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ON GRADIENT RICCI SOLITONS WITH SYMMETRY

PETER PETERSEN AND WILLIAM WYLIE

Abstract. We study gradient Ricci solitons with maximal symmetry. First we show that there are no non-trivial homogeneous gradient Ricci solitons. Thus the most symmetry one can expect is an isometric cohomogeneity one group action. Many examples of cohomogeneity one gradient solitons have been constructed. However, we apply the main result in [21] to show that there are no noncompact cohomogeneity one shrinking gradient solitons with nonnegative curvature.

1. Introduction

The goal of this paper is to study how symmetries can yield rigidity of a gradient Ricci soliton together with weaker conditions than we used in [21]. Recall that a Ricci soliton is a Riemannian metric together with a vector field (M, g, X) that satisfies

$$\mathrm{Ric} + \frac{1}{2} L_X g = \lambda g.$$

It is called shrinking when $\lambda > 0$, steady when $\lambda = 0$, and expanding when $\lambda < 0$. In case $X = \nabla f$ the equation can also be written as

$$\mathrm{Ric} + \mathrm{Hess} f = \lambda g$$

and is called a gradient (Ricci) soliton. A gradient soliton is rigid if it is isometric to a quotient of $N \times \mathbb{R}^k$ where N is an Einstein manifold and $f = \frac{\lambda}{2}|x|^2$ on the Euclidean factor. Throughout this paper we will also assume that our metrics have bounded curvature. Shi's estimates for the Ricci flow then imply that all the derivatives of curvature are also bounded (see Chapter 6 of [6]).

First we show that all gradient solitons with maximal symmetry are rigid.

Theorem 1.1. *All homogeneous gradient Ricci solitons are rigid.*

This is in sharp contrast to the more general Ricci solitons that exist on many Lie groups and other homogeneous spaces see [1, 14, 15]. It also shows that the maximal amount of symmetry we can expect on a nontrivial gradient soliton is a cohomogeneity 1 action that leaves f invariant. Particular cases, such as the rotationally symmetric case on $\mathbb{R}^n$ and the $U(n)$ invariant case on certain Kähler manifolds, have been studied extensively and many interesting examples have been found, see e.g. [4, 5, 7, 9, 11, 12, 13, 26, 27]. In particular, Kotschwar [13] has shown that the only rotationally symmetric shrinking gradient soliton metrics on S^n, $\mathbb{R}^n$, and $S^{n-1} \times \mathbb{R}$ are the rigid ones and Feldman, Ilmanen, and Knopf [9] have proven that the only $U(n)$ invariant shrinking soliton on $\mathbb{C}^n$ is the flat metric. No curvature assumption is required for these results. On the other hand, there are non-rigid complete noncompact $U(n)$ invariant gradient shrinking solitons [7, 9, 27]. These examples show that some other assumption is necessary in general to prove rigidity. Here we show that nonnegative curvature suffices.

Theorem 1.2. *All complete noncompact shrinking gradient solitons of cohomogeneity 1 with nonnegative Ricci curvature and* $\sec(E, \nabla f) \geq 0$ *are rigid.*

Recently Naber [16], building on work of Ni and Wallach [19], has shown that every 4-dimensional complete shrinking soliton with nonnegative curvature operator is rigid. (This was proven in dimensions 2 and 3 by Hamilton [10] and Perelman [20] respectively.) Theorem 1.2 offers further evidence this result extends to higher dimensions. In fact, in the proof all we use about cohomogeneity one is a much weaker condition on f we call rectifiability which we will discuss in section 3. (Also recall that the work of Böhm and Wilking [3] implies that every compact shrinking gradient Ricci soliton with nonnegative curvature operator is a quotient of the round sphere.) For other recent results concerning the classification of gradient shrinking solitons see [8, 17, 18, 22, 23, 25].

The famous Bryant soliton (see [11]) and the examples in [5] show that there are non-rigid rotationally symmetric steady and expanding gradient solitons with positive curvature operator.

2. Killing Fields on Gradient Solitons

In this section we establish a splitting theorem involving Killing fields on a gradient soliton which leads to Theorem 1.1. The main observation is the following.

Proposition 1. *If X is a Killing field on a gradient soliton, then $\nabla D_X f$ is parallel. Moreover, if $\lambda \neq 0$ and $\nabla D_X f = 0$ then also $D_X f = 0$.*

Proof. We have that $L_X g = 0$, thus $L_X \mathrm{Ric} = 0$ and hence

$$\begin{aligned} 0 &= L_X \mathrm{Hess} f \\ &= \mathrm{Hess} L_X f \\ &= \mathrm{Hess} D_X f \end{aligned}$$

this proves the first claim.

Next note that if $\nabla D_X f = 0$, then $D_X f$ is constant. Thus $f \circ \gamma_X(t) : \mathbb{R} \to \mathbb{R}$ is onto if γ_X is an integral curve for X and $D_X f$ doesn't vanish.

On the other hand recall that the soliton equation implies that

$$\mathrm{scal} + |\nabla f|^2 - 2\lambda f = \mathrm{const}$$

So if the scalar curvature is bounded we see that f must either be bounded from below or above and hence $D_X f = 0$. □

This shows that either $D_X f = 0$ or the metric splits off a Euclidean factor. One might worry that the soliton structure may not also split, however the next lemma shows this is not an issue.

Lemma 2.1. *If a gradient soliton splits $(M, g) = (M_1 \times M_2, g_1 + g_2)$ as a Riemannian product, then $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$ also splits in such a way that each (M_i, g_i, f_i) is a soliton*

$$\mathrm{Ric}_{g_i} + \mathrm{Hess} f_i = \lambda g_i$$

Proof. Use the $(1,1)$ version of the soliton equation

$$\mathrm{Ric} + \nabla\nabla f = \lambda I$$

to see that the operator $E \to \nabla_E \nabla f$ preserves the manifold splitting as the Ricci curvature preserves the splitting. This can be used to first split the gradient ∇f.

To see how, use local coordinates x^j such $x^1, ..., x^m$ are coordinates on M_1 and $x^{m+1}, ..., x^n$ coordinates on M_2. The splitting of the metric then implies that

$$\nabla_{\partial_i} \partial_j = 0$$

if $i \leq m$ and $j \geq m+1$ or $i \geq m = 1$ and $j \leq m$. If we write $\nabla f = \alpha^j \partial_j$, then

$$\begin{aligned} \nabla_{\partial_i} \nabla f &= \nabla_{\partial_i} \alpha^j \partial_j \\ &= \left(\partial_i \alpha^j\right) \partial_j + \alpha^j \nabla_{\partial_i} \partial_j. \end{aligned}$$

If we assume that $i \leq m$ then

$$\begin{aligned} \nabla_{\partial_i} \nabla f &\in TM_1, \\ \alpha^j \nabla_{\partial_i} \partial_j &\in TM_1 \end{aligned}$$

showing that $\partial_i \alpha^j = 0$ for $j \geq m+1$. Similarly $\partial_i \alpha^j = 0$ when $i \geq m+1$ and $j \leq m$. This shows that

$$\nabla f = X_1 + X_2$$

where X_i are vector fields on M_i. We then see that

$$X_i = \nabla f_i$$

where

$$\begin{aligned} f_1(x_1) &= f(x_1, q), \\ f_2(x_2) &= f(p, x_2) - f(p, q) \end{aligned}$$

for some fixed point $(p, q) \in M_1 \times M_2$. □

Note that the splitting of the metric implies

$$R(\cdot, \nabla f) \nabla f = R_1(\cdot, \nabla f_1) \nabla f_1 + R_2(\cdot, \nabla f_2) \nabla f_2$$

So if, say, M_2 is flat then the radial curvatures of M and M_1 are the same.

This implies the reduction result alluded to above.

Corollary 1. *If X is a Killing field on a gradient soliton, then either $D_X f = 0$ or we have an isometric splitting $M = N \times \mathbb{R}$ where N is a gradient soliton with the same radial curvatures as M.*

Intuitively, Corollary 1 says that if the metric of a gradient soliton has some symmetry, then the only way f can break the symmetry is by splitting off a Gaussian factor. With this fact we can prove the result for homogeneous solitons.

Theorem 2.2. *All homogeneous gradient solitons are rigid.*

Proof. In case the soliton is steady this is a consequence of the scalar curvature being constant and hence M is Ricci flat.

When the soliton is expanding or shrinking split $M = N \times \mathbb{R}^k$ such that N doesn't have any flat de Rham factors. If G acts transitively on M it also acts transitively on each of the two factors as isometries preserve the flat de Rham factor.

The previous lemma and corollary now tell us that all Killing fields on N must leave f_1 invariant. Thus N can't be homogeneous unless f_1 is trivial. □

3. Rectifiability

In this section we prove the result for cohomogeneity one and more general rectifiable gradient solitons.

We say that a function u is *rectifiable* if it can be written as $u = h(r)$ where r is a distance function. It is easy to check that a function is rectifiable if and only if its gradient ∇u has constant length on the level sets of u. We will say that a gradient soliton (M, g, f) is rectifiable if the function f is rectifiable on (M, g).

It is easy to see that a gradient soliton with a cohomogeneity 1 group action that leaves f invariant is rectifiable. Assume that G is such a isometric group action. This gives us a distance function

$$r : M \to M/G \subset \mathbb{R}$$

(locally if G is noncompact) and $f = h(r)$ as f is constant on the orbits of the action. Similarly the scalar curvature is also rectifiable with respect to r.

We note the following interesting properties of rectifiable solitons.

Proposition 2. *If (M, g, f) is a rectifiable gradient soliton with $f = h(r)$ then* scal*, Δf, and Δr are also rectifiable. In particular,* $\mathrm{Ric}(\nabla f, \nabla f) = 0$ *if and only if (M, g) has constant scalar curvature.*

Proof. If f is rectifiable, then $|\nabla f|$ is also rectifiable so the equation

$$\mathrm{scal} + |\nabla f|^2 - 2\lambda f = \mathrm{const}$$

implies that the scalar curvature is rectifiable.

Tracing the soliton equation then gives

$$\mathrm{scal} = \lambda n - \Delta f,$$

so Δf is rectifiable. Since f is rectifiable we can write

$$\Delta f = h''(r) + h'(r)\Delta r$$

so Δf rectifiable implies that Δr is also rectifiable.

Now since scal and f are rectifiable $\nabla \mathrm{scal} = \mathrm{Ric}(\nabla f)$ is proportional to ∇f, proving the last statement. □

The main result from [21] now shows that a rectifiable gradient soliton is rigid if and only if it is radially flat. We note that, in the case of cohomogeneity one, radial flatness, even without the soliton equation, is already quite restrictive.

Theorem 3.1. *A radially flat cohomogeneity 1 space coming from a compact action is a flat bundle.*

Proof. Let $r : M \to \mathbb{R}$ be the distance function coming from the quotient $M \to M/G$. It is smooth except at the singular orbits. The singular orbits correspond to the minimum and/or maximum of r if they exist.

Let $S_r = \nabla\nabla r$, then

$$\nabla_{\nabla r} S_r + S_r^2 = 0$$

This means that S_r is completely determined by the singular orbits where $S_r \to 0$ on vectors tangent to the singular orbit and $S_r \to \infty$ on vectors normal to the singular orbit and perpendicular to ∇r.

If there are no singular orbits, then $S_r = 0$ is the only possibility as all other solutions blow up in finite time going forwards or backwards. Thus the space splits.

If r has a minimum set, then solutions that start out being zero stay zero, while the other solutions that start out being ∞ decay to zero. As they never become zero the space is noncompact. We see that the space must then be a flat bundle $N \times_\Gamma \mathbb{R}^k$ where N/Γ is the singular orbit. □

We now turn our attention to proving rigidity for rectifiable shrinking solitons with nonnegative radial curvature.

Proposition 3. *Let (M, g) be a Riemannian manifold and $r : M \to [0, \infty)$ a proper distance function that is smooth outside a compact set. If $\sec(E, \nabla r) \geq 0$, then r is convex outside a compact set.*

Proof. Define $S_r = \nabla\nabla r$ and use that it solves the equation

$$\nabla_{\nabla r} S_r = -S_r^2 - R(\cdot, \nabla r)\nabla r.$$

As $E \to R(E, \nabla r)\nabla r$ is assumed to be nonnegative we see that if S_r has a negative eigenvalue somewhere, then it will go to $-\infty$ before r reaches infinity. This contradicts that r is smooth. □

Lemma 3.2. *Let (M, g, f) be a noncompact nontrivial shrinking gradient soliton with rectifiable and proper f. If the radial curvatures, $\sec(E, \nabla f)$ are nonnegative, then f is convex at infinity.*

Proof. Since f is rectifiable: $f = h(r)$, where $r : M \to [0, \infty)$ is a distance function that is smooth outside a compact set. Since f and r have proportional gradients, $\nabla f = h'\nabla r$, our curvature assumption guarantees that r is convex at infinity.

First note that the equation

$$\text{scal} + |\nabla f|^2 - 2\lambda f = \text{const}$$

shows that $|\nabla f| \to \infty$ as scal is bounded and f is proper, i.e., $|f| \to \infty$. In particular $h' > 0$ outside a compact set.

Define $S_f = \nabla\nabla f$ and $S_r = \nabla\nabla r$, they are related by

$$\begin{aligned} S_f &= \nabla\nabla f \\ &= h'' dr \otimes \nabla r + h'\nabla\nabla r \\ &= h'' dr \otimes \nabla r + h' S_r \end{aligned}$$

The soliton equation shows that

$$\text{Ric}(\nabla r, \nabla r) + h'' = \lambda$$

Since $\text{Ric}(\nabla r, \nabla r)$ is nonnegative this shows that $h'' \leq \lambda$.

Next we claim that $\text{Ric}(\nabla r) \to 0$ as $r \to \infty$. This follows from the formula

$$\begin{aligned} \text{Ric}(\nabla r) &= \pm\frac{\text{Ric}(\nabla f)}{|\nabla f|} \\ &= \pm\frac{1}{2}\frac{\nabla\text{scal}}{|\nabla f|} \end{aligned}$$

where we note that ∇scal is bounded and $|\nabla f| \to \infty$ at infinity.

Thus

$$S_f(\nabla r) = h''\nabla r \sim \lambda\nabla r$$

at infinity. This proves that outside some large compact set $h'' \geq \lambda/2$ and $h' > 0$. Thus f is convex outside a compact set. □

Theorem 3.3. *A complete, noncompact, rectifiable, shrinking gradient soliton with nonnegative radial sectional curvature, and nonnegative Ricci curvature is rigid.*

Proof. Let $f = h(r)$. Since we have a shrinking gradient Ricci soliton with bounded nonnegative curvature f is proper [20]. Therefore, the previous lemmas show that f and r are proper and convex outside a compact set. This implies that $\mathrm{Ric} \leq \lambda g$ outside a compact set. Define $\Delta_f = \Delta - D_{\nabla f}$ to be the f-Laplacian, then (see [21])

$$\Delta_f \mathrm{scal} = \mathrm{tr}\left(\mathrm{Ric} \circ (\lambda I - \mathrm{Ric})\right)$$

So $\mathrm{Ric} \leq \lambda g$ outside a compact set implies

$$\Delta_f \mathrm{scal} \geq 0$$

outside a set $\Omega_R = \{x \in M : r \leq R\}$. We also know that scal is increasing along gradient curves for ∇f as

$$D_{\nabla f}\mathrm{scal} = 2\mathrm{Ric}\left(\nabla f, \nabla f\right) \geq 0.$$

If

$$s_R = \min_{p \in \partial\Omega_R} \mathrm{scal}_p$$

then the function

$$u = \max\left\{\mathrm{scal}, s_R\right\}$$

satisfies

$$\Delta_f u \geq 0$$

From Theorem 4.2 in [22] it follows that u is constant (also see [24]). This shows that $\mathrm{scal} = s_R$ on $M - \Omega_R$. Since (M, g) is analytic (see [2]) the scalar curvature is constant on all of M. This in turn shows that $\mathrm{Ric}\left(\nabla f, \nabla f\right) = 0$ everywhere and hence $\sec(E, \nabla f) \geq 0$ implies that (M, g) is radially flat. The main theorem from [21] then shows that M is rigid. □

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