\{\lambda \in \mathbb{C} : r_2^-(S) \leq |\lambda| \leq r_3^+(S)\} \subset \sigma_S(e_0). \tag{4.18}$$

Indeed, from (4.17) we can assume that $r_2^-(S) < r_3^+(S)$. By writing $\widetilde{e_0}(\lambda) = \sum_{n\in\mathbb{Z}} A_n(\lambda)e_n$, $\lambda \in \rho_S(e_0)$, we get from the equation

$$(S - \lambda)\widetilde{e_0}(\lambda) = e_0,\ \lambda \in \rho_S(e_0),$$

that for every $\lambda \in \rho_S(e_0)$, we have

$$A_{-1}(\lambda)\omega_{-1} - \lambda A_0(\lambda) = 1, \tag{4.19}$$

and

$$A_n(\lambda)\omega_n - \lambda A_{n+1}(\lambda) = 0,\ n \in \mathbb{Z},\ n \neq -1. \tag{4.20}$$

And so, from (4.20) we see that for every $\lambda \in \rho_S(e_0)$, we have

$$\frac{\lambda^n}{\beta_n}A_n(\lambda) = A_0(\lambda),\ \text{and}\ A_{-n}(\lambda) = \omega_{-1}\beta_{-n}\lambda^{n-1}A_{-1}(\lambda),\ n \geq 1. \tag{4.21}$$

Now, suppose that there is $\lambda_0 \in \rho_S(e_0)$ such that $r_2^-(S) < |\lambda_0| < r_3^+(S)$. So, there are two subsequences $(n_k)_{k\geq 0}$, and $(m_k)_{k\geq 0}$ such that for every $k \geq 0$, we have

$$|\lambda_0|^{n_k} \leq \beta_{n_k},\ \text{and}\ \frac{1}{\beta_{-m_k}} \leq |\lambda_0|^{m_k}.$$

Therefore, it follows from (4.21) that for every $k \geq 0$, we have

$$|A_0(\lambda_0)| \leq |A_{n_k}(\lambda_0)|,\ \text{and}\ |A_{-m_k}(\lambda_0)| \geq \omega_{-1}|A_{-1}(\lambda_0)|.$$

Since $\lim\limits_{n\to+\infty} A_n(\lambda_0) = \lim\limits_{n\to+\infty} A_{-n}(\lambda_0) = 0$, it follows that $A_0(\lambda_0) = A_{-1}(\lambda_0) = 0$, which is a contradiction to (4.19). Thus, the desired inclusion (4.18) holds. By combining (4.16), and (4.18) we get

$$\sigma_S(e_0) = \{\lambda \in \mathbb{C} : r_2^-(S) \leq |\lambda| \leq r_3^+(S)\}. \tag{4.22}$$

Finally, let $x = \sum_{n\in\mathbb{Z}} a_n e_n \in \mathcal{H}_0\backslash\{0\}$, and let n_0 be the smallest integer n for which $a_n \neq 0$. So, there is a non-zero polynomial p such that $x = p(S)e_{n_0}$. If $r_2^-(S) > r_3^+(S)$, then from the fact that $\sigma_S(p(S)e_{n_0}) \subset \sigma_S(e_{n_0})$ and (4.13), we see that

$$\sigma_S(x) = \{\lambda \in \mathbb{C} : r_2^-(S) \leq |\lambda| \leq r_3^+(S)\} = \emptyset.$$

So, assume now that $r_2^-(S) \leq r_3^+(S)$. If $\sigma_p(S) = \emptyset$, then by theorem 1.1, we have

$$\sigma_S(x) = \sigma_S(e_{n_0}) = \{\lambda \in \mathbb{C} : r_2^-(S) \leq |\lambda| \leq r_3^+(S)\}.$$

If $\sigma_p(S) \neq \emptyset$, then, in view of theorem 9(ii) of [15] and the fact that $r_2^-(S) \leq r_3^+(S)$, we have $r_2^-(S) = r_3^+(S)$ and $\sigma_p(S) = \{\lambda \in \mathbb{C} : |\lambda| = r_3^+(S)\}$. Therefore, by theorem 1.1, we have

$$\sigma_S(e_{n_0}) = \sigma_S(x) \cup Z_S(p, e_{n_0}).$$

As $Z_S(p, e_{n_0})$ is a finite set, and $\sigma_S(e_{n_0}) = \{\lambda \in \mathbb{C} : |\lambda| = r_3^+(S)\}$, we see that

$$\sigma_S(x) = \sigma_S(e_{n_0}) = \{\lambda \in \mathbb{C} : |\lambda| = r_3^+(S)\}.$$

And the desired conclusion holds. $\square$

For every non-zero $x = \sum_{n\in\mathbb{Z}} a_n e_n \in \mathcal{H}$, we set

$$R_\omega^-(x) := \limsup_{n\to+\infty} \Big|\frac{a_{-n}}{\beta_{-n}}\Big|^{\frac{1}{n}}, \text{ and } R_\omega^+(x) := \liminf_{n\to+\infty} \Big|\frac{\beta_n}{a_n}\Big|^{\frac{1}{n}}.$$

Note that $0 \leq R_\omega^-(x) \leq r_3^-(S)$ and $r_2^+(S) \leq R_\omega^+(x) \leq +\infty$ for every non-zero $x \in \mathcal{H}$.

Theorem 4.7. *Assume that $r_2^-(S) \leq r_3^+(S)$. For every non-zero $x \in \mathcal{H}$, the following statements hold.*

(*a*) *If $R_\omega^-(x) < r_2^-(S)$ and $r_3^+(S) < R_\omega^+(x)$, then*

$$\sigma_S(x) = \{\lambda \in \mathbb{C} : r_2^-(S) \leq |\lambda| \leq r_3^+(S)\}.$$

(*b*) *Otherwise,*

$$\{\lambda \in \mathbb{C} : \max(R_\omega^-(x), r_2^-(S)) < |\lambda| < \min(R_\omega^+(x), r_3^+(S))\} \subset \sigma_S(x).$$

Proof. Recall again that the assumption $r_2^-(S) \leq r_3^+(S)$ means that S has the single-valued extension property (see proposition 4.1). Let $x = \sum_{n\in\mathbb{Z}} a_n e_n$ be a non-zero element of $\mathcal{H}$.

(*a*) Assume that $R_\omega^-(x) < r_2^-(S)$, and $r_3^+(S) < R_\omega^+(x)$. Keep in mind that $\sigma_S(e_0) = \{\lambda \in \mathbb{C} : r_2^-(S) \leq |\lambda| \leq r_3^+(S)\}$, and

$$\widetilde{e_0}(\lambda) = \begin{cases} -\sum_{n=0}^{+\infty} \frac{\beta_n}{\lambda^{n+1}} e_n & \text{if } |\lambda| > r_3^+(S) \\ \\ \sum_{n=1}^{+\infty} \lambda^{n-1} \beta_{-n} e_{-n} & \text{if } |\lambda| < r_2^-(S) \end{cases}$$

Now, consider the following analytic $\mathcal{H}$–valued function, defined on the open annulus $\{\lambda \in \mathbb{C} : R_\omega^-(x) < |\lambda| < R_\omega^+(x)\}$ by

$$f(\lambda) := \sum_{n\in\mathbb{Z}} \frac{a_n}{\beta_n} \lambda^n.$$

Let R^+ and R^- be two reals such that

$$r_3^+(S) < R^+ < R_\omega^+(x), \text{ and } R_\omega^-(x) < R^- < r_2^-(S).$$

Let Γ^+ (resp. Γ^-) be the circle centered at the origin and of radius R^+ (resp. R^-). Suppose that Γ^+ (resp. Γ^-) is oriented anti-clockwise (resp. clockwise) direction. We have

$$\begin{aligned} f(S,e_0) &= \frac{-1}{2\pi i}\oint_{\lambda\in\Gamma^+\cup\Gamma^-} f(\lambda)\widetilde{e_0}(\lambda)d\lambda \\ &= \sum_{n=0}^{+\infty}\left[\frac{1}{2\pi i}\oint_{\lambda\in\Gamma^+}\frac{f(\lambda)}{\lambda^{n+1}}d\lambda\right]\beta_n e_n \\ &\quad+\sum_{n=1}^{+\infty}\left[\frac{-1}{2\pi i}\oint_{\lambda\in\Gamma^-} f(\lambda)\lambda^{n-1}d\lambda\right]\beta_{-n}e_{-n} \\ &= x. \end{aligned}$$

And so, as in the last part of the proof of proposition 4.6, we get from theorem 1.1 that

$$\sigma_S(x) = \sigma_S(e_0) = \{\lambda\in\mathbb{C} : r_2^-(S) \le |\lambda| \le r_3^+(S)\}.$$

(b) Suppose that $\max(R_\omega^-(x), r_2^-(S)) < \min(R_\omega^+(x), r_3^+(S))$, otherwise there is nothing to prove. By writing $\widetilde{x}(\lambda) = \sum_{n\in\mathbb{Z}} A_n(\lambda)e_n$, $\lambda\in\rho_S(x)$, we get from the equation

$$(S-\lambda)\widetilde{x}(\lambda) = x,\ \lambda\in\rho_S(x),$$

that for every $\lambda\in\rho_S(x)$, we have

$$A_n(\lambda)\omega_n - \lambda A_{n+1}(\lambda) = a_{n+1},\ \forall n\in\mathbb{Z}. \tag{4.23}$$

In particular, for every $\lambda\in\rho_S(x)$, we have

$$A_{-1}(\lambda)\omega_{-1} - \lambda A_0(\lambda) = a_0. \tag{4.24}$$

And so, by a simple calculation, we get from (4.23) that, for every $\lambda\in\rho_S(x)$ and $n\ge 1$,

$$\frac{\lambda^n}{\beta_n}A_n(\lambda) = A_0(\lambda) - \frac{a_1}{\beta_1} - \frac{a_2}{\beta_2}\lambda - \dots - \frac{a_n}{\beta_n}\lambda^{n-1}, \tag{4.25}$$

and

$$A_{-n}(\lambda) = \frac{\lambda^{n-1}}{\omega_{-2}\dots\omega_{-n}}A_{-1}(\lambda) + \frac{\lambda^{n-2}a_{-1}}{\omega_{-2}\dots\omega_{-n}} + \dots + \frac{a_{-n+1}}{\omega_{-n}}. \tag{4.26}$$

Note that for every $\lambda\in\rho_S(x)$, $\lambda\ne 0$, (4.26) can be reformulated as follows.

$$A_{-n}(\lambda) = \beta_{-n}\lambda^{n-1}\left[\omega_{-1}A_{-1}(\lambda) + \frac{a_{-1}}{\beta_{-1}\lambda} + \frac{a_{-2}}{\beta_{-2}\lambda^2} + \dots + \frac{a_{-n+1}}{\beta_{-n+1}\lambda^{n-1}}\right]. \tag{4.27}$$

Now, suppose that

$$O := \{\lambda\in\mathbb{C} : \max(R_\omega^-(x), r_2^-(S)) < |\lambda| < \min(R_\omega^+(x), r_3^+(S))\}\cap\rho_S(x) \ne \emptyset.$$

Fix $\lambda \in O$. There are two subsequences $(n_k)_{k\geq 0}$, and $(m_k)_{k\geq 0}$ such that for every $k \geq 0$, we have

$$|\lambda|^{n_k} \leq \beta_{n_k}, \text{ and } \frac{1}{\beta_{-m_k}} \leq |\lambda|^{m_k}.$$

And so, from (4.25) and (4.27), we get, respectively,

$$|A_{n_k}(\lambda)| \geq |A_0(\lambda) - \frac{a_1}{\beta_1} - \frac{a_2}{\beta_2}\lambda - ... - \frac{a_{n_k}}{\beta_{n_k}}\lambda^{n_k-1}|,$$

and

$$|A_{-m_k}(\lambda)| \geq |\omega_{-1}A_{-1}(\lambda) + \frac{a_{-1}}{\beta_{-1}\lambda} + \frac{a_{-2}}{\beta_{-2}\lambda^2} + ... + \frac{a_{-m_k+1}}{\beta_{-m_k+1}\lambda^{m_k-1}}|.$$

As $\lim\limits_{n\to+\infty} A_n(\lambda) = \lim\limits_{n\to+\infty} A_{-n}(\lambda) = 0$, and both series $\sum\limits_{n\geq 1} \frac{a_n}{\beta_n}\lambda^{n-1}$ and $\sum\limits_{n\leq -1} \frac{a_n}{\beta_n}\lambda^n$ converge, it follows that

$$A_0(\lambda) = \sum_{n=1}^{+\infty} \frac{a_n}{\beta_n}\lambda^{n-1}, \tag{4.28}$$

and

$$\omega_{-1}A_{-1}(\lambda) = -\sum_{n\leq -1} \frac{a_n}{\beta_n}\lambda^n. \tag{4.29}$$

So, from (4.24), (4.28), and (4.29), we see that $\sum\limits_{n\in\mathbb{Z}} \frac{a_n}{\beta_n}\lambda^n = 0$ for every $\lambda \in O$; therefore, $a_n = 0$ for every $n \in \mathbb{Z}$. This is a contradiction to the fact that $x \neq 0$. And, the proof is complete. $\square$

Note that the importance of theorem 4.7 occurs when $r_2^-(S) > 0$. In this case, we computed the local spectrum of S at most points in $\mathcal{H}$. The next result deals with the case $r_2^-(S) = 0$. For every $k \in \mathbb{Z}$, we write

$$\mathcal{H}_k^+ = \bigvee\{e_n : n \geq k\} \text{ and } \mathcal{H}_k^- = \bigvee\{e_n : n < k\},$$

where "$\bigvee$" denotes the closed linear span.

Theorem 4.8. *Assume that $r_2^-(S) = 0$. Let $k \in \mathbb{Z}$, then for every non-zero $x \in \mathcal{H}_k^+$, the following statements hold.*

(a) *If $r_3^+(S) < R_\omega^+(x)$, then $\sigma_S(x) = \{\lambda \in \mathbb{C} : |\lambda| \leq r_3^+(S)\}$.*
(b) *If $r_3^+(S) \geq R_\omega^+(x)$, then $\{\lambda \in \mathbb{C} : |\lambda| \leq R_\omega^+(x)\} \subset \sigma_S(x)$.*

Proof. Since $\mathcal{H}_k^+$ and $\mathcal{H}_k^-$ are invariant subspaces of S and S^* respectively, we note that $S^+ := S_{|\mathcal{H}_k^+}$ and $S^- := S^*_{|\mathcal{H}_k^-}$ are unilateral weighted shift operators with corresponding weight sequences $\omega^+ := (\omega_n)_{n\geq k}$ and $\omega^- := (\omega_n)_{n<k}$. We have

$$r_i(S^\pm) = r_i^\pm(S), \text{ and } r(S^\pm) = r^\pm(S), \ (i = 1, 2, 3).$$

Moreover, for every $x \in \mathcal{H}_k^+$, we have

$$R_\omega^+(x) = R_{\omega^+}(x).$$

We note also that, since $r_2(S^-) = r_2^-(S) = 0$, S^{-*} has the single-valued extension property. As S has the following matrix representation on $\mathcal{H} = \mathcal{H}_k^+ \oplus \mathcal{H}_k^-$:

$$S = \begin{bmatrix} S^+ & * \\ 0 & S^{-*} \end{bmatrix},$$

it follows from lemma 2.4 that $\sigma_S(x) = \sigma_{S^+}(x)$ for every $x \in \mathcal{H}_k^+$. In view of theorem 3.5, the proof is complete. □

4.2. **Which bilateral weighted shift operators satisfy Dunford's condition** (C)**?** The following result gives a necessary condition for the bilateral weighted shift operator S to satisfy Dunford's condition (C).

Theorem 4.9. *Assume that S satisfies Dunford's condition (C).*

(a) If S is not invertible, then

$$r_1(S) = r_2^-(S) = 0 \leq r_3^+(S) = r(S). \tag{4.30}$$

(b) If S is invertible, then

$$r_1(S) = \frac{1}{r(S^{-1})} = r_2^-(S) \leq r_3^+(S) = r(S). \tag{4.31}$$

Proof. Note that, since S satisfies Dunford's condition (C), S has the single-valued extension property. By proposition 4.1, we have $r_2^-(S) \leq r_3^+(S)$. Now, let $F := \{\lambda \in \mathbb{C} : r_2^-(S) \leq |\lambda| \leq r_3^+(S)\}$; it follows from proposition 4.6 that $\mathcal{H}_S(F)$ contains a dense subspace. As $\mathcal{H}_S(F)$ is a closed subspace, we have $\mathcal{H}_S(F) = \mathcal{H}$; so, $\sigma_S(x) \subset F$ for every $x \in \mathcal{H}$. Since, $\sigma(S) = \bigcup\limits_{x\in\mathcal{H}} \sigma_S(x) \subset F$ (see proposition 1.3.2 of [10]), the desired conclusion holds from theorem 5 of [15]. □

Remark 4.10. An interesting special case occurs when the sequences $([\beta_n]^{\frac{1}{n}})_{n\geq1}$ and $\left([\frac{1}{\beta_{-n}}]\right)_{n\geq1}$ converge respectively to a and b. By combining proposition 4.3, and theorem 4.9, we see that

1. If S is not invertible, then the following statement are equivalent.
   - (a) S has fat local spectra, that is $\sigma_S(x) = \sigma(S)$ for every non-zero $x \in \mathcal{H}$.
   - (b) S satisfies Dunford's condition (C).
   - (c) $b = 0 \leq a = r(S)$.

2. If S is invertible and its spectrum is not a circle, then following statement are equivalent.
   - (a) S has fat local spectra.
   - (b) S satisfies Dunford's condition (C).
   - (c) $b = \frac{1}{r(S^{-1})} < a = r(S)$.

As the unilateral case, we note that in general setting if S is not invertible (resp. invertible) and satisfies (4.30) (resp. (4.31)), then $\sigma_S(x) = \sigma(S)$ for all non-zero x in a dense subset of $\mathcal{H}$ (see proposition 4.6, theorem 4.7 and theorem 4.8). Thus, we conjecture that

1. If S is not invertible, then the following statements are equivalent.
   (a) S has fat local spectra.
   (b) S satisfies Dunford's condition (C).
   (c) $r_1(S) = r_2^-(S) = 0 \leq r_3^+(S) = r(S)$.
2. If S is invertible and its spectrum is not a circle, then the following statements are equivalent.
   (a) S has fat local spectra.
   (b) S satisfies Dunford's condition (C).
   (c) $r_1(S) = r_2^-(S)$, and $r_3^+(S) = r(S)$.

Note that if the spectrum of a bilateral weighted shift operator S is a circle, then S may have fat local spectra (see example 4.19) as well S may satisfy Dunford's condition (C) and without fat local spectra. This is the case for example of any non quasi-nilpotent decomposable bilateral weighted shift operator (see corollary 4.16). But, we do not know in general, if a bilateral weighted shift operator S satisfies always Dunford's condition (C) whenever its spectrum is a circle.

4.3. **Which bilateral weighted shift operators possess Bishop's property (β)?** For an operator $T \in \mathcal{L}(\mathcal{H})$, we shall denote

$$q(T) := \min\{|\lambda| : \lambda \in \sigma(T)\}.$$

Note that

$$q(T) = \begin{cases} 0 & \text{if } T \text{ is not invertible} \\ \frac{1}{r(T^{-1})} & \text{if } T \text{ is invertible.} \end{cases}$$

Now, suppose that S possesses Bishop's property (β). It is shown in theorem 3.10 of [8] that

(a) If S is invertible then S satisfies both (4.32) and (4.33).

(b) If S is not invertible then S satisfies (4.33) and $q(S) = r_2^-(S) = 0$.

On the other hand, it is shown in theorem 2.7 of [13] that

$$q(S) = r_1(S) = r_2^-(S) = r_3^-(S),$$

and

$$r_2^+(S) = r_3^+(S) = r^+(S) = r(S).$$

Therefore, the authors deduce from proposition 4.3 that either S has fat local spectra or $r_1(S) = r(S)$. Using some ideas from both proofs given in [8] and [13], we refine these results as follows.

Theorem 4.11. *If S possesses Bishop's property (β), then*

$$q(S) = r_1^-(S) = r_2^-(S) = r_3^-(S) = r^-(S), \tag{4.32}$$

and

$$r_1^+(S) = r_2^+(S) = r_3^+(S) = r^+(S) = r(S). \tag{4.33}$$

In order to prove this theorem, we shall use the following result from [13].

Proposition 4.12. *Assume that $T \in \mathcal{L}(\mathcal{H})$ is an injective operator. If T possesses Bishop's property (β) then for every $x \in T^{\infty}\mathcal{H}$, the sequence $(\|T^{-n}x\|^{-\frac{1}{n}})_{n\geq 1}$ is convergent and* $\lim\limits_{n\to+\infty} \|T^{-n}x\|^{-\frac{1}{n}} = \min\{|\lambda| : \lambda \in \sigma_T(x)\}$.

Proof of theorem 4.11. Let $\mathcal{H}_0^+ := \bigvee\{e_k : k \geq 0\}$ and let $S^+ := S_{|\mathcal{H}_0^+}$. It is clear that S^+ is a unilateral weighted shift operator with Bishop's property (β). Hence, according theorem 3.8, we have $r_1(S^+) = r(S^+)$. As

$$r_i(S^+) = r_i^+(S),\ r(S^+) = r^+(S),\ (i = 1, 2, 3),$$

and

$$r_1^+(S) \leq r_2^+(S) \leq r_3^+(S) \leq r^+(S),$$

we have

$$r_1^+(S) = r_2^+(S) = r_3^+(S) = r^+(S). \tag{4.34}$$

Note that, since S satisfies Dunford's condition (C), we have $r_3^+(S) = r(S)$ (see theorem 4.9). Therefore, from (4.34), we see that

$$r_1^+(S) = r_2^+(S) = r_3^+(S) = r^+(S) = r(S).$$

Since $r(S) = \max(r^-(S), r^+(S))$, we have

$$r^-(S) \leq r_1^+(S) = r_2^+(S) = r_3^+(S) = r^+(S) = r(S). \tag{4.35}$$

If S is invertible, then S^{-1} possesses also Bishop's property (β) (see theorem 3.3.9 of [10]). As,

$$r^{\pm}(S^{-1}) = \frac{1}{r_1^{\mp}(S)},\ r_1^{\pm}(S^{-1}) = \frac{1}{r^{\mp}(S)} \text{ and } r_{2,3}^{\pm}(S^{-1}) = \frac{1}{r_{3,2}^{\mp}(S)},$$

the desired conclusion holds by applying (4.35) to S^{-1}.

Now, assume that S is not invertible. Since S satisfies Dunford's condition (C), it follows from (4.30) that $r_1(S) = r_2^-(S) = 0$. Hence,

$$q(S) = r_1(S) = r_1^-(S) = r_2^-(S) = 0. \tag{4.36}$$

In view of (4.35) and (4.36), we get from theorem 7 of [15] that

$$\sigma_{ap}(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq r^-(S)\} \cup \{\lambda \in \mathbb{C} : |\lambda| = r(S)\}.$$

Since S is a rationally cyclic injective operator, it follows from proposition 2.2 that

$$\begin{aligned} \Re(S^*) &= \overline{\sigma(S)\backslash\sigma_{ap}(S)} \\ &= \{\lambda \in \mathbb{C} : r^-(S) < |\lambda| < r(S)\}. \end{aligned}$$

On the other hand, we always have

$$\Re(S^*) = \{\lambda \in \mathbb{C} : r_3^-(S) < |\lambda| < r_2^+(S)\}.$$

Therefore,

$$r_3^-(S) = r^-(S). \tag{4.37}$$

Since the sequence $(\|S^{-n}e_0\|^{-\frac{1}{n}})_{n\geq 1}$ is convergent (see proposition 4.12), we have

$$r_2^-(S) = r_3^-(S). \tag{4.38}$$

Thus, the desired result follows from (4.36), (4.37) and (4.38). □

Conversely, we have the following result.

Theorem 4.13. *Assume that S satisfies (4.32) and (4.33). The following statements hold.*

(*a*) *If S is not invertible, then either S possesses Bishop's property (β) or $\sigma_\beta(S) = \{\lambda \in \mathbb{C} : |\lambda| = r(S)\}$.*

(*b*) *If S is invertible, then either S possesses Bishop's property (β) or $\sigma_\beta(S)$ equals either one of the circles $\{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{r(S^{-1})}\}$ and $\{\lambda \in \mathbb{C} : |\lambda| = r(S)\}$ or the union of both of them.*

Proof. (*a*) Assume that S is not invertible. If we set $\mathcal{H}_0^+ = \bigvee\{e_k : k \geq 0\}$ and $\mathcal{H}_0^- = \bigvee\{e_k : k < 0\}$, then S has the following matrix representation on $\mathcal{H} = \mathcal{H}_0^+ \oplus \mathcal{H}_0^-$:

$$S = \begin{bmatrix} S^+ & * \\ 0 & S^{-*} \end{bmatrix},$$

where $S^+ = S_{|\mathcal{H}_0^+}$ and $S^- = S^*_{|\mathcal{H}_0^-}$ are unilateral weighted shift operators with corresponding weight sequences $\omega^+ := (\omega_n)_{n\geq 0}$ and $\omega^- := (\omega_n)_{n<0}$. Since $r(S^-) = r^-(S) = 0$ (see (4.32)), S^- is quasi-nilpotent. In particular, S^{-*} possesses Bishop's property (β). As $r_1(S^+) = r_1^+(S)$, $r(S^+) = r^+(S)$ and $r_1^+(S) = r^+(S) = r(S)$ (see (4.33)), we have $r_1(S^+) = r(S^+) = r(S)$. By theorem 3.8, we see that either S^+ possesses Bishop's property (β) or $\sigma_\beta(S^+) = \{\lambda \in \mathbb{C} : |\lambda| = r^+(S)\}$. Now, the desired conclusion follows from lemma 2.3.

(*b*) Assume that S is invertible and satisfies (4.32) and (4.33). It follows from theorem 7 of [15] that

$$\sigma_{ap}(S) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{r(S^{-1})}\} \cup \{\lambda \in \mathbb{C} : |\lambda| = r(S)\}.$$

By proposition 2.1, and lemma 2.5, the result is established. □

Remark 4.14. As in the unilateral case, we note that if $r_1(S) = r(S)$, then S possesses Bishop's property (β) or $\sigma_\beta(S) = \{\lambda \in \mathbb{C} : |\lambda| = r(S)\}$. But, if S possesses Bishop's property (β), then $r_1(S)$ and $r(S)$ need not be equal as the following example shows. Indeed, let S be the bilateral weighted shift with the corresponding weight $(\omega_n)_{n\in\mathbb{Z}}$ given by

$$\omega_n = \begin{cases} 2 & \text{if } n \geq 0 \\ \\ 1 & \text{if } n < 0 \end{cases}$$

It is clear that S is a hyponormal operator; in particular, S possesses Bishop's property (β). On the other hand, we have $r_1(S) = 1 < r(S) = 2$.

Remark 4.15. In view of remark 4.10, and theorem 4.11, one can construct examples of bilateral weighted shift operators without Bishop's property (β) which satisfy Dunford's condition (C) (see also example 4.19).

We give here a simple proof of the well known following result.

Corollary 4.16. *If S is decomposable then either S is quasi-nilpotent or it is invertible and its spectrum is a circle. In particular, $r_1(S) = r(S)$.*

Proof. If S is decomposable, then both S and S^* possess Bishop's property (β). Since S^* is also a bilateral weighted shift and satifies

$$r_i^{\pm}(S^*) = r_i^{\mp}(S) \text{ and } r^{\pm}(S^*) = r^{\mp}(S), \ (i = 1, 2, 3),$$

the desired result holds by applying theorem 4.11 to S and S^*. □

The next two propositions give a sufficient condition on the weight $(\omega_n)_{n\in\mathbb{Z}}$ so that the corresponding bilateral weighted shift S is decomposable.

Proposition 4.17. *If $m(S) = w(S)$, then S is decomposable.*

Proof. Without loss of generality, assume that $m(S) = w(S) = 1$. As in the proof of proposition 3.9, for every $\lambda \notin \sigma(S) = \{\mu \in \mathbb{C} : |\mu| = 1\}$ and for every $x \in \mathcal{H}$, we have

$$|1 - |\lambda|| \|x\| \leq \|(S - \lambda)x\|.$$

Therefore, it follows from theorem 1.7.2 of [10] that S is decomposable. □

Proposition 4.18. *If $(\omega_n)_{n\in\mathbb{Z}}$ is a periodic sequence, then S is decomposable.*

Proof. Suppose that $(\omega_n)_{n\in\mathbb{Z}}$ is a periodic sequence of period k. So, as in the proof of proposition 3.11, we have $\frac{1}{\omega_0 \ldots \omega_{k-1}} S^k$ is an isometry which is clearly invertible. By, proposition 1.6.7 of [10], S^k is decomposable. And so, it follows from theorem 3.3.9 of [10] that S is decomposable. □

Recall that the weight $(\omega_n)_{n\in\mathbb{Z}}$ is said to be *almost periodic* if there is a periodic positive sequence $(p_n)_{n\in\mathbb{Z}}$ such that $\lim\limits_{|n|\to+\infty} (\omega_n - p_n) = 0$. Note that, if k is the period of $(p_n)_{n\in\mathbb{Z}}$, then

$$r_1(S) = r(S) = (p_0 \ldots p_{k-1})^{\frac{1}{k}}.$$

So, one may ask if S is decomposable when the weight $(\omega_n)_{n\in\mathbb{Z}}$ is almost periodic. It turn out that this is not true as the following example shows.

Example 4.19. Consider the following real-valued continuous function on $\mathbb{R}$

$$\varphi(x) := \exp\left(\frac{|x| \sin\left[\log\log\log(|x| + e^3)\right]}{\log(|x| + e)} \right),$$

and set

$$\omega_n = \exp\left(\varphi(n+1) - \varphi(n)\right), \ n \in \mathbb{Z}.$$

Since $\lim_{|x|\to+\infty} \varphi'(x) = 0$, we have

$$\lim_{|n|\to+\infty} \omega_n = 1.$$

Hence, $(\omega_n)_{n\in\mathbb{Z}}$ is in particular almost periodic and

$$r_1(S) = r(S) = 1.$$

On the other hand, it shown in [4] that $(\omega_n)_{n\in\mathbb{Z}}$ satisfies all the hypotheses of theorem 2 of [4]. Therefore, it follows from the proof of theorem 2 of [4] that

$$\sigma_S(x) = \sigma_{S^*}(x) = \sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \text{ for every non-zero } x \in \mathcal{H}.$$

This shows that both S and S^* have hat local spectra, and then satisfy Dunford's condition (C), but neither S nor S^* possesses Bishop's property (β).

5. Examples and comments

The *quasi-nilpotent part* of an operator $T \in \mathcal{L}(\mathcal{H})$ is the set

$$\mathcal{H}_0(T) = \{x \in \mathcal{H} : \lim_{n\to+\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

It is a linear subspace of $\mathcal{H}$, generally not closed. The operator T is said to have *property* (Q) if $\mathcal{H}_0(T-\lambda)$ is closed for every $\lambda \in \mathbb{C}$. Note that Dunford's condition (C) implies property (Q) and in its turn property (Q) implies the single-valued extension property. In [1], the authors have shown, by a convolution operator of group algebras and a direct sum of unilateral weighted shits, that the above implications are not reversed in general. The purpose of this section is to provide elementrary results in order to produce simpler counter-examples showing that property (Q) is strictly intermediate to Dunford's condition (C) and the single-valued extension property.

Throughout the remainder of this paper, we suppose that S is a unilateral weighted shift operator on $\mathcal{H}$ and the weights ω_n, $(n \geq 0)$, are non-negative. Note that S is injective if and only if none of the weights is zero.

It is shown in proposition 17 of [2] that if S is an injective unilateral weighted shift operator, then $\mathcal{H}_0(S) + \operatorname{ran}(S)$ is dense in $\mathcal{H}$ if and only if $r_3(S) = 0$. However, in view of (3.3) and the fact that $r_S(e_n) = r_3(S)$ for all $n \geq 0$, one can see that either $\mathcal{H}_0(S) = \{0\}$ or $\mathcal{H}_0(S)$ is dense in $\mathcal{H}$. More precisely, we have

Proposition 5.1. *If S is an injective unilateral weighted shift operator, then $\mathcal{H}_0(S)$ is dense in $\mathcal{H}$ if and only if $r_3(S) = 0$.*

The following two propositions enable us to produce examples of operators which have the single-valued extension property and without property (Q).

Proposition 5.2. *The following assertions are equivalent.*

(a) *S^* is quasi-nilpotent.*
(b) *S^* is decomposable.*

(c) *S^* possesses Bishop's property (β).*
(d) *S^* satisfies Dunford's condition (C).*
(e) *S^* has property (Q).*
(f) *$\mathcal{H}_0(S^*)$ is closed.*

Proof. It suffices to show that the implication $(f) \Rightarrow (a)$ holds. For every $n \geq 0$, we have ${S^*}^{n+1} e_n = 0$. This shows that $e_n \in \mathcal{H}_0(S^*)$ for every $n \geq 0$. Since $\mathcal{H}_0(S^*)$ is closed, we have $\mathcal{H}_0(S^*) = \mathcal{H}$. And so, it follows from theorem 1.5 of [17] that S^* is quasi-nilpotent. □

A similar proof of proposition 5.2 yields the following result.

Proposition 5.3. *If infinitely many weights ω_n are zero, then the following statments are equivalent.*
(a) *S is quasi-nilpotent.*
(b) *S is decomposable.*
(c) *S possesses Bishop's property (β).*
(d) *S satisfies Dunford's condition (C).*
(e) *S has property (Q).*
(f) *$\mathcal{H}_0(S)$ is closed.*

Example 5.4. Note that if S is injective, then S^* has the single-valued extension property if and only if $r_2(S) = 0$. Thus, in view of proposition 5.2, we see that if $0 = r_2(S) < r(S)$, then S^* has the single-valued extension property but not property (Q).

It remains to produce a positive bounded weight sequence $(\omega_n)_{n\geq 0}$ such that the corresponding weighted shift operator S satisfies $0 = r_2(S) < r(S)$. Let $(k_i)_{i\geq 1}$ be the sequence given by

$$k_1 = 1 \text{ and } k_{i+1} = (i+1)k_i + 1, \ \forall i \geq 1.$$

Let $(\omega_n)_{n\geq 0}$ be the weight given by

$$\omega_n = \begin{cases} \frac{1}{2^{ik_i}} & \text{if } n = k_i - 1 \text{ for some } i \geq 1 \\ \\ 2 & \text{otherwise} \end{cases}$$

It is easy to check that $\beta_n \leq 1$ for every $n \geq 0$, and that

$$\beta_{k_i - 1} = 1, \text{ and } \beta_{k_i} = \frac{1}{2^{ik_i}} \text{ for every } i \geq 1.$$

This shows that $0 = r_2(S) < 1 = r_3(S) \leq r(S)$. Therefore, S^* has the single-valued extension property and without property (Q).

Example 5.5. Let $(\omega_n)_{n\geq 0}$ be the weight given by

$$\omega_n = \begin{cases} 0 & \text{if } n \text{ is a square of an integer} \\ \\ 1 & \text{otherwise} \end{cases}$$

It is easy to see that $\|S^n\| = 1$ for every $n \geq 1$. This shows that S is not quasi-nilpotent. And so, S has not property (Q) (see Proposition 5.3). On the other hand, S has the single-valued extension property since $\sigma_p(S) = \{0\}$.

If infinitely many weights ω_n are zero, then it is seen in proposition 5.3 that S enjoy property (Q) if and only if it is quasi-nilpotent. When only finitely many weights ω_n are zero, the corresponding weighted shift S is a direct sum of finite-dimensional nilpotent operator S_1 and an injective unilateral weighted shift operator S_2. Therefore, S has property (Q) if and only if S_2 has property (Q). Thus in studying which unilateral weighted shift operator enjoy property (Q) we may assume that none of the weights ω_n is zero.

Proposition 5.6. *If S is injective, then the following statments are equivalent.*

(a) *S has property (Q).*
(b) *Either S is quasi-nilpotent or $r_3(S) > 0$.*

Proof. Since S has the single-valued extension property, we have $\mathcal{H}_0(S-\lambda) = \mathcal{H}_S(\{\lambda\})$ for every $\lambda \in \mathbb{C}$.

Assume that $r_3(S) > 0$. For every non-zero $x \in \mathcal{H}$, we have $\sigma_S(x)$ is a connected set containing 0 (see corollary 3.2) and $r_3(S) \leq r_S(x) \leq r(S)$ (see (3.3)). This implies that $\mathcal{H}_0(S-\lambda) = \mathcal{H}_S(\{\lambda\}) = \{0\}$ for every $\lambda \in \mathbb{C}$. Therefore, S has property (Q) and the implication $(b) \Rightarrow (a)$ holds.

Now, suppose that S has property (Q) and $r_3(S) = 0$. From proposition 5.1 we see that $\mathcal{H}_0(S)$ is a dense subspace of $\mathcal{H}$. As $\mathcal{H}_0(S)$ is closed, we have $\mathcal{H}_0(S) = \mathcal{H}$. By theorem 1.5 of [17], S is quasi-nilpotent. This finishes the proof. ☐

To separate Dunford's condition (C) and property (Q) we need to produce a positive bounded weight sequence $(\omega_n)_{n\geq 0}$ such that the corresponding weighted shift operator S satisfies $0 < r_3(S) < r(S)$ (see theorem 3.7 and proposition 5.6).

Example 5.7. Let $(C_k)_{k\geq 0}$ be a sequence of successive disjoint segments covering the set $\mathbb{N}$ of non-negative integers such that each segment C_k contains k^2 elements. Let $k \in \mathbb{N}$, for $n \in C_k$ we set

$$\omega_n = \begin{cases} 2 & \text{if } n \text{ is one of the first } k^{th} \text{ terms of } C_k \\ \\ 1 & \text{otherwise} \end{cases}$$

It is easy to see that $r_1(S) = r_3(S) = 1 < r(S) = 2$. Therefore, S has property (Q) but not Dunford's condition (C).

The original idea of this construction is due to W. C. Ridge [14].

Finally, we would like to point out that

(a) proposition 5.1 remain valid for an injective bilateral weighted shift operator S by replacing $r_3(S)$ with $r_3^+(S)$.

(*b*) if S is a non injective bilateral weighted shift operator then S is a direct sum of a unilateral weighted shift operator and a backward weighted shift operator. Therefore, the local spectral properties of S can be deduced from the preceding results.

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