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Testing for Structural Change of a Time Trend Regression in Panel Data

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Abstract

In this paper, we propose two classes of test statistics for detecting a break at an unknown date in panel data models with time trend. The first one is the fluctuation test of Ploberger-Kramer-Kontrus (1989). The second one is based on the mean and exponential Wald statistics of Andrew and Ploberger (1994) and maximum Wald statistic of Andrew (1993). We derive the limiting distributions of the proposed tests and tabulate the critical values. Asymptotic results were derived $I(0)$, $I(1)$ and nearly $I(1)$ error terms. We also show that these tests have non-trivial local power only when the error terms are $I(0)$.

1 Introduction

Testing for structural changes has been an important research topic in nonstationary time series econometrics. Recent issues of the *Journal of Business and Statistics* and *Journal of Econometrics* are devoted to such studies, e.g., Chu and White (1992); Hansen (1992); Gregory and Hansen (1996); Campos, Ericsson and Hendry (1996). Kao and Ross (1995) extended the dynamic cumulative sum (CUSUM) test of Kramer, Ploberger and Alt (1988) to the model where serial correlation is present. None of these papers has looked at the tests in the context of the panel data except Han and Park (1989) and Hansen (1999). Han and Park (1989) proposed a CUSUM and a CUSUM of squares tests for panel data models. Hansen (1999) developed methods for testing the threshold effects in panel data. In a recent paper, though not a panel context, Bai, Lumsdaine and Stock (1998) developed methods for testing and constructing asymptotically valid confidence intervals for the date of a single break in multivariate time series, including $I(0)$, $I(1)$ and deterministically trending regressors. They showed there are substantial gains by using multivariate time series which have a common break date.

*An electronic version of the paper in postscript format can be retrieved from http://websyr.edu/~cdkao. Gauss programs which replicate the Monte Carlo studies are also available this website. Address correspondence to: Chihwa Kao, Center for Policy Research, 426 Eggers Hall, Syracuse University, Syracuse, NY 13244-1020; e-mail: cdkao@maxwell.syr.edu.

This paper is a first step in understanding how to test for structural changes when nonstationary panel data is being used. In this paper, we propose two classes of test statistics for detecting a break at an unknown date in panel data models with time trend. The first one is the fluctuation test of Ploberger-Kramer-Kontrus (1989). The second one is based on the mean and exponential Wald statistics of Andrew and Ploberger (1994) and maximum Wald statistic of Andrew (1993). We derive the limiting distributions of the proposed tests and tabulate the critical values. Asymptotic results were derived for $I(0)$, $I(1)$ and nearly $I(1)$ error terms. We also show that these tests have non-trivial local power only when the error terms are $I(0)$.

This paper contributes to the literature of testing for structural changes in two ways. First, we extend the fluctuations tests of Chu and White (1992) and Wald tests of Vogelsang (1997) to panel models. Second, it provides a serious study of the finite sample properties of the proposed tests.

The organization of the paper is as follows. Section 2 introduces the model and test statistics. The limiting distributions of the proposed test statistics with an $I(0)$ error term under the null hypothesis are established. Section 3 gives the limiting distributions of the test statistics under the null hypothesis with an $I(1)$ error term. In Section 4, we discuss the limiting distributions of the test statistics under the null hypothesis when the error is nearly $I(1)$. Section 5 establishes the limiting distributions of test statistics under local alternatives. In section 6 we derive the limiting distributions of the test statistics under both the null hypothesis and the local alternatives for a polynomial trend model. In Section 7 we summarize the findings. All proofs are in the Appendix.

A word on notation. We use $\stackrel{d}{\to}$ to denote convergence in distribution, $\stackrel{p}{\to}$ to denote convergence in probability, $[x]$ to denote the largest integer $\leq x$, and $I(0)$ and $I(1)$ to signify a time series that is integrated of order zero and one, respectively.

2 The Model and the Tests

Consider the following simple linear trend with one-way error component model

$$y_{it} = \alpha + \beta_t x_t + u_{it}, \tag{1}$$

$$u_{it} = \mu_i + v_{it},$$

$i = 1, ..., N$, $t = 1, ...T$, where $\{y_{it}\}$ are 1×1, β is the slope parameters, $x_t = \frac{t}{T}$, $\{\mu_i\}$ are the unobservable individual effects with $\mu_i \sim iid\left(0, \sigma_\mu^2\right)$, and $\{v_{it}\}$ are AR(1) stationary disturbance terms with

$$v_{it} = \rho v_{it-1} + \varepsilon_{it}, |\rho| < 1, \tag{2}$$

where $\varepsilon_{it} \sim iid\left(0, \sigma_\varepsilon^2\right)$. The μ_i are assumed to be independent of v_{it} and $v_{it} \sim \left(0, \sigma_v^2\right), t = 2, ..., T$, where $\sigma_v^2 = \frac{\sigma_\varepsilon^2}{1-\rho^2}$. We assume $v_{i1} = \sum_{j=0}^{[\kappa T]} \rho^j \varepsilon_{i1-j}$, where κ is a parameter that governs the variance of the initial condition. When $\kappa = 0$, v_{i1} is $O_p(1)$. When $\kappa > 0$, v_{i1} is $O_p(1)$ when v_{it} is $I(0)$ but is $O_p(T^{1/2})$ when v_{it} is $I(1)$.

The problem of interest is to test the changes in the parameter β where the change points are unknown. For testing the null hypothesis

$$H_0 : \beta_t = \beta \text{ for all } t. \tag{3}$$

The estimator to be considered is the recursive OLS

$$\widehat{\beta}_k = \frac{\sum_{i=1}^N \left[\sum_{t=1}^k (x_t - \overline{x}_k) y_{it}\right]}{\sum_{i=1}^N \sum_{t=1}^k (x_t - \overline{x}_k)^2}, \tag{4}$$

where

$$\overline{x}_k = \frac{1}{k} \sum_{t=1}^k x_t.$$

Following Ploberger et al. (1989) and Chu and White (1992), the null hypothesis is rejected if $\widehat{\beta}_k$ fluctuate too much, i.e., the null hypothesis is rejected if

$$\max_{i=1,\dots,k} \left|\widehat{\beta}_k - \widehat{\beta}_T\right|$$

is too large. Define the test statistic be

$$T_1 = \sup_{1-[Tr^*]\le k \le T-[Tr^*]} \left|\sqrt{NT}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3 \left(\widehat{\beta}_k - \widehat{\beta}_T\right)\right|, \tag{5}$$

where

$$\sigma_0^2 = \frac{\sigma_\varepsilon^2}{(1-\rho)^2} \tag{6}$$

and r^* is the fraction of trimming, usually taken to be either 0.15 or 0.01. All limits in Theorems 1-3, 5, 7-8, 10, and Lemma 1 are taken as $T \to \infty$ for a fixed N except Theorems 4, 6, 9 which are taken as $T \to \infty$ followed by $N \to \infty$ sequentially. Also all the convergences in all the theorems are uniform convergence in r. We then prove the following theorem:

Theorem 1 *Under H_0.*

$$\sqrt{NT}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3 \left(\widehat{\beta}_k - \widehat{\beta}_T\right) \xrightarrow{d} G_0(r),$$

where

$$G_0(r) = G(r) - r^3 G(1).$$

and

$$G(r) = rW(r) - 2\int_0^r W(s)ds.$$

Theorem 1 provides the limiting distribution of the test statistic in (5). Since

$$\begin{aligned} P(T_1 > c) &\rightarrow P\left(\sup_{r^* \leq r \leq 1-r^*} |G_0(r)| > c\right) \\ &= P\left(\sup_{s^* \leq s \leq 1-s^*} |W_0(s)| > \sqrt{3}c\right) \end{aligned} \tag{7}$$

under H_0, where $W_0(s)$ is a standard Brownian bridge. Note $P\left(\sup_{s^* \leq s \leq 1-s^*} |W_0(s)| > \sqrt{3}c\right)$ is well known (e.g., Chu and White, 1992). Some useful critical values for are .708 (10%), .784 (5%), and .940 (1%).

Consider the alternative hypothesis that there is only one change point k, i.e.,

$$H_1 : \beta_t = \begin{cases} \beta_1 & \text{for } t = 1, ..., k \\ \beta_2 & \text{for } t = k+1, ..., T \end{cases} \tag{8}$$

Let $W(k)$ be the Wald statistic for testing $\beta_1 = \beta_2$:

$$\begin{aligned} \mathrm{W}(k) &= \frac{1}{\widehat{\sigma}_v^2}\left(\widehat{\beta}_{1k} - \widehat{\beta}_{2k}\right)' \left[\left(\sum_{i=1}^N \sum_{t=1}^k (x_t - \overline{x}_{1k})^2\right)^{-1} + \left(\sum_{i=1}^N \sum_{t=k+1}^T (x_t - \overline{x}_{2k})^2\right)^{-1}\right]^{-1} \left(\widehat{\beta}_{1k} - \widehat{\beta}_{2k}\right) \\ &= \frac{1}{\widehat{\sigma}_v^2} \frac{\left(\widehat{\beta}_{1k} - \widehat{\beta}_{2k}\right)^2}{\left[\left(\sum_{i=1}^N \sum_{t=1}^k (x_t - \overline{x}_{1k})^2\right)^{-1} + \left(\sum_{i=1}^N \sum_{t=k+1}^T (x_t - \overline{x}_{2k})^2\right)^{-1}\right]}, \end{aligned} \tag{9}$$

where

$$\widehat{\sigma}_v^2 = \frac{1}{NT}\sum_{i=1}^N \sum_{t=1}^T (u_{it} - \overline{u}_i)^2,$$

$$\widehat{\beta}_{1k} = \frac{\sum_{i=1}^N \left[\sum_{t=1}^k (x_t - \overline{x}_{1k})y_{it}\right]}{\sum_{i=1}^N \sum_{t=1}^k (x_t - \overline{x}_{1k})^2},$$

$$\widehat{\beta}_{2k} = \frac{\sum_{i=1}^N \left[\sum_{t=k+1}^T (x_t - \overline{x}_{2k})y_{it}\right]}{\sum_{i=1}^N \sum_{t=k+1}^T (x_t - \overline{x}_{2k})^2},$$

$$\overline{x}_{1k} = \frac{1}{k}\sum_{t=1}^k x_t,$$

and

$$\overline{x}_{2k} = \frac{1}{T-k}\sum_{t=k+1}^T x_t.$$

Remark 1 *Under (8), (1) can be written as*

$$\begin{aligned} y_{it} &= \alpha + \beta_1 x_t I(t \leq k) + \beta_2 x_t I(t > k) + u_{it} \\ &= \alpha + x_t^{'}(k)\,\beta^* + u_{it} \end{aligned} \tag{10}$$

where $\beta^* = (\beta_1, \beta_2)^{'}$ *and*

$$x_i(k) = \begin{cases} x_t I(t \leq k) \\ x_t I(t > k) \end{cases}.$$

Let

$$\begin{aligned} \overline{x}_t(k) &= \frac{1}{T}\sum_{t=1}^{T} x_t\,(k) \\ &= \begin{cases} \frac{1}{T}\sum_{t=1}^{T} x_t I(t \leq k) \\ \frac{1}{T}\sum_{t=1}^{T} x_t I(t > k) \end{cases}. \end{aligned}$$

Then

$$\begin{aligned} \widehat{\sigma}_v^2 &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(u_{it} - \overline{u}_i)^2 \\ &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[(y_{it} - \overline{y}_i) - (x_t(k) - \overline{x}_t(k))^{'}\,\widehat{\beta}^*\right]^2. \end{aligned}$$

Then we have the following theorem:

Theorem 2 *Under* H_0 :

$$\widehat{\sigma}_v^2 \xrightarrow{p} \sigma_v^2$$

and

$$\mathrm{W}(k) \xrightarrow{d} \frac{3\sigma_0^2}{\sigma_v^2} Q_1\,(r)$$

where

$$Q_1(r) = \left\{ \frac{G(r)(1-r)^3 - r^3\left[G(1) - G(r) + W(r) - rW(1)\right]}{\left[r^3(1-r)^3\left[(1-r)^3 + r^3\right]\right]^{\frac{1}{2}}} \right\}^2.$$

Define

$$\mathrm{W}_1(k) = \frac{\widetilde{\sigma}_v^2}{3\sigma_0^2}\mathrm{W}(k)$$

where

$$\widetilde{\sigma}_v^2 = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(u_{it} - \overline{u}_i)^2$$

is the estimate of σ_v^2 under $H_{0.}$ Following Vogelsang (1997), we consider three statistics: $sup\mathrm{W}_1(k)$, $Mean\mathrm{W}_1(k)$, and $Exp\mathrm{W}_1(k)$, where

$$sup\mathrm{W}_1(k) = \sup_{[Tr^*]\leq k\leq T-[Tr^*]} \mathrm{W}_1(k),$$

$$Mean\mathrm{W}_1(k) = \frac{1}{T}\sum_{k=[Tr^*]}^{T-[Tr^*]} \mathrm{W}_1(k),$$

and

$$Exp\mathrm{W}_1(k) = \log\left(\frac{1}{T}\sum_{k=[Tr^*]}^{T-[Tr^*]} \exp\left(\frac{1}{2}\mathrm{W}_1(k)\right)\right).$$

Using the continuous mapping theorem we then have the following corollary:

Corollary 1 *Under H_0 :*

1. $sup\mathrm{W}_1(k) \xrightarrow{d} \sup_{r^*\leq r\leq 1-r^*} Q_1(r),$
2. $Mean\mathrm{W}_1(k) \xrightarrow{d} \int_{r^*}^{1-r^*} Q_1(r)dr,$
3. $Exp\mathrm{W}_1(k) \xrightarrow{d} \log\left(\int_{r^*}^{1-r^*} \exp\left(\frac{1}{2}Q_1(r)\right)dr\right).$

Remark 2 *1. We write time trend in the form of $x_t = \frac{t}{T}$ in (1) to avoid a scaling matrix that would otherwise be required when deriving limiting distributions.*

2. In practice, we need to replace σ_0^2 by a consistent estimator, $\widehat{\sigma}_0^2$.(e.g., Andrews, 1991). The limiting distributions of T_1 and $\mathrm{W}_1(k)$ will be unchanged if we replace σ_0^2 by $\widehat{\sigma}_0^2$.

3. The results in this session will not change if we replace an AR(1) assumption in (2) by a more general process, e.g.,

$$v_{it} = \psi(L)\varepsilon_{it} = \sum_{j=0}^{\infty}\psi_j\varepsilon_{it-j},$$

where $\sum_{j=0}^{\infty} j\left|\psi_j\right| < \infty$ and $\varepsilon_{it} \sim iid\left(0,\sigma_\varepsilon^2\right)$. Then $\sigma_0^2 = \sigma_\varepsilon^2\psi^2(1)$. Note $\psi(1) = \frac{1}{1-\rho}$ if v_{it} is assumed to be an AR(1) in (2).

4. Also the results of this section will not change if we replace iid by martingale difference sequence (MDF) for ε_{it}.

5. *The homogeneity assumption made for σ_ε^2 and ρ across i can be relaxed by allowing $\sigma_{\varepsilon i}^2$ and ρ_i to differed for different i. The limiting distributions of T_1 and $\mathrm{W}_1(k)$ will be unchanged if we define σ_0^2 and σ_v^2 as follows*

$$\sigma_0^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\sigma_{\varepsilon i}^2}{(1-\rho_i)^2}$$

and

$$\sigma_v^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\sigma_{\varepsilon i}^2}{1-\rho_i^2}.$$

3 The Limiting Distribution of the Test Statistics when $\rho = 1$

Model (2) is restrictive because it excludes v_{it} to be $I(1)$. We investigated the asymptotic properties of the two test statistics, T_1 and $\mathrm{W}_1(k)$, in Section 2. In this section v_{it} is $I(1)$. We will show that the previous conclusions in Section 2 are substantially altered when v_{it} is $I(1)$. Define

$$T_2 = \sup_{1\le k\le T-1} \left| \sqrt{NT^{-1}} \frac{1}{6\sigma_\varepsilon} \left(\frac{k}{T}\right)^3 \left(\widehat{\beta}_k - \widehat{\beta}_T\right) \right|. \tag{11}$$

Theorem 3 *Under H_0 and $v_{it} = v_{it-1} + \varepsilon_{it}$ then*

$$\sqrt{NT^{-1}} \frac{1}{6\sigma_\varepsilon} \left(\frac{k}{T}\right)^3 \left(\widehat{\beta}_k - \widehat{\beta}_T\right) \xrightarrow{d} H_0(r),$$

where

$$H_0(r) = H(r) - r^3 H(1),$$

$$H(r) = 2\int_0^r s\left[W(s) + \widetilde{W}(\kappa)\right] ds - r\int_0^r \left[W(s) + \widetilde{W}(\kappa)\right] ds,$$

and

$$H(1) = 2\int_0^1 s\left[W(s) + \widetilde{W}(\kappa)\right] ds - \int_0^1 \left[W(s) + \widetilde{W}(\kappa)\right] ds.$$

From Theorem 2 we know that the limiting distribution of T_2 differs from T_1. Next we present the limiting distribution of $W_T(k)$ when $\rho = 1$.

Theorem 4 *Under H_0 and $v_{it} = v_{it-1} + \varepsilon_{it}$ then*

$$\frac{1}{T}\widehat{\sigma}_v^2 \xrightarrow{p} \sigma_\varepsilon^2 \left(\frac{1}{6} + \kappa\right)$$

and

$$\frac{1}{T}\mathrm{W}(k) \xrightarrow{d} \frac{18}{1+6\kappa} Q_2(r)$$

where

$$Q_2(r)=\left\{\frac{H(r)\left(1-r\right)^3-\left[H(1)-H(r)-r\int_0^1 W(s)ds+\int_0^r W(s)ds\right]r^3}{\left[r^3\left(1-r\right)^3\left[\left(1-r\right)^3+r^3\right]\right]^{1/2}}\right\}^2$$

Define $\mathrm{W}_2(k)=\frac{(1+6\kappa)}{18}\frac{1}{T}\mathrm{W}(k)$. Then we have following corollary:

Corollary 2 *Under* H_0 :

1. $sup\mathrm{W}_2(k)\xrightarrow{d}\sup_{r^*\leq r\leq 1-r^*}Q_2(r)$,
2. $Mean\mathrm{W}_2(k)\xrightarrow{d}\int_{r^*}^{1-r^*}Q_2(r)dr$,
3. $Exp\mathrm{W}_2(k)\xrightarrow{d}\log\left(\int_{r^*}^{1-r^*}\exp\left(\frac{1}{2}Q_2(r)\right)dr\right)$.

Remark 3 *In practice, we need to replace* σ_ε^2 *and* κ *by their consistent estimator,* $\widehat{\sigma}_\varepsilon^2$ *and* $\widehat{\kappa}$. *The limiting distributions of* T_2 *and* $\mathrm{W}_2(k)$ *will be unchanged if we replace* σ_ε^2 *and* κ *by* $\widehat{\sigma}_\varepsilon^2$ *and* $\widehat{\kappa}$. $\widehat{\sigma}_\varepsilon^2$ *can be formed by*

$$\widehat{\sigma}_\varepsilon^2=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{it}-y_{it-1}\right)^2.$$

κ *can be estimated using the following relationship: under* H_0 *we have*

$$\frac{1}{T}\widehat{\sigma}_v^2\xrightarrow{p}\sigma_\varepsilon^2\left(\frac{1}{6}+\kappa\right).$$

That is

$$\widehat{\kappa}=\frac{1}{T}\frac{\widetilde{\sigma}_v^2}{\widehat{\sigma}_\varepsilon^2}-\frac{1}{6}.$$

Moon and Phillips (1999) also proposed a consistent estimator for κ.

Some of the critical values of the proposed test statistics in Sections 2 and 3 are tabulated in Table 1.

4 Nearly I(1) Errors

In recent years, there has been considerable interest in the asymptotic properties of the estimation and inference of β in (1) when ρ is close to one in the time-series (i.e., when $N=1$) econometrics literature. In this section we assume $\rho=1+c/T$ in (2), i.e., the v_{it} follows a local-to-unit or a nearly $I(1)$ process. The asymptotics for the test statistic T_1 are given in the following theorem:

Theorem 5 *Under H_0 and $v_{it} = \rho v_{it-1} + \varepsilon_{it}$, $\rho = 1 + c/T$, then*

$$\sqrt{NT^{-1}}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k - \widehat{\beta}_T\right) \xrightarrow{d} H_c(r) - r^3 H_c(1),$$

where

$$H_c(r) = 2\int_0^r s\left[W_c(s) + e^{sc}\tilde{W}_c(\kappa)\right]ds - r\int_0^r \left[W_c(s) + e^{sc}\tilde{W}_c(\kappa)\right]ds$$

and

$$H_c(1) = 2\int_0^1 s\left[W_c(s) + e^{sc}\tilde{W}_c(\kappa)\right]ds - \int_0^1 \left[W_c(s) + e^{sc}\tilde{W}_c(\kappa)\right]ds.$$

Next we present the limiting distribution of the Wald Statistic when $\rho = 1 + c/T$.

Theorem 6 *Under H_0 and $v_{it} = \rho v_{it-1} + \varepsilon_{it}$, $\rho = 1 + c/T$, then*

$$\frac{1}{T}\mathrm{W}(k) \xrightarrow{d} \frac{18c}{c + 3\kappa\left(e^{2c} - 1\right)}Q_c(r),$$

where

$$Q_c(r) = \left\{\frac{H_c(r)\left(1-r\right)^3 - \left[H_c(1) - H_c(r) - r\int_0^1\left[W_c(s) + e^{sc}\tilde{W}_c(\kappa)\right]ds + \int_0^r\left[W_c(s) + e^{sc}\tilde{W}_c(\kappa)\right]ds\right]r^3}{\left[r^3\left(1-r\right)^3\left[\left(1-r\right)^3 + r^3\right]\right]^{1/2}}\right\}^2.$$

Remark 4 *Note*

$$\lim_{c\to 0}\left(H_c(r) - r^3 H_c(1)\right) = H(r) - r^3 H(1)$$

and

$$\lim_{c\to 0}\left(\frac{18c}{c + 3\kappa\left(e^{2c} - 1\right)}Q_c(r)\right) = \frac{18}{1 + 6\kappa}Q_2(r).$$

5 Local Asymptotic Power

Consider the local alternative:

$$\beta_t^{(T)} = \beta + \frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right), \tag{12}$$

where g is an arbitrary function defined on $[0,1]$. Define

$$y_{it}^{(T)} = \alpha + \beta_t^{(T)}x_t + u_{it}$$

and let

$$\widehat{\beta}_k^{(T)} = \frac{\sum_{i=1}^N\sum_{t=1}^k(x_t - \overline{x}_k)y_{it}^{(T)}}{\sum_{i=1}^N\sum_{t=1}^k(x_t - \overline{x}_k)^2}$$

be the OLS estimator under the local alternative (12). Similarly, let

$$T_1^{(T)} = \sup_{1\le k\le T-1}\left|\sqrt{NT}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)} - \widehat{\beta}_T^{(T)}\right)\right|,$$

$$T_2^{(T)} = \sup_{1\le k\le T-1}\left|\sqrt{NT^{-1}}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)} - \widehat{\beta}_T^{(T)}\right)\right|,$$

$$\mathrm{W}^{(T)}(k) = \frac{1}{\widehat{\sigma}_v^2}\frac{\left(\widehat{\beta}_{1k}^{(T)} - \widehat{\beta}_{2k}^{(T)}\right)^2}{\left[\left(\sum_{i=1}^N\sum_{t=1}^k (x_t - \overline{x}_{1k})^2\right)^{-1} + \left(\sum_{i=1}^N\sum_{t=k+1}^T (x_t - \overline{x}_{2k})^2\right)^{-1}\right]},$$

$$\mathrm{W}_1^{(T)}(k) = \frac{\widehat{\sigma}_v^2}{3\sigma_0^2}\mathrm{W}^{(T)}(k),$$

and

$$\mathrm{W}_2^{(T)}(k) = \frac{(1+6\kappa)}{18}\frac{1}{T}\mathrm{W}^{(T)}(k)$$

be the corresponding test statistics under local alternative.

Theorem 7 *Under the local alternatives (12),*

1. *If* $|\rho| < 1$, *then*

$$\sqrt{NT}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)} - \widehat{\beta}_T^{(T)}\right) \xrightarrow{d} G_0(r) + \frac{2}{\sigma_0}h_0(r),$$

where

$$h_0(r) = h(r) - r^3 h(1),$$

$$h(r) = \int_0^r s^2 g(s)ds - \frac{1}{2}r\int_0^r sg(s)ds,$$

and

$$h(1) = \int_0^1 s^2 g(s)ds - \frac{1}{2}\int_0^1 sg(s)ds.$$

2. *If* $\rho = 1$, *then*

$$\sqrt{NT^{-1}}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)} - \widehat{\beta}_T^{(T)}\right) \xrightarrow{d} H_0(r),$$

3. *If* $\rho = 1 + \frac{c}{T}$, *then*

$$\sqrt{NT^{-1}}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)} - \widehat{\beta}_T^{(T)}\right) \xrightarrow{d} H_c(r) - r^3 H_c(1).$$

If $h_0(r) = 0$ we obtain the distribution under the null. Next we consider the behavior of $W(k)$ under sequences of local alternatives.

Theorem 8 *Under the local alternatives (12),*

1. *If* $|\rho| < 1$, *then*

$$W^{(T)}(k) \xrightarrow{d} \frac{3\sigma_0^2}{\sigma_v^2} \left\{ [Q_1(r)]^{\frac{1}{2}} + \frac{2}{\sigma_0} [Q_3(r)]^{\frac{1}{2}} \right\}^2,$$

where

$$Q_3(r) = \left\{ \frac{(1-r)^3 h(r) - r^3 \left[h(1) - h(r) - \frac{1}{2} r \int_0^1 sg(s)ds + \frac{1}{2} \int_0^r sg(s)ds \right]}{\left[r^3 (1-r)^3 \left[(1-r)^3 + r^3 \right] \right]^{\frac{1}{2}}} \right\}^2.$$

2. *If* $\rho = 1$, *then*

$$\frac{1}{T} W_T(k) \xrightarrow{d} \frac{18}{1+6\kappa} Q_2(r),$$

3. *If* $\rho = 1 + \frac{c}{T}$, *then*

$$\frac{1}{T} W^{(T)}(k) \xrightarrow{d} \frac{18c}{c + 3\kappa \left(e^{2c} - 1 \right)} Q_c(r).$$

Remark 5 *The asymptotic local power of* $T_2^{(T)}$ *and* $W_2^{(T)}$ *are equal to the size of the tests.*

6 Polynomial Trend

Consider the panel polynomial regression

$$y_{it} = \alpha + \beta_{1t} x_t + \cdots + \beta_{pt} x_t^p + u_{it}. \tag{13}$$

The null hypothesis is $\beta_{1t} = \beta_1$, $\beta_{2t} = \beta_2, \ldots,$ and $\beta_{pt} = \beta_p$. Let $W_{pT}(k)$ be the Wald statistic for testing the null hypothesis.

$$\begin{aligned} W_{pT}(k) &= (Rb-q)' \left[var(Rb-q) \right]^{-1} (Rb-q) \\ &= \left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)} \right)' \left[\widehat{\sigma}_v^2 R(Var(b))R' \right]^{-1} \left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)} \right) \\ &= \frac{1}{\widehat{\sigma}_v^2} \left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)} \right)' \left[R(Var(b))R' \right]^{-1} \left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)} \right), \end{aligned}$$

where

$$\begin{aligned} b &= \left(\widehat{\beta}_{1(1k)}, \widehat{\beta}_{2(1k)}, \ldots, \widehat{\beta}_{p(1k)}, \widehat{\beta}_{1(2k)}, \widehat{\beta}_{2(2k)}, \ldots, \widehat{\beta}_{p(2k)} \right)' \\ &= \left(\underline{\widehat{\beta}}_{(1k)}, \underline{\widehat{\beta}}_{(2k)} \right)' \end{aligned}$$

is a $2p \times 1$ vector,

$$Var\,(b) = \begin{bmatrix} \left[\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t - \overline{x}_{(1k)})'(\underline{x}_t - \overline{x}_{(1k)})\right]^{-1} & 0_{p\times p} \\ 0_{p\times p} & \left[\sum_{i=1}^{N}\sum_{t=k+1}^{T}(\underline{x}_t - \overline{x}_{(2k)})'(\underline{x}_t - \overline{x}_{(2k)})\right]^{-1} \end{bmatrix}$$

is a $2p \times 2p$ matrix, and

$$R = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots & 0 & -1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 \end{bmatrix}$$

is a $p \times 2p$ matrix. Also,

$$\widehat{\sigma}_v^2 = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(u_{it} - \overline{u}_i)^2\,,$$

$$\underline{\widehat{\beta}}_{(1k)} = \left[\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t - \overline{x}_{(1k)})'(\underline{x}_t - \overline{x}_{(1k)})\right]^{-1}\left[\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t - \overline{x}_{(1k)})'y_{it}\right],$$

$$\underline{\widehat{\beta}}_{(2k)} = \left[\sum_{i=1}^{N}\sum_{t=k+1}^{T}(\underline{x}_t - \overline{x}_{(2k)})'(\underline{x}_t - \overline{x}_{(2k)})\right]^{-1}\left[\sum_{i=1}^{N}\sum_{t=k+1}^{T}(\underline{x}_t - \overline{x}_{(2k)})'y_{it}\right],$$

where

$$\underline{x}_t = (x_t, \ldots, x_t^p)\ \ ,$$

$$\overline{x}_{(1k)} = \left(\frac{1}{k}\sum_{t=1}^{k}x_t, \cdots, \frac{1}{k}\sum_{t=1}^{k}x_t^p\right)\ \ ,$$

and

$$\overline{x}_{(2k)} = \left(\frac{1}{T-k}\sum_{t=k+1}^{T}x_t, \ldots, \frac{1}{T-k}\sum_{t=k+1}^{T}x_t^p\right).$$

More specifically, we calculate the Wald statistic for the model (13) with $p = 2:$

$$y_{it} = \alpha + \beta_{1t}x_t + \beta_{2t}x_t^2 + u_{it}.$$

Thus we have

$$W_p(k) = \frac{1}{\widehat{\sigma}_v^2}\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)}\right)'\left[R(Var(b))R'\,\right]^{-1}\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)}\right),$$

where

$$\begin{aligned} b &= \left(\widehat{\beta}_{1(1k)}, \widehat{\beta}_{2(1k)}, \widehat{\beta}_{1(2k)}, \widehat{\beta}_{2(2k)}\right)' \\ &= \left(\underline{\widehat{\beta}}_{(1k)}, \underline{\widehat{\beta}}_{(2k)}\right)' \end{aligned}$$

is a 4×1 vector,

$$Var\left(b\right) = \left[\begin{array}{cc} \left[\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t - \overline{x}_{(1k)})'(\underline{x}_t - \overline{x}_{(1k)})\right]^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & \left[\sum_{i=1}^{N}\sum_{t=k+1}^{T}(\underline{x}_t - \overline{x}_{(2k)})'(\underline{x}_t - \overline{x}_{(2k)})\right]^{-1} \end{array} \right]$$

is a 4×4 matrix, and

$$R = \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array} \right]$$

is a 2×4 matrix,

$$\underline{x}_t = \left(x_t, x_t^2\right),$$

$$\overline{x}_{(1k)} = \left(\frac{1}{k}\sum_{t=1}^{k} x_t, \frac{1}{k}\sum_{t=1}^{k} x_t^2\right),$$

and

$$\overline{x}_{(2k)} = \left(\frac{1}{T-k}\sum_{t=k+1}^{T} x_t, \frac{1}{T-k}\sum_{t=k+1}^{T} x_t^2\right).$$

Theorem 9 *Under H_0 we have the following:*

1. *If $|\rho| < 1$, then*
$$W_p(k) \xrightarrow{d} \frac{\sigma_0^2}{\sigma_v^2} P_1\left(r\right),$$

2. *If $\rho = 1$, then*
$$\frac{1}{T} W_p(k) \xrightarrow{d} \frac{24}{(1+6\kappa)} P_2\left(r\right),$$

3. *If $\rho = 1 + \frac{c}{T}$, then*
$$\frac{1}{T} W_p(k) \xrightarrow{d} \frac{24c}{\left(c + 3\kappa\left(e^{2c} - 1\right)\right)} P_c\left(r\right),$$

where $P_1\left(r\right)$, $P_2\left(r\right)$, and $P_c\left(r\right)$ are given in the Appendix.

Next we consider the behavior of $W_p(k)$ under sequences of local alternatives.

Theorem 10 *Under the local alternatives (12),*

1. *If $|\rho| < 1$, then*
$$W_p^{(T)}\left(k\right) \xrightarrow{d} \frac{\sigma_0^2}{\sigma_v^2}\left[P_1\left(r\right) + \frac{1}{\sigma_0^2} R_1\left(r\right)\right],$$

2. *If $\rho = 1$, then*
$$\frac{1}{T} W_p^{(T)}(k) \xrightarrow{d} \frac{24}{(1+6\kappa)} P_2\left(r\right),$$

3. If $\rho = 1 + \frac{c}{T}$, then

$$\frac{1}{T} W_p^{(T)}(k) \xrightarrow{d} \frac{24c}{(c + 3\kappa\,(e^{2c} - 1))} P_c\,(r)\,,$$

where $P_1\,(r)$, $P_2\,(r)$, $P_c\,(r)$, and $R_1\,(r)$ are given in the Appendix.

7 Conclusion

In this paper, we propose two classes of test statistics for detecting a break at an unknown date in panel data models with time trend. We derive the limiting distributions of the proposed tests and tabulate the critical values. Asymptotic results were derived $I(0)$, $I(1)$ and nearly $I(1)$ error terms. We also show that these tests have non-trivial local power only when the error terms are $I(0)$.

Appendix

A Proof of Theorem 1

Proof. Note that $\widehat{\beta}_k - \widehat{\beta}_T$ can be written as

$$\widehat{\beta}_k - \widehat{\beta}_T = \left(\widehat{\beta}_k - \beta\right) - \left(\widehat{\beta}_T - \beta\right).$$

Under H_0 we know

$$\begin{aligned} \sqrt{NT}\left(\widehat{\beta}_k - \beta\right) &= \frac{\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[\sum_{t=1}^{k} (x_t - \overline{x}_k) u_{it}\right]}{\frac{1}{NT} N \sum_{t=1}^{k} (x_t - \overline{x}_k)^2} \\ &= \frac{\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[\frac{1}{\sqrt{T}} \sum_{t=1}^{k} (x_t - \overline{x}_k) v_{it}\right]}{\frac{1}{T} \sum_{t=1}^{k} (x_t - \overline{x}_k)^2}. \end{aligned}$$

Note

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^{k} (x_t - \overline{x}_k)^2 &= \frac{k}{T} \frac{1}{k} \sum_{t=1}^{k} (x_t - \overline{x}_k)^2 \\ &= \frac{k}{T} \frac{1}{12} \frac{k^2 - 1}{T^2} \to \frac{1}{12} r^3 \end{aligned}$$

as $T \to \infty$ and

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{k} (x_t - \overline{x}_k) v_{it} = \frac{1}{\sqrt{T}} \sum_{t=1}^{k} (\frac{t}{T} - \frac{1}{k} \sum_{t=1}^{k} \frac{t}{T}) v_{it}$$

$$
\begin{aligned}
&= \frac{1}{T^{3/2}}\sum_{t=1}^{k}(t-\frac{k\left(k+1\right)}{2k})v_{it} \\
&= \frac{1}{T^{3/2}}\left[\sum_{t=1}^{k}tv_{it}-\frac{k+1}{2}\left(\sum_{t=1}^{k}v_{it}\right)\right] \\
&= \frac{1}{T^{3/2}}\left[\sum_{t=1}^{k}tv_{it}-\frac{k}{2}\left(\sum_{t=1}^{k}v_{it}\right)-\frac{1}{2}\sum_{t=1}^{k}v_{it}\right] \\
&\xrightarrow{d} \sigma_0\left[rW_i(r)-\int_0^r W_i(s)ds\right]-\frac{1}{2}\sigma_0 rW_i(r) \\
&= \frac{1}{2}\sigma_0\left[rW_i(r)-2\int_0^r W_i(r)dr\right]
\end{aligned}
$$

uniformly in r as $T\to\infty$ since (e.g., Lemma 1.1, Chu and White, 1992)

$$
\frac{1}{T^{3/2}}\sum_{t=1}^{k}tv_{it}\xrightarrow{d}\sigma_0\left[rW_i(r)-\int_0^r W_i(s)ds\right],
$$

$$
\frac{1}{T^{3/2}}\frac{k}{2}\left(\sum_{t=1}^{k}v_{it}\right)=\frac{1}{2T^{1/2}}\frac{k}{T}\left(\sum_{t=1}^{k}v_{it}\right)\xrightarrow{d}\frac{1}{2}\sigma_0 rW_i(r),
$$

uniformly in r and

$$
\frac{1}{T^{3/2}}\frac{1}{2}\sum_{t=1}^{k}v_{it}=o_p(1)
$$

where

$$
\sigma_0^2=\sigma_\varepsilon^2\left(\frac{1}{1-\rho}\right)^2.
$$

Now

$$
\begin{aligned}
&\sqrt{NT}\left(\widehat{\beta}_k-\beta\right) \\
&= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{k}(x_t-\overline{x}_k)v_{it}\right]}{\frac{1}{T}\sum_{t=1}^{k}(x_t-\overline{x}_k)^2} \\
&= 12\left(\frac{k}{T}\right)^{-3}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{k}(x_t-\overline{x}_k)v_{it}\right]+o_p(1).
\end{aligned}
$$

For a fixed N as $T\to\infty$ we have

$$
12\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{\sqrt{T}}\sum_{t=1}^{k}(x_t-\overline{x}_k)v_{it}\xrightarrow{d}\frac{6\sigma_0}{\sqrt{N}}\sum_{i=1}^{N}\left[rW_i(r)-2\int_0^r W_i(s)ds\right]=\frac{6\sigma_0}{\sqrt{N}}\sum_{i=1}^{N}G_i(r)
$$

uniformly in r, where $G_i(r)=rW_i(r)-2\int_0^r W_i(s)ds$ is a Gaussian process with zero mean and variance, $\frac{1}{3}r^2$, i. e., for each r

$$
G_i(r)\sim \mathrm{N}(0,\frac{1}{3}r^2).
$$

Hence

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}G_i(r)\sim \mathrm{N}(0,\frac{1}{3}r^2)$$

for all N. Finally,

$$\sqrt{NT}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\beta\right)\xrightarrow{d}G(r).$$

Similarly,

$$\sqrt{NT}\frac{1}{6\sigma_0}\left(\widehat{\beta}_T-\beta\right)\xrightarrow{d}G(1).$$

It follows that

$$\begin{aligned}\sqrt{NT}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\widehat{\beta}_T\right) &= \sqrt{NT}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\beta\right)-\sqrt{NT}\left(\frac{k}{T}\right)^3\frac{1}{6\sigma_0}\left(\widehat{\beta}_T-\beta\right)\\ \xrightarrow{d}G(r)-r^3G(1) &= G_0(r)\end{aligned}$$

uniformly in r as T$\to\infty$ for all N proving Theorem 1. ■

B Proof of Theorem 2

Proof. First we note that

$$\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)=\left(\widehat{\beta}_{1k}-\beta\right)-\left(\widehat{\beta}_{2k}-\beta\right).$$

Then under H_0

$$\begin{aligned}&\sqrt{NT}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)\\ =&\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}\sqrt{NT}\left(\widehat{\beta}_{1k}-\beta\right)\\ \sqrt{NT}\left(\widehat{\beta}_{2k}-\beta\right)\end{bmatrix}\\ =&\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}\frac{\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left[\sum_{t=1}^{k}(x_t-\overline{x}_{1k})v_{it}\right]}{\frac{1}{T}\sum_{t=1}^{k}(x_t-\overline{x}_{1k})^2}\\ \frac{\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left[\sum_{t=k+1}^{T}(x_t-\overline{x}_{2k})v_{it}\right]}{\frac{1}{T}\sum_{t=k+1}^{T}(x_t-\overline{x}_{2k})^2}\end{bmatrix}\\ =&\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}12r^{-3}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left[\sum_{t=1}^{k}(x_t-\overline{x}_{1k})v_{it}\right]\\ 12\left(1-r\right)^{-3}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left[\sum_{t=k+1}^{T}(x_t-\overline{x}_{2k})v_{it}\right]\end{bmatrix}+o_p(1)\\ \xrightarrow{d}&\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}6\sigma_0r^{-3}G(r) & 0\\ 0 & 6\sigma_0\left(1-r\right)^{-3}\left(G(1)-G(r)+W(r)-rW(1)\right)\end{bmatrix}\\ =&6\sigma_0\left[\frac{G(r)}{r^3}-\frac{G(1)-G(r)+W(r)-rW(1)}{\left(1-r\right)^3}\right] \qquad (14)\end{aligned}$$

uniformly in r since

$$\frac{1}{T}\sum_{t=k+1}^{T}(x_t-\overline{x}_{2k})^2 = \frac{T-k}{T}\frac{1}{T-k}\sum_{t=k+1}^{T}(x_t-\overline{x}_{2k})^2 \to \frac{1}{12}(1-r)^3,$$

and

$$\begin{bmatrix} 12r^{-3}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left[\sum_{t=1}^{k}(x_t-\overline{x}_{1k})v_{it}\right] \\ 12(1-r)^{-3}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left[\sum_{t=k+1}^{T}(x_t-\overline{x}_{2k})v_{it}\right] \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 6\sigma_0 r^{-3} & 0 \\ 0 & 6\sigma_0(1-r)^{-3} \end{bmatrix}\begin{bmatrix} G(r) & 0 \\ 0 & G(1)-G(r)+W(r)-rW(1) \end{bmatrix}.$$

uniformly in r as $T\to\infty$ for all N. Note

$$\begin{aligned} & \left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{k}(x_t-\overline{x}_{1k})^2\right)^{-1}+\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(x_t-\overline{x}_{2k})^2\right)^{-1} \\ = & \left(\frac{1}{T}\sum_{t=1}^{k}(x_t-\overline{x}_{1k})^2\right)^{-1}+\left(\frac{1}{T}\sum_{t=k+1}^{T}(x_t-\overline{x}_{2k})^2\right)^{-1} \\ \to & \left(\frac{1}{12}r^3\right)^{-1}+\left(\frac{1}{12}(1-r)^3\right)^{-1} \\ = & \frac{12}{r^3}+\frac{12}{(1-r)^3}. \end{aligned} \tag{15}$$

It is easy to see that under H_0 that

$$\begin{aligned} \widehat{\sigma}_v^2 &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[(y_{it}-\overline{y}_i)-(x_t(k)-\overline{x}_t(k))'\widehat{\beta}^*\right]^2 \\ &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[(y_{it}-\overline{y}_i)-\widehat{\beta}(x_t-\overline{x})\right]^2+o_p(1)\xrightarrow{p}\sigma_v^2. \end{aligned} \tag{16}$$

.(14) - (16) and the continuous mapping theorem give

$$\begin{aligned} \mathrm{W}(k) &= \frac{1}{\widehat{\sigma}_v^2}\frac{\left[\sqrt{NT}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)\right]^2}{\left[\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{k}(x_t-\overline{x}_{1k})^2\right)^{-1}+\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(x_t-\overline{x}_{2k})^2\right)^{-1}\right]} \\ &\xrightarrow{d} \frac{1}{\sigma_v^2}\frac{\left\{6\sigma_0\left[\frac{G(r)}{r^3}-\frac{G(1)-G(r)+W(r)-rW(1)}{(1-r)^3}\right]\right\}^2}{\frac{12}{r^3}+\frac{12}{(1-r)^3}} \\ &= \frac{3\sigma_0^2}{\sigma_v^2}\frac{\left[\frac{G(r)}{r^3}-\frac{G(1)-G(r)+W(r)-rW(1)}{(1-r)^3}\right]^2}{\frac{1}{r^3}+\frac{1}{(1-r)^3}} \end{aligned}$$

$$= \frac{3\sigma_0^2}{\sigma_v^2} \frac{\left[\frac{G(r)(1-r)^3-[G(1)-G(r)+W(r)-rW(1)]r^3}{r^3(1-r)^3}\right]^2}{\frac{(1-r)^3+r^3}{r^3(1-r)^3}}$$

$$= \frac{3\sigma_0^2}{\sigma_v^2} \frac{\left[G(r)(1-r)^3 - r^3\left[G(1)-G(r)+W(r)-rW(1)\right]\right]^2}{r^3(1-r)^3\left[(1-r)^3+r^3\right]}$$

$$= \frac{3\sigma_0^2}{\sigma_v^2} \left\{ \frac{G(r)(1-r)^3 - r^3\left[G(1)-G(r)+W(r)-rW(1)\right]}{\left[r^3(1-r)^3\left[(1-r)^3+r^3\right]\right]^{\frac{1}{2}}} \right\}^2$$

uniformly in r proving Theorem 2. ■

C Proof of Theorem 3

Proof. Note that $v_{it} = \sum_{j=0}^{t} \varepsilon_{ij}$ so

$$\begin{aligned}
\frac{1}{T^{3/2}} \sum_{t=1}^{k} (x_t - \overline{x}_k) u_{it} &= \frac{1}{T^{3/2}} \sum_{t=1}^{k} (x_t - \overline{x}_k) v_{it} \\
&= \frac{1}{T^{3/2}} \sum_{t=1}^{k} \left(\frac{t}{T} - \frac{1}{k}\sum_{t=1}^{k}\frac{t}{T}\right) v_{it} \\
&= \frac{1}{T^{5/2}} \left[\sum_{t=1}^{k} t v_{it} - \frac{(k+1)}{2}\left(\sum_{t=1}^{k} v_{it}\right)\right] \\
&= \frac{1}{T^{5/2}} \left[\sum_{t=1}^{k} t v_{it} - \frac{k}{2}\left(\sum_{t=1}^{k} v_{it}\right) - \frac{1}{2}\sum_{t=1}^{k} v_{it}\right] \\
&\xrightarrow{d} \sigma_\varepsilon \int_0^r s\left[W(s) + \widetilde{W}(\kappa)\right] ds - \frac{\sigma_\varepsilon}{2} r \int_0^r \left[W(s) + \widetilde{W}(\kappa)\right] ds \\
&= \frac{\sigma_\varepsilon}{2}\left[2\int_0^r s\left[W(s) + \widetilde{W}(\kappa)\right] ds - r\int_0^r \left[W(s) + \widetilde{W}(\kappa)\right] ds\right]
\end{aligned}$$

since

$$\frac{1}{T^{5/2}} \sum_{t=1}^{k} t v_{it} \xrightarrow{d} \sigma_\varepsilon \int_0^r s\left[W(s) + \widetilde{W}(\kappa)\right] ds,$$

$$\begin{aligned}
\frac{1}{T^{5/2}} \frac{k}{2} \sum_{t=1}^{k} v_{it} &= \frac{1}{2}\frac{k}{T}\frac{1}{T^{3/2}} \sum_{t=1}^{k} v_{it} \\
&\xrightarrow{d} \frac{\sigma_\varepsilon}{2} r \int_0^r \left[W(s) + \widetilde{W}(\kappa)\right] ds,
\end{aligned}$$

and

$$\frac{1}{T^{5/2}} \sum_{t=1}^{k} v_{it} = o_p(1)$$

uniformly in r. Then

$$\begin{aligned}
\sqrt{NT^{-1}}\left(\widehat{\beta}_k-\beta\right) &= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{k}(x_t-\overline{x}_k)u_{it}\right]}{\frac{1}{NT}N\sum_{t=1}^{k}(x_t-\overline{x}_k)^2} \\
&= 12\left(\frac{k}{T}\right)^{-3}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{k}(x_t-\overline{x}_k)v_{it}\right]+o_p(1) \\
&\xrightarrow{d} 12r^{-3}\frac{\sigma_\varepsilon}{2}\left[2\int_0^r s\left[W(s)+\widetilde{W}(\kappa)\right]ds-r\int_0^r\left[W(s)+\widetilde{W}(\kappa)\right]ds\right] \\
&= 6\sigma_\varepsilon r^{-3}\left[2\int_0^r s\left[W(s)+\widetilde{W}(\kappa)\right]ds-r\int_0^r\left[W(s)+\widetilde{W}(\kappa)\right]ds\right].
\end{aligned}$$

uniformly in r as $T\to\infty$ for all N. It follows that

$$\sqrt{NT^{-1}}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\beta\right)\xrightarrow{d}2\int_0^r s\left[W(s)+\widetilde{W}(\kappa)\right]ds-r\int_0^r\left[W(s)+\widetilde{W}(\kappa)\right]ds=H(r).$$

In a similar fashion, we have

$$\sqrt{NT^{-1}}\frac{1}{6\sigma_\varepsilon}\left(\widehat{\beta}_T-\beta\right)\xrightarrow{d}2\int_0^1 s\left[W(s)+\widetilde{W}(\kappa)\right]ds-\int_0^1\left[W(s)+\widetilde{W}(\kappa)\right]ds=H(1).$$

uniformly in r as $T\to\infty$ for all N. We therefore prove the Theorem 3, i.e.,

$$\sqrt{NT^{-1}}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k-\widehat{\beta}_T\right)\xrightarrow{d}H(r)-r^3H(1)=H_0(r).$$

uniformly in r as $T\to\infty$ for all N. ■

D Proof of Theorem 4

Proof. Consider $\widehat{\sigma}_v^2$ under H_0 :

$$\begin{aligned}
\widehat{\sigma}_v^2 &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[(y_{it}-\overline{y}_i)-(x_t(k)-\overline{x}_t(k))'\,\widehat{\beta}^*\right]^2 \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[(y_{it}-\overline{y}_i)-\widehat{\beta}\,(x_t-\overline{x})\right]^2+o_p(1) \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[(v_{it}-\overline{v}_i)+\beta\,(x_t-\overline{x})-\widehat{\beta}\,(x_t-\overline{x})\right]^2 \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[(v_{it}-\overline{v}_i)-\left(\widehat{\beta}-\beta\right)(x_t-\overline{x})\right]^2 \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[(v_{it}-\overline{v}_i)^2-2\left(\widehat{\beta}-\beta\right)(v_{it}-\overline{v}_i)+\left(\widehat{\beta}-\beta\right)^2(x_t-\overline{x})^2\right]
\end{aligned}$$

$$= \quad \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(v_{it}-\overline{v}_i)^2 - \frac{1}{NT}\frac{\left(\sum_{i=1}^{N}\sum_{t=1}^{T}(x_t-\overline{x})\, v_{it}\right)^2}{\sum_{i=1}^{N}\sum_{t=1}^{T}(x_t-\overline{x})^2}$$

Recall that

$$\frac{1}{N^{1/2}}\frac{1}{T^{3/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}(x_t-\overline{x})\, v_{it} \xrightarrow{d} \frac{\sigma_\varepsilon}{2}\left[2\int_0^1 s\left[W(s)+\widetilde{W}(\kappa)\right] ds - \int_0^1 \left[W(s)+\widetilde{W}(\kappa)\right] ds\right] = O_p(1) \quad (17)$$

and

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(x_t-\overline{x})^2 \to \frac{1}{12} = O(1). \quad (18)$$

For a fixed N, we have

$$\begin{aligned}\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}(v_{it}-\overline{v}_i)^2 &= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T^2}\sum_{t=1}^{T}(v_{it}-\overline{v}_i)^2\right)\\ &\xrightarrow{d} \frac{1}{N}\sum_{i=1}^{N}\sigma_\varepsilon^2\left\{\int_0^1\left[W(s)+\widetilde{W}(\kappa)\right]^2 ds - \left\{\int_0^1\left[W(s)+\widetilde{W}(\kappa)\right] ds\right\}^2\right\}\end{aligned}$$

Then

$$\frac{1}{N}\sum_{i=1}^{N}\sigma_\varepsilon^2\left\{\int_0^1\left[W(s)+\widetilde{W}(\kappa)\right]^2 ds - \left\{\int_0^1\left[W(s)+\widetilde{W}(\kappa)\right] ds\right\}^2\right\} \xrightarrow{p} \sigma_\varepsilon^2\left(\frac{1}{6}+\kappa\right)$$

as $N\to\infty$ using Lemma A.1 in Kao (1999). Then

$$\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}(v_{it}-\overline{v}_i)^2 \xrightarrow{p} \sigma_\varepsilon^2\left(\frac{1}{6}+\kappa\right) \quad (19)$$

as $T\to\infty$ and $N\to\infty$ sequentially. Similarly,

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T^{3/2}}\sum_{t=1}^{T}(x_t-\overline{x})\, v_{it} \xrightarrow{p} 0 \quad (20)$$

as $T\to\infty$ and $N\to\infty$ sequentially. Combining (14) - (20) gives

$$\begin{aligned}\frac{1}{T}\widehat{\sigma}_v^2 &= \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}(v_{it}-\overline{v}_i)^2 - \frac{1}{NT^2}\frac{\left(\sum_{i=1}^{N}\sum_{t=1}^{T}(x_t-\overline{x})\, v_{it}\right)^2}{\sum_{i=1}^{N}\sum_{t=1}^{T}(x_t-\overline{x})^2}\\ &= \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}(v_{it}-\overline{v}_i)^2 - \frac{\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T^{3/2}}\sum_{t=1}^{T}(x_t-\overline{x})\, v_{it}\right)^2}{\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(x_t-\overline{x})^2}\\ &\xrightarrow{p} \sigma_\varepsilon^2\left(\frac{1}{6}+\kappa\right) = \frac{\sigma_\varepsilon^2(1+6\kappa)}{6}.\end{aligned}$$

From the proof of Theorem 3 we know that

$$\begin{aligned}&\sqrt{NT^{-1}}\left(\widehat{\beta}_{1k}-\beta\right)\\ &\xrightarrow{d} 6\sigma_\varepsilon r^{-3}H(r)\end{aligned}$$

and

$$\begin{aligned}
&\sqrt{NT^{-1}}\left(\widehat{\beta}_{2k}-\beta\right) \\
&\xrightarrow{d} 6\sigma_{\varepsilon}\left(1-r\right)^{-3}\left[H(1)-H(r)-r\int_0^1 W(s)ds+\int_0^r W(s)ds\right]
\end{aligned}$$

uniformly in r. Then

$$\begin{aligned}
&\sqrt{NT^{-1}}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right) \\
= &\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{NT^{-1}}\left(\widehat{\beta}_{1k}-\beta\right) \\ \sqrt{NT^{-1}}\left(\widehat{\beta}_{2k}-\beta\right) \end{bmatrix} \\
&\xrightarrow{d} \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 6\sigma_{\varepsilon}r^{-3}H(r) & 0 \\ 0 & 6\sigma_{\varepsilon}\left(1-r\right)^{-3}\begin{pmatrix} H(1)-H(r)- \\ r\int_0^1 W(s)ds+\int_0^r W(s)ds \end{pmatrix} \end{bmatrix}+o_p(1) \\
= &6\sigma_{\varepsilon}\left[\frac{H(r)}{r^3}-\frac{H(1)-H(r)-r\int_0^1 W(s)ds+\int_0^r W(s)ds}{\left(1-r\right)^3}\right]
\end{aligned}$$

uniformly in r. Also

$$\begin{aligned}
&\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{k}(x_t-\overline{x}_{1k})^2\right)^{-1}+\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(x_t-\overline{x}_{2k})^2\right)^{-1} \\
\rightarrow &\frac{12}{r^3}+\frac{12}{\left(1-r\right)^3}
\end{aligned}$$

uniformly in r. Hence

$$\begin{aligned}
\frac{1}{T}\mathrm{W}(k) &= \frac{1}{\frac{1}{T}\widehat{\sigma}_v^2}\frac{\left[\sqrt{NT^{-1}}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)\right]^2}{\left[\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{k}(x_t-\overline{x}_{1k})^2\right)^{-1}+\left(\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(x_t-\overline{x}_{2k})^2\right)^{-1}\right]} \\
&\xrightarrow{d} \frac{6}{\sigma_{\varepsilon}^2\left(1+6\kappa\right)}\frac{\left\{6\sigma_{\varepsilon}\left(\frac{H(r)}{r^3}-\frac{H(1)-H(r)-r\int_0^1 W(s)ds+\int_0^r W(s)ds}{\left(1-r\right)^3}\right)\right\}^2}{\frac{12}{r^3}+\frac{12}{\left(1-r\right)^3}} \\
&= \frac{18}{\left(1+6\kappa\right)}\left\{\frac{H(r)\left(1-r\right)^3-\left(H(1)-H(r)-r\int_0^1 W(s)ds+\int_0^r W(s)ds\right)r^3}{\left(r^3\left(1-r\right)^3\left(\left(1-r\right)^3+r^3\right)\right)^{1/2}}\right\}^2
\end{aligned}$$

uniformly in r proving Theorem 4. ■

E Proof of Theorem 5

Proof. Note that $v_{it} = \rho v_{it-1} + \varepsilon_{it}$ with $\rho = 1 + c/T$, so

$$\begin{aligned}
\frac{1}{T^{3/2}}\sum_{t=1}^{k}(x_t - \overline{x}_k)u_{it} &= \frac{1}{T^{3/2}}\sum_{t=1}^{k}(x_t - \overline{x}_k)v_{it} \\
&= \frac{1}{T^{3/2}}\sum_{t=1}^{k}(\frac{t}{T} - \frac{1}{k}\sum_{t=1}^{k}\frac{t}{T})v_{it} \\
&= \frac{1}{T^{5/2}}\left[\sum_{t=1}^{k} t v_{it} - \frac{(k+1)}{2}\left(\sum_{t=1}^{k} v_{it}\right)\right] \\
&= \frac{1}{T^{5/2}}\left[\sum_{t=1}^{k} t v_{it} - \frac{k}{2}\left(\sum_{t=1}^{k} v_{it}\right) - \frac{1}{2}\sum_{t=1}^{k} v_{it}\right] \\
&\xrightarrow{d} \sigma_\varepsilon \int_0^r s\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right] ds - \frac{\sigma_\varepsilon}{2} r \int_0^r \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right] ds \\
&= \frac{\sigma_\varepsilon}{2}\left[2\int_0^r s\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right] ds - r\int_0^r \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right] ds\right]
\end{aligned}$$

uniformly in r since

$$\frac{1}{T^{5/2}}\sum_{t=1}^{k} t v_{it} \xrightarrow{d} \sigma_\varepsilon \int_0^r s\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right] ds,$$

$$\begin{aligned}
\frac{1}{T^{5/2}}\frac{k}{2}\sum_{t=1}^{k} v_{it} &= \frac{1}{2}\frac{k}{T}\frac{1}{T^{3/2}}\sum_{t=1}^{k} v_{it} \\
&\xrightarrow{d} \frac{\sigma_\varepsilon}{2} r \int_0^r \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right] ds,
\end{aligned}$$

and

$$\frac{1}{T^{5/2}}\sum_{t=1}^{k} v_{it} = o_p(1)$$

uniformly in r. Then

$$\begin{aligned}
\sqrt{NT^{-1}}\left(\widehat{\beta}_k - \beta\right) &= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{k}(x_t - \overline{x}_k)u_{it}\right]}{\frac{1}{NT}N\sum_{t=1}^{k}(x_t - \overline{x}_k)^2} \\
&= 12\left(\frac{k}{T}\right)^{-3}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\frac{1}{T^{3/2}}\sum_{t=1}^{k}(x_t - \overline{x}_k)v_{it}\right] + o_p(1) \\
&\xrightarrow{d} 12 r^{-3}\frac{\sigma_\varepsilon}{2}\left[2\int_0^r s\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right] ds - r\int_0^r \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right] ds\right] \\
&= 6\sigma_\varepsilon r^{-3}\left[2\int_0^r s\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right] ds - r\int_0^r \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right] ds\right].
\end{aligned}$$

uniformly in r as $T \to \infty$ for all N. It follows that

$$\sqrt{NT^{-1}}\frac{1}{6\sigma_{\varepsilon}}\left(\frac{k}{T}\right)^{3}\left(\widehat{\beta}_{k}-\beta\right)\xrightarrow{d}2\int_{0}^{r}s\left[W_{c}(s)+e^{sc}\widetilde{W}_{c}(\kappa)\right]ds-r\int_{0}^{r}\left[W_{c}(s)+e^{sc}\widetilde{W}_{c}(\kappa)\right]ds=H_{c}(r).$$

In a similar fashion, we have

$$\sqrt{NT^{-1}}\frac{1}{6\sigma_{\varepsilon}}\left(\widehat{\beta}_{T}-\beta\right)\xrightarrow{d}2\int_{0}^{1}s\left[W_{c}(s)+e^{sc}\widetilde{W}_{c}(\kappa)\right]ds-\int_{0}^{1}\left[W_{c}(s)+e^{sc}\widetilde{W}_{c}(\kappa)\right]ds=H_{c}(1).$$

uniformly in r as $T \to \infty$ for all N. We therefore prove Theorem 7, i.e.,

$$\sqrt{NT^{-1}}\frac{1}{6\sigma_{\varepsilon}}\left(\frac{k}{T}\right)^{3}\left(\widehat{\beta}_{k}-\widehat{\beta}_{T}\right)\xrightarrow{d}H_{c}(r)-r^{3}H_{c}(1).$$

uniformly in r as $T \to \infty$ for all N. ■

F Proof of Theorem 6

Proof. Consider

$$\begin{aligned}
\widehat{\sigma}_{v}^{2} &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[\left(y_{it}-\overline{y}_{i}\right)-\widehat{\beta}\left(x_{t}-\overline{x}\right)\right]^{2} \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[\left(v_{it}-\overline{v}_{i}\right)+\beta\left(x_{t}-\overline{x}\right)-\widehat{\beta}\left(x_{t}-\overline{x}\right)\right]^{2} \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[\left(v_{it}-\overline{v}_{i}\right)-\left(\widehat{\beta}-\beta\right)\left(x_{t}-\overline{x}\right)\right]^{2} \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[\left(v_{it}-\overline{v}_{i}\right)^{2}-2\left(\widehat{\beta}-\beta\right)\left(v_{it}-\overline{v}_{i}\right)+\left(\widehat{\beta}-\beta\right)^{2}\left(x_{t}-\overline{x}\right)^{2}\right] \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(v_{it}-\overline{v}_{i}\right)^{2}-\frac{1}{NT}\frac{\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_{t}-\overline{x}\right)v_{it}\right)^{2}}{\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_{t}-\overline{x}\right)^{2}} \qquad (21)
\end{aligned}$$

Recall that

$$\frac{1}{N^{1/2}}\frac{1}{T^{3/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_{t}-\overline{x}\right)v_{it}\xrightarrow{d}\frac{\sigma_{\varepsilon}}{2}\left[2\int_{0}^{1}s\left[W_{c}(s)+e^{sc}\widetilde{W}_{c}(\kappa)\right]ds-\int_{0}^{1}\left[W_{c}(s)+e^{sc}\widetilde{W}_{c}(\kappa)\right]ds\right]=O_{p}(1)$$

and

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_{t}-\overline{x}\right)^{2}\rightarrow\frac{1}{12}=O(1). \qquad (22)$$

For a fixed N, we have

$$\frac{1}{NT^{2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(v_{it}-\overline{v}_{i}\right)^{2} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T^{2}}\sum_{t=1}^{T}\left(v_{it}-\overline{v}_{i}\right)^{2}\right) \qquad (23)$$

$$\xrightarrow{d} \frac{1}{N}\sum_{i=1}^{N}\sigma_{\varepsilon}^{2}\left\{\int_{0}^{1}\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]^2 ds-\left\{\int_{0}^{1}\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right] ds\right\}^2\right\} \tag{24}$$

Then

$$\frac{1}{N}\sum_{i=1}^{N}\sigma_{\varepsilon}^{2}\left\{\int_{0}^{1}\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]^2 ds-\left\{\int_{0}^{1}\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right] ds\right\}^2\right\}\xrightarrow{p}\sigma_{\varepsilon}^{2}\left(\frac{1}{6}+\frac{\kappa\left(e^{2c}-1\right)}{2c}\right)$$

as $T\to\infty$. It follows that (e.g., Lemma A.1, Kao, 1999)

$$\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(v_{it}-\overline{v}_i\right)^2\xrightarrow{p}\sigma_{\varepsilon}^{2}\left(\frac{1}{6}+\frac{\kappa\left(e^{2c}-1\right)}{2c}\right) \tag{25}$$

as $T\to\infty$ and $N\to\infty$ sequentially. Similarly,

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\left(x_t-\overline{x}\right)v_{it}\xrightarrow{p}0 \tag{26}$$

as $T\to\infty$ and $N\to\infty$ sequentially. Combining (21) - (26) gives

$$\begin{aligned}
\frac{1}{T}\widehat{\sigma}_v^2 &= \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(v_{it}-\overline{v}_i\right)^2-\frac{1}{NT^2}\frac{\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_t-\overline{x}\right)v_{it}\right)^2}{\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_t-\overline{x}\right)^2}\\
&= \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(v_{it}-\overline{v}_i\right)^2-\frac{\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\left(x_t-\overline{x}\right)v_{it}\right)^2}{\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(x_t-\overline{x}\right)^2}\\
&\xrightarrow{p}\sigma_{\varepsilon}^{2}\left(\frac{1}{6}+\frac{\kappa\left(e^{2c}-1\right)}{2c}\right)=\frac{\sigma_{\varepsilon}^{2}\left(c+3\kappa\left(e^{2c}-1\right)\right)}{6c}.
\end{aligned}$$

From the proof of Theorem 5 we know that

$$\sqrt{NT^{-1}}\left(\widehat{\beta}_{1k}-\beta\right)\xrightarrow{d}6\sigma_{\varepsilon}r^{-3}H_c(r)$$

and

$$\begin{aligned}
&\sqrt{NT^{-1}}\left(\widehat{\beta}_{2k}-\beta\right)\\
&\xrightarrow{d}6\sigma_{\varepsilon}\left(1-r\right)^{-3}\left[H_c(1)-H_c(r)-r\int_{0}^{1}\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds+\int_{0}^{r}\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds\right]
\end{aligned}$$

uniformly in r. Then

$$\begin{aligned}
&\sqrt{NT^{-1}}\left(\widehat{\beta}_{1k}-\widehat{\beta}_{2k}\right)\\
=&\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}\sqrt{NT^{-1}}\left(\widehat{\beta}_{1k}-\beta\right)\\ \sqrt{NT^{-1}}\left(\widehat{\beta}_{2k}-\beta\right)\end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&\xrightarrow{d} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 6\sigma_\varepsilon r^{-3} H_c(r) & 0 \\ 0 & 6\sigma_\varepsilon (1-r)^{-3} \begin{pmatrix} H_c(1) - H_c(r) - \\ r\int_0^1 \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa) \right] ds \\ + \int_0^r \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa) \right] ds \end{pmatrix} \end{bmatrix} \\
&+o_p(1) \\
= \quad & 6\sigma_\varepsilon \left[\frac{H_c(r)}{r^3} - \frac{\left[H_c(1) - H_c(r) - r\int_0^1 \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa) \right] ds + \int_0^r \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa) \right] ds \right]}{(1-r)^3} \right]
\end{aligned}
$$

uniformly in r. Also

$$
\begin{aligned}
& \left(\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{k} (x_t - \overline{x}_{1k})^2 \right)^{-1} + \left(\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=k+1}^{T} (x_t - \overline{x}_{2k})^2 \right)^{-1} \\
\rightarrow \quad & \frac{12}{r^3} + \frac{12}{(1-r)^3}.
\end{aligned}
$$

uniformly in r. Hence

$$
\begin{aligned}
& \frac{1}{T} \mathrm{W}(k) \\
= \quad & \frac{1}{\frac{1}{T}\widehat{\sigma}_v^2} \frac{\left[\sqrt{NT^{-1}} \left(\widehat{\beta}_{1k} - \widehat{\beta}_{2k} \right) \right]^2}{\left[\left(\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{k} (x_t - \overline{x}_{1k})^2 \right)^{-1} + \left(\frac{1}{NT} \sum_{i=1}^{N} \sum_{t=k+1}^{T} (x_t - \overline{x}_{2k})^2 \right)^{-1} \right]} \\
\xrightarrow{d} \quad & \frac{6c}{\sigma_\varepsilon^2 \left(c + 3\kappa \left(e^{2c} - 1 \right) \right)} \frac{\left\{ 6\sigma_\varepsilon \left[\frac{H_c(r)}{r^3} - \frac{\left\{ H_c(1) - H_c(r) - r\int_0^1 \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa) \right] ds + \int_0^r \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa) \right] ds \right\}}{(1-r)^3} \right] \right\}^2}{\frac{12}{r^3} + \frac{12}{(1-r)^3}} \\
= \quad & \frac{18c}{\left(c + 3\kappa \left(e^{2c} - 1 \right) \right)} \left\{ \frac{H_c(r)(1-r)^3 - \begin{pmatrix} H_c(1) - H_c(r) - r\int_0^1 \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa) \right] ds + \\ \int_0^r \left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa) \right] ds \end{pmatrix} r^3}{\left[r^3 (1-r)^3 \left[(1-r)^3 + r^3 \right] \right]^{1/2}} \right\}^2
\end{aligned}
$$

uniformly in r proving Theorem 6. ■

G Proof of Theorem 7

Proof. The model under the alternative is

$$
\begin{aligned}
y_{it}^{(T)} &= \alpha + \beta_t^{(T)} x_t + u_{it} \\
&= \alpha + \beta x_t + \frac{1}{\sqrt{T}} g\left(\frac{t}{T} \right) x_t + u_{it}.
\end{aligned}
$$

Note

$$
\begin{aligned}
\widehat{\beta}_k^{(T)} &= \frac{\sum_{i=1}^N \sum_{t=1}^k (x_t - \overline{x}_k) y_{it}^{(T)}}{N \sum_{t=1}^k (x_t - \overline{x}_k)^2} \\
&= \frac{\sum_{i=1}^N \sum_{t=1}^k (x_t - \overline{x}_k) \left[\alpha + \beta x_t + \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right) x_t + u_{it}\right]}{N \sum_{t=1}^k (x_t - \overline{x}_k)^2} \\
&= \frac{\sum_{i=1}^N \sum_{t=1}^k (x_t - \overline{x}_k) \left[\alpha + \beta x_t + \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right) x_t + u_{it}\right]}{N \sum_{t=1}^k (x_t - \overline{x}_k)^2} \\
&= \beta + \frac{\sum_{i=1}^N \sum_{t=1}^k (x_t - \overline{x}_k) \left[\frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right) x_t + u_{it}\right]}{N \sum_{t=1}^k (x_t - \overline{x}_k)^2} \\
&= \beta + \frac{\sum_{i=1}^N \sum_{t=1}^k (x_t - \overline{x}_k) x_t \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)}{N \sum_{t=1}^k (x_t - \overline{x}_k)^2} + \frac{\sum_{i=1}^N \sum_{t=1}^k (x_t - \overline{x}_k) v_{it}}{N \sum_{t=1}^k (x_t - \overline{x}_k)^2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sqrt{NT}\left(\widehat{\beta}_k^{(T)} - \beta\right) \\
&= \frac{\frac{1}{\sqrt{N}} \sum_{i=1}^N \frac{1}{\sqrt{T}} \sum_{t=1}^k (x_t - \overline{x}_k) x_t \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)}{\frac{1}{NT} N \sum_{t=1}^k (x_t - \overline{x}_k)^2} + \frac{\frac{1}{\sqrt{N}} \sum_{i=1}^N \frac{1}{\sqrt{T}} \sum_{t=1}^k (x_t - \overline{x}_k) v_{it}}{\frac{1}{NT} N \sum_{t=1}^k (x_t - \overline{x}_k)^2} \\
&= 12\left(\frac{k}{T}\right)^{-3} \left[\frac{1}{\sqrt{N}} \sum_{i=1}^N \frac{1}{\sqrt{T}} \sum_{t=1}^k (x_t - \overline{x}_k) x_t \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right) + \frac{1}{\sqrt{N}} \sum_{i=1}^N \frac{1}{\sqrt{T}} \sum_{t=1}^k (x_t - \overline{x}_k) v_{it}\right] + o_p(1)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\sqrt{NT}\left(\widehat{\beta}_T^{(T)} - \beta\right) \\
&= 12 \frac{1}{\sqrt{N}} \sum_{i=1}^N \frac{1}{\sqrt{T}} \sum_{t=1}^T (x_t - \overline{x}_k) x_t \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right) + \frac{1}{\sqrt{N}} \sum_{i=1}^N \frac{1}{\sqrt{T}} \sum_{t=1}^T (x_t - \overline{x}_k) v_{it} + o_p(1)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\sqrt{NT}\left(\widehat{\beta}_k^{(T)} - \widehat{\beta}_T^{(T)}\right) \\
&= \frac{1}{\sqrt{N}} \sum_{i=1}^N \left[12\left(\frac{k}{T}\right)^{-3} \frac{1}{\sqrt{T}} \sum_{t=1}^k (x_t - \overline{x}_k) v_{it} - \frac{12}{\sqrt{T}} \sum_{t=1}^T (x_t - \overline{x}_k) v_{it}\right] \\
&\quad + \frac{1}{\sqrt{N}} \sum_{i=1}^N \left[12\left(\frac{k}{T}\right)^{-3} \frac{1}{\sqrt{T}} \sum_{t=1}^k (x_t - \overline{x}_k) x_t \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right) - \frac{12}{\sqrt{T}} \sum_{t=1}^T (x_t - \overline{x}_k) x_t \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)\right] + o_p(1)
\end{aligned}
$$

From Theorem 1 we know that

$$
\frac{1}{6\sigma_0}\left[\frac{1}{\sqrt{T}} \sum_{t=1}^k (x_t - \overline{x}_k) v_{it} - \left(\frac{k}{T}\right)^3 \frac{1}{\sqrt{T}} \sum_{t=1}^T (x_t - \overline{x}_k) v_{it}\right] \xrightarrow{d} G_0(r)
$$

uniformly in r. It is easy to show that

$$\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{k}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)-\left(\frac{k}{T}\right)^3\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)\right]$$
$$\rightarrow \int_0^r s^2g(s)ds-\frac{1}{2}r\int_0^r sg(s)ds-r^3\left(\int_0^1 s^2g(s)ds-\frac{1}{2}\int_0^1 sg(s)ds\right)$$

uniformly in r since (e.g., Bai, 1996, p. 609)

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=1}^{k}(x_t-\bar{x}_k)\,x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)\\
=&\ \frac{1}{T}\sum_{t=1}^{k}x_t^2g\left(\frac{t}{T}\right)-\frac{1}{T}\left(\sum_{t=1}^{k}x_tg\left(\frac{t}{T}\right)\right)\left(\frac{1}{k}\sum_{t=1}^{k}x_t\right)\\
=&\ \frac{1}{T}\sum_{t=1}^{k}x_t^2g\left(\frac{t}{T}\right)-\frac{1}{T}\left(\sum_{t=1}^{k}x_tg\left(\frac{t}{T}\right)\right)\left(\frac{k}{T}\frac{1}{k^2}\sum_{t=1}^{k}t\right)\\
\rightarrow&\ \int_0^r\frac{d\frac{1}{3}s^3}{ds}g(s)ds-\frac{1}{2}r\int_0^r\frac{d\frac{1}{2}s^2}{ds}g(s)ds\\
=&\ \int_0^r s^2g(s)ds-\frac{1}{2}r\int_0^r sg(s)ds
\end{aligned}$$

and

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(x_t-\bar{x}_k)\,x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)\rightarrow\int_0^1 s^2g(s)ds-\frac{1}{2}\int_0^1 sg(s)ds$$

uniformly in s. Therefore for a fixed N

$$\begin{aligned}
&\sqrt{NT}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\hat{\beta}_k^{(T)}-\hat{\beta}_T^{(T)}\right)\\
&\xrightarrow{d}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left\{G_0(r)+\frac{12}{6\sigma_0}\left[\left(\int_0^r s^2g(s)ds-\frac{1}{2}r\int_0^r sg(s)ds\right)-r^3\left(\int_0^1 s^2g(s)ds-\frac{1}{2}\int_0^1 sg(s)ds\right)\right]\right\}\\
=&\ \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left\{G_0(r)+\frac{2}{\sigma_0}\left[\left(\int_0^r s^2g(s)ds-\frac{1}{2}r\int_0^r sg(s)ds\right)-r^3\left(\int_0^1 s^2g(s)ds-\frac{1}{2}\int_0^1 sg(s)ds\right)\right]\right\}
\end{aligned}$$

uniformly in r as $T\rightarrow\infty$. Then

$$\sqrt{NT}\frac{1}{6\sigma_0}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)}-\widehat{\beta}_T^{(T)}\right)\xrightarrow{d}$$
$$G_0(r)+\frac{2}{\sigma_0}\left[\left(\int_0^r s^2g(s)ds-\frac{1}{2}r\int_0^r sg(s)ds\right)-r^3\left(\int_0^1 s^2g(s)ds-\frac{1}{2}\int_0^1 sg(s)ds\right)\right]$$

uniformly in r for all N proving (a).

The proofs of (b) and (c) are similar to (a) with a different speed.

$$\sqrt{NT^{-1}}\left(\widehat{\beta}_k^{(T)}-\beta\right)$$

$$= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)}{\frac{1}{NT}N\sum_{t=1}^{k}(x_t-\overline{x}_k)^2}+\frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)v_{it}}{\frac{1}{NT}N\sum_{t=1}^{k}(x_t-\overline{x}_k)^2}$$

$$= 12\left(\frac{k}{T}\right)^{-3}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)v_{it}\right]+o_p(1)$$

Similarly

$$\sqrt{NT^{-1}}\left(\widehat{\beta}_T^{(T)}-\beta\right)$$

$$= 12\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}(x_t-\overline{x}_k)v_{it}+o_p\left(1\right)$$

which implies that

$$\sqrt{NT^{-1}}\left(\widehat{\beta}_k^{(T)}-\widehat{\beta}_T^{(T)}\right)$$

$$= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[12\left(\frac{k}{T}\right)^{-3}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)v_{it}-\frac{12}{T^{\frac{3}{2}}}\sum_{t=1}^{T}(x_t-\overline{x}_k)v_{it}\right]$$

$$+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[12\left(\frac{k}{T}\right)^{-3}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)-\frac{12}{T^{\frac{3}{2}}}\sum_{t=1}^{T}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)\right]+o_p\left(1\right)$$

From Theorem 3 we know that

$$\frac{1}{6\sigma_\varepsilon}\left[\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)v_{it}-\left(\frac{k}{T}\right)^3\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}(x_t-\overline{x}_k)v_{it}\right]\xrightarrow{d}H_0(r)$$

uniformly in r. It is easy to show that

$$\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)-\frac{12}{T^{\frac{3}{2}}}\sum_{t=1}^{T}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)=o_p(1)$$

Then

$$\sqrt{NT^{-1}}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)}-\widehat{\beta}_T^{(T)}\right)\xrightarrow{d}H_0(r)$$

uniformly in r for all N proving (b).

To prove (c):

$$\sqrt{NT^{-1}}\left(\widehat{\beta}_k^{(T)}-\beta\right)$$

$$= \frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)}{\frac{1}{NT}N\sum_{t=1}^{k}(x_t-\overline{x}_k)^2}+\frac{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)v_{it}}{\frac{1}{NT}N\sum_{t=1}^{k}(x_t-\overline{x}_k)^2}$$

$$= 12\left(\frac{k}{T}\right)^{-3}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)v_{it}\right]+o_p(1)$$

Similarly

$$
\begin{aligned}
& \sqrt{NT^{-1}}\left(\widehat{\beta}_T^{(T)}-\beta\right) \\
= \; & 12\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}(x_t-\overline{x}_k)v_{it}+o_p(1)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \sqrt{NT^{-1}}\left(\widehat{\beta}_k^{(T)}-\widehat{\beta}_T^{(T)}\right) \\
= \; & \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[12\left(\frac{k}{T}\right)^{-3}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)v_{it}-\frac{12}{T^{\frac{3}{2}}}\sum_{t=1}^{T}(x_t-\overline{x}_k)v_{it}\right] \\
& +\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[12\left(\frac{k}{T}\right)^{-3}\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)-\frac{12}{T^{\frac{3}{2}}}\sum_{t=1}^{T}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)\right]+o_p(1)
\end{aligned}
$$

From Theorem 5 we know that

$$
\frac{1}{6\sigma_\varepsilon}\left[\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)v_{it}-\left(\frac{k}{T}\right)^3\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{T}(x_t-\overline{x}_k)v_{it}\right]\xrightarrow{d}H_c(r)-r^3H_c(1)
$$

uniformly in r. It is easy to show that

$$
\frac{1}{T^{\frac{3}{2}}}\sum_{t=1}^{k}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)-\frac{12}{T^{\frac{3}{2}}}\sum_{t=1}^{T}(x_t-\overline{x}_k)x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)=o_p(1)
$$

Then

$$
\sqrt{NT^{-1}}\frac{1}{6\sigma_\varepsilon}\left(\frac{k}{T}\right)^3\left(\widehat{\beta}_k^{(T)}-\widehat{\beta}_T^{(T)}\right)\xrightarrow{d}H_c(r)-r^3H_c(1)
$$

uniformly in r for all N proving (c). ■

H Proof of Theorem 8

Proof. Under the local alternative we have

$$
\begin{aligned}
& \sqrt{NT}\left(\widehat{\beta}_{1k}^{(T)}-\beta\right) \\
= \; & 12\left(\frac{k}{T}\right)^{-3}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{\sqrt{T}}\sum_{t=1}^{k}(x_t-\overline{x}_{1k})x_t\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{\sqrt{T}}\sum_{t=1}^{k}(x_t-\overline{x}_{1k})v_{it}\right]+o_p(1) \\
& \xrightarrow{d} 6r^{-3}\left[\sigma_0G(r)+2h(r)\right]
\end{aligned}
$$

and

$$
\sqrt{NT}\left(\widehat{\beta}_{2k}^{(T)}-\beta\right)\xrightarrow{d}6(1-r)^{-3}\left[\sigma_0\left[G(1)-G(r)-rW(1)+W(r)\right]+2h(1)-2h(r)-r\int_0^1 sg(s)ds+\int_0^r sg(s)ds\right],
$$

uniformly in r where

$$G(r) = rW(r) - 2\int_0^r W(r)dr$$

and

$$h(r) = \int_0^r s^2 g(s)ds - \frac{1}{2}r\int_0^r sg(s)ds.$$

Part (a) holds using the proof of Theorem 2 with $G(r)$ and $G(1) - G(r) - rW(1) + W(r)$ replaced by

$$G(r) + 2\frac{h(r)}{\sigma_0}$$

and

$$G(1) - G(r) - rW(1) + W(r) + \frac{2h(1) - 2h(r) - r\int_0^1 sg(s)ds + \int_0^r sg(s)ds}{\sigma_0},$$

respectively. Then under the alternative

$$\begin{aligned} &\mathrm{W}(k) \\ &\xrightarrow{d} \frac{3\sigma_0^2}{\sigma_v^2}\left\{\frac{\left(G(r) + 2\frac{h(r)}{\sigma_0}\right)(1-r)^3 - r^3\left(\begin{array}{c} G(1) - G(r) - rW(1) + W(r) \\ +\frac{2}{\sigma_0}\left(h(1) - h(r) - \frac{1}{2}r\int_0^1 sg(s)ds + \frac{1}{2}\int_0^r sg(s)ds\right)\end{array}\right)}{\left[r^3(1-r)^3\left[(1-r)^3 + r^3\right]\right]^{\frac{1}{2}}}\right\}^2 \\ &= \frac{3\sigma_0^2}{\sigma_v^2}\left\{\frac{\left(\begin{array}{c}\left[(1-r)^3 + r^3\right]G(r) \\ -r^3\left[G(1) - rW(1) + W(r)\right]\end{array}\right)}{\left[r^3(1-r)^3\left[(1-r)^3 + r^3\right]\right]^{\frac{1}{2}}} + \frac{2}{\sigma_0}\frac{\left(\begin{array}{c}\left[(1-r)^3 + r^3\right]h(r) \\ -r^3\left[h(1) - \frac{1}{2}r\int_0^1 sg(s)ds + \frac{1}{2}\int_0^r sg(s)ds\right]\end{array}\right)}{\left[r^3(1-r)^3\left[(1-r)^3 + r^3\right]\right]^{\frac{1}{2}}}\right\}^2 \end{aligned}$$

uniformly in r.

Part (b) holds using the proof of Theorem 4.

Part (c) holds using the proof of Theorem 5. ■

Lemma 1 *When $|\rho| < 1$, we have*

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N \frac{1}{\sqrt{T}}\sum_{t=1}^k x_t^2 v_{it} \xrightarrow{d} r^2 W(r) - r\int_0^r W(s)ds - \int_0^r sW(s)ds.$$

Proof. Note that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^k x_t^2 v_{it} \quad = \quad \frac{1}{T^{\frac{5}{2}}}\sum_{t=1}^k t^2 v_{it}$$

$$
\begin{aligned}
&= \frac{1}{T^{\frac{5}{2}}}\sum_{t=1}^{k} k^2 v_{it} - \frac{1}{T^{\frac{5}{2}}}\sum_{t=1}^{k-1}\left(k^2 - t^2\right) v_{it} \\
&= \frac{1}{T^{\frac{5}{2}}}\sum_{t=1}^{k} k^2 v_{it} - \frac{1}{T^{\frac{5}{2}}}\left(k\sum_{t=1}^{k-1}(k-t)\, v_{it} + \sum_{t=1}^{k-1}(k-t)\, t v_{it}\right) \\
&= \frac{1}{T^{\frac{5}{2}}}\sum_{t=1}^{k} k^2 v_{it} - \frac{1}{T^{\frac{5}{2}}}\left(k\sum_{j=2}^{k}\sum_{t=1}^{j-1} v_{it} + \sum_{j=2}^{k}\sum_{t=1}^{j-1} t v_{it}\right) \\
&\xrightarrow{d} r^2 W(r) - r\int_0^r W(s)ds - \int_0^r sW(s)ds.
\end{aligned}
$$

For a fixed N as $T \to \infty$ we have

$$
\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{\sqrt{T}}\sum_{t=1}^{k} x_t^2 v_{it} \xrightarrow{d} \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(r^2 W(r) - r\int_0^r W(s)ds - \int_0^r sW(s)ds\right).
$$

Hence

$$
\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(r^2 W(r) - r\int_0^r W(s)ds - \int_0^r sW(s)ds\right) \xrightarrow{d} r^2 W(r) - r\int_0^r W(s)ds - \int_0^r sW(s)ds
$$

as $N \to \infty$. ■

I Proof of Theorem 9

Proof. We will need to prove a lemma before we can prove Theorem 9. Now we can prove Theorem 9. First we note that

$$
\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)}\right) = \left(\underline{\widehat{\beta}}_{(1k)} - \underline{\beta}\right) - \left(\underline{\widehat{\beta}}_{(2k)} - \underline{\beta}\right)
$$

Define

$$
F(r) = 2r^2 W(r) - 3r\int_0^r W(s)ds - 3\int_0^r sW(s)ds.
$$

Then under H_0 and $|\rho| < 1$ we have

$$
\begin{aligned}
&\sqrt{NT}\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)}\right) \\
&= \sqrt{NT}\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\beta}\right) - \sqrt{NT}\left(\underline{\widehat{\beta}}_{(2k)} - \underline{\beta}\right) \\
&= \left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t - \overline{\underline{x}}_{(1k)})'(\underline{x}_t - \overline{\underline{x}}_{(1k)})\right]^{-1}\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t - \overline{\underline{x}}_{(1k)})' v_{it}\right] \\
&\quad - \left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(\underline{x}_t - \overline{\underline{x}}_{(2k)})'(\underline{x}_t - \overline{\underline{x}}_{(2k)})\right]^{-1}\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(\underline{x}_t - \overline{\underline{x}}_{(2k)})' v_{it}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\xrightarrow{d} \begin{bmatrix} \frac{1}{12}r^3 & \frac{1}{12}r^4 \\ \frac{1}{12}r^4 & \frac{4}{45}r^5 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2}\sigma_0 G(r) \\ \frac{1}{3}\sigma_0 F(r) \end{bmatrix} \\
&- \begin{bmatrix} \frac{(1-r)^3}{12} & \frac{(1-r)^3(1+r)}{12} \\ \frac{(1-r)^3(1+r)}{12} & \frac{\left(4+7r+r^2\right)(1-r)^3}{45} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2}\sigma_0\left(G(1)-G(r)+W(r)-rW(1)\right) \\ \frac{1}{3}\sigma_0\left[F(1)-F(r)+(1+r)W(r)-r(1+r)W(1)\right] \end{bmatrix} \\
= \ & \frac{\sigma_0}{r^4} \begin{bmatrix} \frac{-12A}{C} \\ \frac{30B}{rC} \end{bmatrix}.
\end{aligned}
$$

where

$$
\begin{aligned}
A = G(r)\left[-2r\left(45r^5-117r^4+108r^3-8r^2-20r+4\right)\right] \\
+F(r)\left[60r^5-150r^4+130r^3-10r^2-25r+5\right]+W(r)\left[-r^4\left(3r^2-4r-3\right)\right] \\
+G(1)\left[2r^4\left(r^2+7r+4\right)\right]+F(1)\left[-5r^4(r+1)\right]+W(1)\left[r^5\left(3r^2-4r-3\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
B = G(r)\left[-3r\left(12r^5-30r^4+1-5r-2r^2+26r^3\right)\right] \\
+F(r)\left[24r^5-62r^4+52r^3-4r^2-10r+2\right] \\
+W(r)\left[r^5(r+1)\right]+G(1)\left[3r^5(r+1)\right]+F(1)\left[-2r^5\right]+W(1)\left[-r^6(r+1)\right],
\end{aligned}
$$

and

$$
C = 11r^5-31r^4+26r^3-2r^2-5r+1.
$$

Now,

$$
\begin{aligned}
&NT\left(Var(b)\right) \\
= &\begin{bmatrix} \left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t-\overline{\underline{x}}_{(1k)})'(\underline{x}_t-\overline{\underline{x}}_{(1k)})\right]^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & \left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(\underline{x}_t-\overline{\underline{x}}_{(2k)})'(\underline{x}_t-\overline{\underline{x}}_{(2k)})\right]^{-1} \end{bmatrix} \\
\to &\begin{bmatrix} \begin{bmatrix} \frac{1}{12}r^3 & \frac{1}{12}r^4 \\ \frac{1}{12}r^4 & \frac{4}{45}r^5 \end{bmatrix}^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & \begin{bmatrix} \frac{(1-r)^3}{12} & \frac{(1-r)^3(1+r)}{12} \\ \frac{(1-r)^3(1+r)}{12} & \frac{\left(4+7r+r^2\right)(1-r)^3}{45} \end{bmatrix}^{-1} \end{bmatrix} \\
= &\begin{bmatrix} \begin{bmatrix} \frac{192}{r^3} & -\frac{180}{r^4} \\ -\frac{180}{r^4} & \frac{180}{r^5} \end{bmatrix} & 0_{2\times 2} \\ 0_{2\times 2} & \begin{bmatrix} 48\frac{4+7r+r^2}{11r^5-31r^4+26r^3-2r^2-5r+1} & -180\frac{1+r}{11r^5-31r^4+26r^3-2r^2-5r+1} \\ -180\frac{1+r}{11r^5-31r^4+26r^3-2r^2-5r+1} & \frac{180}{11r^5-31r^4+26r^3-2r^2-5r+1} \end{bmatrix} \end{bmatrix}
\end{aligned}
$$

Then

$$
NT\left[R(Var(b))R'\right]
$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{192}{r^3} & -\frac{180}{r^4} & 0 & 0 \\ -\frac{180}{r^4} & \frac{180}{r^5} & 0 & 0 \\ 0 & 0 & \frac{48\left(4+7r+r^2\right)}{11r^5-31r^4+26r^3-2r^2-5r+1} & \frac{-180(1+r)}{11r^5-31r^4+26r^3-2r^2-5r+1} \\ 0 & 0 & \frac{-180(1+r)}{11r^5-31r^4+26r^3-2r^2-5r+1} & \frac{180}{11r^5-31r^4+26r^3-2r^2-5r+1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 48\frac{45r^5-117r^4+108r^3-8r^2-20r+4}{r^3(11r^2+2r-1)(r-1)^3} & -180\frac{12r^5-30r^4+26r^3-2r^2-5r+1}{r^4(11r^2+2r-1)(r-1)^3} \\ -180\frac{12r^5-30r^4+26r^3-2r^2-5r+1}{r^4(11r^2+2r-1)(r-1)^3} & 180\frac{12r^5-31r^4+26r^3-2r^2-5r+1}{r^5(11r^2+2r-1)(r-1)^3} \end{bmatrix}.$$

This gives

$$\frac{1}{NT}\left[R(Var(b))R'\right]^{-1} \rightarrow \frac{r^3}{3a}\begin{bmatrix} -\frac{1}{4}c & -\frac{1}{4}rb \\ -\frac{1}{4}rb & -\frac{1}{15}r^2 d \end{bmatrix}$$

where

$$a = 36r^7 - 144r^6 + 228r^5 - 160r^4 + 32r^3 + 16r^2 - 8r + 1,$$

$$b = \left(12r^5 - 30r^4 + 26r^3 - 2r^2 - 5r + 1\right)\left(r^3 - 3r^2 + 3r - 1\right),$$

$$c = \left(12r^5 - 31r^4 + 26r^3 - 2r^2 - 5r + 1\right)\left(r^3 - 3r^2 + 3r - 1\right),$$

and

$$d = \left(45r^5 - 117r^4 + 108r^3 - 8r^2 - 20r + 4\right)\left(r^3 - 3r^2 + 3r - 1\right).$$

Thus we have

$$\begin{aligned} W_p(k) &\xrightarrow{d} \frac{\sigma_0^2}{\sigma_v^2}\frac{1}{3r^5 a}\begin{bmatrix} \frac{-12A}{C} & \frac{30B}{rC} \end{bmatrix}\begin{bmatrix} -\frac{1}{4}c & -\frac{1}{4}rb \\ -\frac{1}{4}rb & -\frac{1}{15}r^2 d \end{bmatrix}\begin{bmatrix} \frac{-12A}{C} \\ \frac{30B}{rC} \end{bmatrix} \\ &= \frac{\sigma_0^2}{\sigma_v^2} 4\frac{\left(-3A^2c + 15ABb - 5B^2 d\right)}{r^5 a C^2} \\ &= \frac{\sigma_0^2}{\sigma_v^2} P_1(r), \end{aligned}$$

where

$$P_1(r) = 4\frac{\left(-3A^2c + 15ABb - 5B^2 d\right)}{r^5 a C^2}.$$

Next we consider the case when $\rho = 1$. Define

$$J(r) = 3\int_0^r s^2\left[W(s) + \widetilde{W}(\kappa)\right] ds - r^2\int_0^r \left[W(s) + \widetilde{W}(\kappa)\right] ds.$$

Then under H_0 we have

$$\begin{aligned} &\sqrt{NT^{-1}}\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)}\right) \\ =\ &\sqrt{NT^{-1}}\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\beta}\right) - \sqrt{NT^{-1}}\left(\underline{\widehat{\beta}}_{(2k)} - \underline{\beta}\right) \end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t-\overline{x}_{(1k)})'(\underline{x}_t-\overline{x}_{(1k)})\right]^{-1}\left[\frac{1}{\sqrt{N}}\frac{1}{T^{\frac{3}{2}}}\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t-\overline{x}_{(1k)})'v_{it}\right] \\
&\quad -\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(\underline{x}_t-\overline{x}_{(2k)})'(\underline{x}_t-\overline{x}_{(2k)})\right]^{-1}\left[\frac{1}{\sqrt{N}}\frac{1}{T^{\frac{3}{2}}}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(\underline{x}_t-\overline{x}_{(2k)})'v_{it}\right] \\
&\quad \xrightarrow{d} \begin{bmatrix} \frac{1}{12}r^3 & \frac{1}{12}r^4 \\ \frac{1}{12}r^4 & \frac{4}{45}r^5 \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{2}\sigma_\varepsilon H(r) \\ \frac{1}{3}\sigma_\varepsilon J(r) \end{bmatrix} \\
&\quad -\begin{bmatrix} \frac{(1-r)^3}{12} & \frac{(1-r)^3(1+r)}{12} \\ \frac{(1-r)^3(1+r)}{12} & \frac{\left(4+7r+r^2\right)(1-r)^3}{45} \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{2}\sigma_\varepsilon\begin{pmatrix} H(1)-H(r)- \\ r\int_0^1 W(s)ds+\int_0^r W(s)ds \end{pmatrix} \\ \frac{1}{3}\sigma_\varepsilon\begin{pmatrix} J(1)-J(r)- \\ r(1+r)\int_0^1 W(s)ds+ \\ (1+r)\int_0^r W(s)ds \end{pmatrix} \end{bmatrix} \\
&= \frac{\sigma_\varepsilon}{r^4}\begin{bmatrix} \frac{-12D}{C} \\ \frac{30E}{rC} \end{bmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
D &= H(r)\left[-2r\left(4-20r+45r^5-117r^4+108r^3-8r^2\right)\right]+J(r)\left[60r^5-150r^4+130r^3-10r^2-25r+5\right]+ \\
&\quad H(1)\left[2r^4\left(4+7r+r^2\right)\right]+J(1)\left[-5r^4(r+1)\right]+ \\
&\quad \left(r^5\left(-3-4r+3r^2\right)\right)\int_0^1 W(s)ds+\left(-r^4\left(-3-4r+3r^2\right)\right)\int_0^r W(s)ds
\end{aligned}
$$

and

$$
\begin{aligned}
E &= H(r)\left[-3r\left(1-5r+12r^5-30r^4+26r^3-2r^2\right)\right]+J(r)\left[24r^5-62r^4+52r^3-4r^2-10r+2\right]+ \\
&\quad H(1)\left[3r^5(r+1)\right]+J(1)\left[-2r^5\right]+\left(-r^6(r+1)\right)\int_0^1 W(s)ds+\left(r^5(r+1)\right)\int_0^r W(s)ds.
\end{aligned}
$$

We know from Theorem 4 that

$$
\frac{1}{T}\widehat{\sigma}_v^2 \xrightarrow{p} \frac{\sigma_\varepsilon^2(1+6\kappa)}{6}.
$$

Hence

$$
\begin{aligned}
\frac{1}{T}Wp(k) &\xrightarrow{d} \frac{6}{(1+6\kappa)\sigma_\varepsilon^2}\frac{\sigma_\varepsilon^2}{3r^5a}\begin{bmatrix} \frac{-12D}{C} & \frac{30E}{rC} \end{bmatrix}\begin{bmatrix} -\frac{1}{4}c & -\frac{1}{4}rb \\ -\frac{1}{4}rb & -\frac{1}{15}r^2d \end{bmatrix}\begin{bmatrix} \frac{-12D}{C} \\ \frac{30E}{rC} \end{bmatrix} \\
&= \frac{24}{(1+6\kappa)}\frac{\left(-3D^2c+15DEb-5E^2d\right)}{r^5aC^2} \\
&= \frac{24}{(1+6\kappa)}P_2(r)
\end{aligned}
$$

where

$$P_2(r) = \frac{\left(-3D^2c + 15DEb - 5E^2d\right)}{r^5aC^2}.$$

Finally we consider the case when $\rho = 1 + \frac{c}{T}$. Define

$$J_c(r) = 3\int_0^r s^2\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds - r^2\int_0^r\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds.$$

Then under H_0 we have

$$\begin{aligned}
&\sqrt{NT^{-1}}\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\widehat{\beta}}_{(2k)}\right)\\
=\;&\sqrt{NT^{-1}}\left(\underline{\widehat{\beta}}_{(1k)} - \underline{\beta}\right) - \sqrt{NT^{-1}}\left(\underline{\widehat{\beta}}_{(2k)} - \underline{\beta}\right)\\
=\;&\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t - \overline{\underline{x}}_{(1k)})'(\underline{x}_t - \overline{\underline{x}}_{(1k)})\right]^{-1}\left[\frac{1}{\sqrt{N}}\frac{1}{T^{\frac{3}{2}}}\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t - \overline{\underline{x}}_{(1k)})'v_{it}\right] -\\
&\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(\underline{x}_t - \overline{\underline{x}}_{(2k)})'(\underline{x}_t - \overline{\underline{x}}_{(2k)})\right]^{-1}\left[\frac{1}{\sqrt{N}}\frac{1}{T^{\frac{3}{2}}}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(\underline{x}_t - \overline{\underline{x}}_{(2k)})'v_{it}\right]\\
\xrightarrow{d}\;&\begin{bmatrix}\frac{1}{12}r^3 & \frac{1}{12}r^4\\ \frac{1}{12}r^4 & \frac{4}{45}r^5\end{bmatrix}^{-1}\begin{bmatrix}\frac{1}{2}\sigma_\varepsilon H_c(r)\\ \frac{1}{3}\sigma_\varepsilon J_c(r)\end{bmatrix} - \begin{bmatrix}\frac{(1-r)^3}{12} & \frac{(1-r)^3(1+r)}{12}\\ \frac{(1-r)^3(1+r)}{12} & \frac{\left(4+7r+r^2\right)(1-r)^3}{45}\end{bmatrix}^{-1}\\
&\begin{bmatrix}\frac{1}{2}\sigma_\varepsilon\begin{pmatrix}H_c(1) - H_c(r) -\\ r\int_0^1\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds +\\ \int_0^r\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds\end{pmatrix}\\ \frac{1}{3}\sigma_\varepsilon\begin{pmatrix}J_c(1) - J_c(r) -\\ r(1+r)\int_0^1\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds +\\ (1+r)\int_0^r\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds\end{pmatrix}\end{bmatrix}\\
=\;&\frac{\sigma_\varepsilon}{r^4}\begin{pmatrix}\frac{-12F}{C}\\ \frac{30G}{rC}\end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
F \;=\;& H_c(r)\left[-2r\left(4 - 20r + 45r^5 - 117r^4 + 108r^3 - 8r^2\right)\right] + J_c(r)\left[60r^5 - 150r^4 + 130r^3 - 10r^2 - 25r + 5\right] +\\
& H_c(1)\left[2r^4\left(4 + 7r + r^2\right)\right] + J_c(1)\left[-5r^4(r+1)\right] + \left(r^5\left(-3 - 4r + 3r^2\right)\right)\int_0^1\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds +\\
& \left(-r^4\left(-3 - 4r + 3r^2\right)\right)\int_0^r\left[W_c(s) + e^{sc}\widetilde{W}_c(\kappa)\right]ds
\end{aligned}$$

and

$$G \;=\; H_c(r)\left[-3r\left(1 - 5r + 12r^5 - 30r^4 + 26r^3 - 2r^2\right)\right] + J_c(r)\left[24r^5 - 62r^4 + 52r^3 - 4r^2 - 10r + 2\right] +$$

$$H_c(1)\left[3r^5(r+1)\right]+J_c(1)\left[2r^5\right]+\left(-r^6(r+1)\right)\int_0^1\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds+$$
$$\left(r^5(r+1)\right)\int_0^r\left[W_c(s)+e^{sc}\widetilde{W}_c(\kappa)\right]ds.$$

From Theorem 6 we know that

$$\frac{1}{T}\widehat{\sigma}_v^2\xrightarrow{p}\frac{\sigma_\varepsilon^2\left(c+3\kappa\left(e^{2c}-1\right)\right)}{6c}.$$

Hence

$$\begin{aligned}\frac{1}{T}W_p(k)&\xrightarrow{d}\frac{6c}{\sigma_\varepsilon^2\left(c+3\kappa\left(e^{2c}-1\right)\right)}\frac{\sigma_\varepsilon^2}{3r^5a}\begin{bmatrix}\frac{-12F}{C}&\frac{30G}{rC}\end{bmatrix}\begin{bmatrix}-\frac{1}{4}c&-\frac{1}{4}rb\\-\frac{1}{4}rb&-\frac{1}{15}r^2d\end{bmatrix}\begin{bmatrix}\frac{-12F}{C}\\\frac{30G}{rC}\end{bmatrix}\\&=\frac{24c}{\left(c+3\kappa\left(e^{2c}-1\right)\right)}\frac{\left(-3F^2c+-15FGb-5G^2d\right)}{r^5aC^2}\\&=\frac{24c}{\left(c+3\kappa\left(e^{2c}-1\right)\right)}P_c(r),\end{aligned}$$

where

$$P_c(r)=\frac{\left(-3F^2c+-15FGb-5G^2d\right)}{r^5aC^2}.$$

■

J Proof of Theorem 10

Proof. Under the local alternative we have

$$\begin{aligned}&\sqrt{NT}\left(\underline{\widehat{\beta}}_{(1k)}^{(T)}-\underline{\beta}\right)\\&=\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t-\underline{\overline{x}}_{(1k)})'(\underline{x}_t-\underline{\overline{x}}_{(1k)})\right]^{-1}\\&\quad\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t-\underline{\overline{x}}_{(1k)})'\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right)+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}(\underline{x}_t-\underline{\overline{x}}_{(1k)})'v_{it}\right]\\&\quad\xrightarrow{d}\begin{bmatrix}\frac{192}{r^3}&-\frac{180}{r^4}\\-\frac{180}{r^4}&\frac{180}{r^5}\end{bmatrix}\begin{bmatrix}\int_0^r s^2g(s)\,ds-\frac{1}{2}r\int_0^r sg(s)\,ds&\int_0^r s^3g(s)\,ds-\frac{1}{2}r\int_0^r s^2g(s)\,ds\\\int_0^r s^3g(s)\,ds-\frac{1}{3}r^2\int_0^r sg(s)\,ds&\int_0^r s^4g(s)\,ds-\frac{1}{3}r^2\int_0^r s^2g(s)\,ds\end{bmatrix}+\\&\qquad\qquad\begin{bmatrix}\frac{192}{r^3}&-\frac{180}{r^4}\\-\frac{180}{r^4}&\frac{180}{r^5}\end{bmatrix}\begin{bmatrix}\frac{1}{2}\sigma_0G(r)\\\frac{1}{3}\sigma_0F(r)\end{bmatrix}\\&=\begin{bmatrix}12\sigma_0\frac{8G(r)r-5F(r)}{r^4}\\-30\sigma_0\frac{3G(r)r-2F(r)}{r^5}\end{bmatrix}+\begin{bmatrix}\frac{192}{r^3}&-\frac{180}{r^4}\\-\frac{180}{r^4}&\frac{180}{r^5}\end{bmatrix}\begin{bmatrix}h_1(r)&h_2(r)\\h_3(r)&h_4(r)\end{bmatrix}\end{aligned}$$

where

$$\begin{bmatrix} h_1(r) & h_2(r) \\ h_3(r) & h_4(r) \end{bmatrix} = \begin{bmatrix} \int_0^r s^2 g(s)\,ds - \frac{1}{2} r \int_0^r s g(s)\,ds & \int_0^r s^3 g(s)\,ds - \frac{1}{2} r \int_0^r s^2 g(s)\,ds \\ \int_0^r s^3 g(s)\,ds - \frac{1}{3} r^2 \int_0^r s g(s)\,ds & \int_0^r s^4 g(s)\,ds - \frac{1}{3} r^2 \int_0^r s^2 g(s)\,ds \end{bmatrix}.$$

We also have

$$\begin{aligned}
&\sqrt{NT}\left(\underline{\widehat{\beta}}_{(2k)}^{(T)} - \underline{\beta}\right) \\
= &\left[\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=k+1}^{T}(\underline{x}_t - \overline{x}_{(2k)})'(\underline{x}_t - \overline{x}_{(2k)})\right]^{-1}\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=k+1}^{Tk}(\underline{x}_t - \overline{x}_{(2k)})'\left(\frac{1}{\sqrt{T}}g\left(\frac{t}{T}\right) + v_{it}\right)\right] \\
&\xrightarrow{d} \begin{bmatrix} \frac{48\left(4+7r+r^2\right)}{C} & \frac{-180(1+r)}{C} \\ \frac{-180(1+r)}{C} & \frac{180}{C} \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} + \\
&\begin{bmatrix} \frac{48\left(4+7r+r^2\right)}{C} & \frac{-180(1+r)}{C} \\ \frac{-180(1+r)}{C} & \frac{180}{C} \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sigma_0\left(G(1) - G(r) + W(r) - rW(1)\right) \\ \frac{1}{3}\sigma_0\left(F(1) - F(r) + (1+r)W(r) - r(1+r)W(1)\right) \end{bmatrix} \\
= &\begin{bmatrix} \frac{12\sigma_0}{C}\begin{pmatrix} 8G(1) - 8G(r) + 3W(r) - 3rW(1) + 14rG(1) - \\ 14G(r)r + 4W(r)r - 4r^2W(1) + 2r^2G(1) - 2r^2G(r) - \\ 3r^2W(r) + 3r^3W(1) - 5rF(1) + 5rF(r) - 5F(1) + 5F(r) \end{pmatrix} \\ \frac{-30\sigma_0}{C}\begin{pmatrix} 3rG(1) - 3G(r)r + W(r)r - r^2W(1) + 3G(1) - \\ 3G(r) + W(r) - rW(1) - 2F(1) + 2F(r) \end{pmatrix} \end{bmatrix} + \\
&\begin{bmatrix} \frac{48\left(4+7r+r^2\right)}{C} & \frac{-180(1+r)}{C} \\ \frac{-180(1+r)}{C} & \frac{180}{C} \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}
\end{aligned}$$

Part (a) holds using the proof of Theorem 9 with $\begin{bmatrix} \frac{1}{2}\sigma_0 G(r) \\ \frac{1}{3}\sigma_0 F(r) \end{bmatrix}$ and

$$\begin{bmatrix} \frac{1}{2}\sigma_0\left(G(1) - G(r) + W(r) - rW(1)\right) \\ \frac{1}{3}\sigma_0\left(F(1) - F(r) + (1+r)W(r) - r(1+r)W(1)\right) \end{bmatrix}$$

replaced by

$$\begin{bmatrix} \frac{1}{2}\sigma_0 G(r) \\ \frac{1}{3}\sigma_0 F(r) \end{bmatrix} + \begin{bmatrix} h_1(r) & h_2(r) \\ h_3(r) & h_4(r) \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1}{2}\sigma_0\left(G(1) - G(r) + W(r) - rW(1)\right) \\ \frac{1}{3}\sigma_0\left(F(1) - F(r) + (1+r)W(r) - r(1+r)W(1)\right) \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

respectively. Then under the local alternative

$$W_p^{(T)}(k) \xrightarrow{d} \frac{\sigma_0^2}{\sigma_v^2}\left[P_1(r) + \frac{1}{\sigma_0^2}R_1(r)\right],$$

where

$$R_1(r) = \begin{bmatrix} \left[\begin{bmatrix} h_1(r) & h_2(r) \\ h_3(r) & h_4(r) \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right]' \begin{bmatrix} -\frac{1}{12}\frac{r^3 c}{a} & -\frac{1}{12}\frac{r^4 b}{a} \\ -\frac{1}{12}\frac{r^4 b}{a} & -\frac{1}{45}\frac{r^5 d}{a} \end{bmatrix} \\ \begin{bmatrix} h_1(r) & h_2(r) \\ h_3(r) & h_4(r) \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} \end{bmatrix},$$

$$e = h_1(1) - h_1(r) - \frac{1}{2} r \int_0^1 s g(s)\, ds + \frac{1}{2}\int_0^r s g(s)\, ds,$$

$$f = h_2(1) - h_2(r) - \frac{1}{2} r \int_0^1 s^2 g(s)\, ds + \frac{1}{2}\int_0^r s^2 g(s)\, ds,$$

$$g = h_3(1) - h_3(r) - \frac{1}{3} r (1+r) \int_0^1 s g(s)\, ds + \frac{1}{3}(1+r)\int_0^r s g(s)\, ds,$$

and

$$h = h_4(1) - h_4(r) - \frac{1}{3} r (1+r) \int_0^1 s^2 g(s)\, ds + \frac{1}{3}(1+r)\int_0^r s^2 g(s)\, ds.$$

Parts (b) and (c) hold using the proof of Theorem 9. ■

References

[1] Andrews, D. W. K. (1991), "Heteroskedasticity and Autocorrelation Consistent Covariance Matrix Estimation," *Econometrica*, 59, 817-858.

[2] Andrews, D. W. K. (1993), "Tests for Parameter Instability and Structural Changes with Unknown Change Point," *Econometrica*, 61, 821-856.

[3] Andrew, D. W. K. and Ploberger, W. (1994), "Optimal Tests When a Nuisance Parameter is Present Only Under the Alternative," *Econometrica*, 62, 1383-1414.

[4] Bai, J. (1996), "Testing for Parameter Constancy in Linear Regressions: An Empirical Distribution Function Approach," *Econometrica*, 64, 597-622.

[5] Bai, J. (1997), "Estimation of a Change Point in Multiple Regression Models," *Review of Economics and Statistics*, 79, 551-563.

[6] Bai, J., Lumsdaine, R. L., and Stock, J. H. (1998), "Testing for and Dating Common Breaks in Multivariate Time Series," *Review of Economic Studies*, 65, 395-432.

[7] Billingsley, P. (1968), *Convergence of Probability Measures*, New York, Wiley.

[8] Campos, J., Ericsson, N. R., and Hendry, D. F. (1996), "Cointegration Tests in the Presence of Structural Breaks," *Journal of Econometrics*, 70, 187-220.

[9] Chu, C.-S., and White, H. (1992), "A Direct Test for Changing Trend," *Journal of Business and Statistics*, 10, 289-299.

[10] Gregory, A. W., and Hansen, B. (1996), "Residual-Based Tests for Cointegration in Models with Regime Shifts," *Journal of Econometrics*, 70, 99-126.

[11] Han, A. K., and Park, D. (1989), "Testing for Structural Change in Panel Data: Application to s Study of U.S. Foreign Trade in Manufacturing Goods," *Review of Economics and Statistics*, 71, 135-142.

[12] Hansen, B. (1992), "Tests for Parameter Instability in Regressions with I(1) Processes," *Journal of Business and Statistics,* 10, 321-336.

[13] Hansen, B. E. (1999), "Threshold Effects in Non-Dynamic Panels: Estimation, Testing and Inference," *Journal of Econometrics*, 93, 345-368.

[14] Kao, C. (1999), "Spurious Regression and Residual-Based Tests for Cointegration in Panel Data," *Journal of Econometrics*, 90, 1-44.

[15] Kao, C., and Ross, S. (1995), "A CUSUM Test in the Linear Regression with Serially Correlated Disturbances," *Econometric Reviews*, 14, 331-346.

[16] Kramer, W., Ploberger, W., and Alt, R. (1988), "Testing for Structural Change in Dynamic Models," *Econometrica*, 56, 1355-1369.

[17] Moon, H. R., and Phillips, P. C. B. (1999), "Estimation of Autoregressive Roots Near Unity Using Panel Data," Manuscript.

[18] Picard, D. (1985), "Testing and Estimating Change-Point in Time Series," *Advances in Applied Probability*, 17, 841-867.

[19] Ploberger, W., Kramer, W., and Kontrus, K. (1989), "A New Test for Structural Stability in the Linear Regression Model," *Journal of Econometrics*, 40, 307-318.

[20] Vogelsang, T. J. (1997), "Wald-Type Tests for Detecting Breaks in the Trend Function of a Dynamic Time Series," *Econometric Theory*, 13, 818-849.

Table 1: Asymptotic Critical Values of the Tests

	T_1	T_2	$sup\text{W}_1$	$Mean\text{W}_1$	$Exp\text{W}_1$	$sup\text{W}_2$	$Mean\text{W}_2$	$Exp\text{W}_2$
1%	.940	.108	4.317	.903	.443	.059	.018	-.344
5%	.784	.083	3.127	.584	.135	.0367	.011	-.349
10%	.708	.071	2.640	.463	.025	.027	.008	-.351