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ASYMPTOTIC INFERENCE IN CENSORED REGRESSION MODELS REVISITED

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Abstract

This paper establishes that regressors in the models with censored dependent variables need not be bounded for the standard asymptotic results to apply. Thus regressors which grow monotonically with the obervation index may be acceptable. It also purports to provide an upper bound on the rate at which regressors may grow.

We show that if $\|x_t\| \le c$ for all t, then $\lambda_{\min} \sum_{t=1}^{T} x_t x_t^T \to \infty$ is a sufficient condition for the consistency and asymptotic normality of the MLE in censored regression models, which are different from those used by Amemiya (1973). For the case of growing regressors, we show that the sufficient conditions for the consistency and asymptotic normality of the MLE are $\|x_t\|^2 = o(\log t)$ and $\lambda_{\min} \sum_{t=1}^{T} x_t x_t^T \ge cT^{\alpha}$, for some $\alpha > 0$ and c > 0, but only for one-half of the parameter space. The admissible growth rate given above implies that the Fisher Information matrix diverges, which seems to be an indispensable requirement for asymptotic inference for the censored regression models. More importantly, it represents a critical upper bound in the Fisher information matrix if it is exceeded monotonically. It also implies that $x_t = (1, t)^T$ in Judge et al. (1985:791) for the Tobit is not admissible.

1. Introduction

Consider the linear regression model

$$y_t = x_t^{'}\beta + \varepsilon_t, \qquad (t = 1,\ldots T) \tag{1}$$

where the ε_t are i.i.d. normal random variables having mean zero and finite variance σ^2, x_t is a k x 1 vector of regressors and β is a k x 1 vector of parameter of interest. However, for each t, instead of observing (y_t, x_t), one observes (z_t, δ_t, x_t), where $z_t = \max\{y_t, 0\}$, and δ_t is an indicator variable taking the value 1 if $y_t > 0$ and 0 if $y_t \le 0$.

The consistency and asymptotic normality of the maximum likelihood estimator (MLE) of (β^T, σ^2) have been studied by Amemiya (1973) under assumptions of $\{x_t\}$, e.g., x_t is bounded, i.e., $\|x_t\| \le c$ for all t, and $\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} x_t x_t^{'} = V$, a positive definite matrix. However, the conditions given by Amemiya (1973) above may be too strong to study the cases where growing regressors are of interest (e.g., time trend model, i.e, $x_t = (1, t)'$, see Judge et al. 1985:791).

The likelihood function of this model is

$$L = \prod_{t=1}^{T} [1 - \Phi(x_t^{'}\beta, \sigma^2)]^{1-\delta_t} [\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_t - x_t^{'}\beta)^2}]^{\delta_t} \tag{2}$$

with

$$\Phi(x_t^{'}\beta, \sigma^2) = \int_{-\infty}^{x_t^{'}\beta} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\lambda/\sigma)^2} d\lambda \tag{3}$$

In section 2, we confirmed Fahrmeir's (1987:103) result that if $\|x_t\| \le c$ for all t, then

$$\lambda_{\min} \sum_{t=1}^{T} x_t x_t^{'} \to \infty \tag{4}$$

is a sufficient condition for the consistency and asymptotic normality of the MLE for the censored model but only for one-half of the parameter space, where $\lambda_{\min} A$ is the smallest eigenvalues of a symmetric matrix A. In Section 3 the case of growing regressors is discussed. We show that the sufficient conditions for the consistency and asymptotic normality of the MLE are

$$\|x_t\|^2 = o(\log t) \tag{5}$$

$$\lambda_{\min} \sum_{t=1}^{T} x_t x_t^{'} \geq cT^{\alpha}, \text{ for some } \alpha > 0 \text{ and } c > 0, \tag{6}$$

for one-half of the parameter space. The summary is given in the Section 4.

Remark 1: In the classical linear regression model with i.i.d errors and the sequence $\{x_t\}$ of regressors, $\lambda_{\min} \sum_{t=1}^{T} x_t x_t^{'} \to \infty$ is necessary and sufficient for weak (Drygas 1976) and strong (Lai, Robbins, and Wei 1979) consistency (also see Amemiya 1985:95).

Remark 2: Note that $\Phi(x_t^{'}\beta, \sigma^2)$ will tend to one or zero if some regressors are growing monotonically to +∞ or -∞. Thus, for large T nearly all response y_t will fall into one category and there will be too less information to draw inference about the relevant parameters. The admissible growth rate in (5) and (6) assure that enough information is available and the asymptotic theory works.

Remark 3: The admissible growth rate, $\|x_t\|^2 = o(\log t)$, given in (5) implies that the Fisher Information matrix diverges, which seems to be an indispensable requirement for asymptotic inference for the censored regression models. More importantly, it represents a critical upper bound in the Fisher information matrix if it is exceeded monotonically

Remark 4: Fahrmeir and Kaufmann (1986) and Gourieroux and Monfort (1981) have discussed the sharp upper bounds on the admissible growth of regressors for logit, probit, cumulative logit, and loglinear, and linear Poisson model.

2. Statistical Inference

Olsen (1978) proved the global concavity of logL in the Tobit model in terms of the transformed parameter $\alpha = \beta / \sigma$ and $h = 1/\sigma$. Without loss of generality (see Remark 7), we assume $\sigma = 1$. The logL in terms of the new parameters can written as (see Olsen 1978)

$$\log L = \sum_0 \log[1 - \Phi(x_t'\beta)] - \frac{1}{2}\sum_1 (y_t - x_t'\beta)^2 \tag{7}$$

where $\sum_0$ is the summation over all observations where $y_t = 0$, $\sum_1$ is the summation over all observations where $y_t > 0$, and T_1 is the number of observations where $y_t > 0$. The score function $s_t(\beta)$ and the information matrix $F_t(\beta)$ are

$$\begin{aligned} s_t(\beta) &= -\sum_0 x_t \frac{\phi(x_t'\beta)}{1-\Phi(x_t'\beta)} + \sum_1 x_t (y_t - x_t'\beta) \\ &= \sum_t x_t \left[\delta_t \left(y_t - x_t'\beta \right) - (1-\delta_t) \left(\frac{\phi(x_t'\beta)}{1-\Phi(x_t'\beta)} \right) \right] \\ &= \sum_t x_t \left\{ \left[\delta_t y_t + (1-\delta_t) \left(x_t'\beta - \frac{\phi(x_t'\beta)}{1-\Phi(x_t'\beta)} \right) \right] - x_t'\beta \right\} \\ &= \sum_t x_t \left(\hat{y}_t - x_t'\beta \right), \text{ say,} \end{aligned} \tag{8}$$

$$F_t(\beta) = \mathrm{cov}_\beta\, s_t(\beta) = E_\beta H_t(\beta) = E_\beta\left[\sum_0 \frac{\phi(x_t'\beta)}{1-\Phi(x_t'\beta)}\left(\frac{\phi(x_t'\beta)}{1-\Phi(x_t'\beta)} - x_t'\beta\right)x_t x_t' + \sum_1 x_t x_t'\right]$$

$$= E_\beta\left[\sum_t \left\{(1-\delta_t)\left[\frac{\phi(x_t'\beta)}{1-\Phi(x_t'\beta)}\left(\frac{\phi(x_t'\beta)}{1-\Phi(x_t'\beta)} - x_t'\beta\right)\right] + \delta_t\right\} x_t x_t'\right]$$

$$= \sum_t \left(\left\{\frac{\phi(x_t'\beta)}{1-\Phi(x_t'\beta)} - x_t'\beta\right\}\phi(x_t'\beta) + \Phi(x_t'\beta)\right) x_t x_t' \tag{9}$$

where $H_t(\beta) = -\dfrac{\partial^2 \log L}{\partial\beta\partial\beta'}$ and $\dfrac{\phi(x_t'\beta)}{1-\Phi(x_t'\beta)} - x_t'\beta > 0$. Therefore $H_t(\beta)$ and $F_t(\beta)$ are positive definite.

Remark 5: $E\hat{y}_t = E\left[\delta_t y_t + (1-\delta_t)\left(x_t'\beta - \dfrac{\phi(x_t'\beta)}{1-\Phi(x_t'\beta)}\right)\right] = x_t'\beta$ and

$Var(\hat{y}_t) = \sigma_t^2 = \sigma^2 - Var(y_t \mid y_t < -x_t'\beta)$. In particular, $\sup_{t\geq 1}\sigma_t^2 \leq \sigma^2 < \infty$ (see James and Smith 1984).

Let $A^{1/2}$ be a unique symmetric positive definite matrix associated with a symmetric positive definite matrix A such that $\left(A^{1/2}\right)^2 = A$. We approach the problem of the distribution of $F_t^{-1/2}s_t$ first, and move to the large sample distribution of $\hat{\beta}$, MLE of β.

Assumption 1: $\|x_t\| \leq c$ for all t,

Assumption 2: $\lambda_{\min}\sum_{t=1}^{T} x_t x_t' \to \infty$.

Although Fahrmeir (1987:103) did not fully present a separate and detailed proof regarding the sufficient condition for the Tobit model with bounded regressors, he indicated that the stated sufficient conditions in Assumption 1 and Assumption 2 can be directly proved by his general results presented in his paper. Here, we merely confirm his conjecture.

Lemma 1: Under Assumption 1 and Assumption 2, the normed score function is asymptotically normal: $F_t^{-1/2} s_t \xrightarrow{d} N(0, I)$.

Proof: We use the Lindeberg-Feller Theorem for triangular arrays. Fix λ with $\lambda^T\lambda = 1$.

$$z_{t,i} = \lambda' F_t^{-1/2} x_i \left(\hat{y}_i - x_i'\beta\right)$$

we have $Ez_{t,i} = 0, \operatorname{var}\sum_{i=1}^{t} z_{t.i} = \operatorname{var}\lambda' F_t^{-1/2} s_t = 1$, i.e., $z_{t,i}$ are independent and $\sum_{i=1}^{t} z_{t,i}$ has mean 0 and variance 1. In order that $z_{t,i}$ obey the central limit theorem, it is sufficient that the Lindeberg condition (see Billingsley 1986:369) is satisfied, i.e., for any $\varepsilon > 0$,

$$\lim_{t\to\infty} \sum_{i=1}^{t} \int_{\{z_{t,i}^2 > \varepsilon^2\}} z_{t,i}^2 dP = 0 . \tag{11}$$

where P is the distribution of $z_{t,i}$. Let $\alpha_{t,i}' = \lambda' F_t^{-1/2} x_i$. By the Cauchy-Schwarz inequality, we have $z_{t,i}^2 \le \alpha_{t,i}'\alpha_{t,i}\left(\hat{y}_i - x_i'\beta\right)^2 = \alpha_{t,i}'\alpha_{t,}e_i^2$, where $e_i = \hat{y}_i - x_i'\beta$. This gives

$$\sum_{i=1}^{t} \int_{\{z_{t,i}^2 > \varepsilon^2\}} z_{t,i}^2 dP \le \sum_{i=1}^{t} \alpha_{t,i}^T \alpha_{t,i} \int_{B(t,i)} e^2 dG_x \tag{12}$$

where G_X is the distribution of e for a given x, and $B(t,\mathrm{i})$ is the set $\left\{e^2 > \dfrac{\varepsilon^2}{\alpha_{t,i}'\alpha_{t,i}}\right\}$. From Assumption 1 and 2, we have $\lambda_{\min} F_t \to \infty$. Define

$$h_c(x) = \sup \int_{\{e^2 > e_c^2\}} e^2 dG_x \tag{13}$$

Under Assumption 1, $\sum_{i=1}^{t} \alpha_{t,i}'\alpha_{t,i} \le K < \infty$ with a constant K. Note that $\sum_{i=1}^{t} \alpha_{t,i}'\alpha_{t,i} \int_{B(t,i)} e^2 dG_x$ is bounded above by $Kh_c(x)$.

$$\sum_{i=1}^{t} \int_{\{z_{t,i}^2 > e^2\}} z_{t,i}^2 dP \le K\ h_c(x) \to 0$$

because $h_c(x) \to 0 \text{ as } c \to \infty$.

Theorem 1: Under Assumption 1 and Assumption 2

(i) $\hat{\beta} \xrightarrow{p} \beta$

(ii) $F_t^{1/2}(\hat{\beta} - \beta) \xrightarrow{d} N(0, I)$

Proof: The proof is similar to the Theorem 4 in Fahrmeir and Kaufmann (1985:364).

Remark 6: A main step is the verification of

$$F_t^{-1/2}(\beta) H_t(\beta) F_t^{-1/2}(\beta) \xrightarrow{p} I. \tag{14}$$

This assertion is equivalent to $\dfrac{\lambda' H_t(\beta)\lambda}{\lambda' F_t(\beta)\lambda} \to 1$, uniformly all $\lambda \neq 0$, i.e., Condition (13) is a continuity condition on $H_t(\beta)$ as well as a convergence condition on the asymptotic relation between $F_t(\beta)$ and $H_t(\beta)$, requiring that the ratio between observed information $H_t(\beta)$ and expected information $F_t(\beta)$ converges to one.

Remark 7: The above results hold even if σ^2 is unknown. Note that

$$\lambda_{\min} F_t \geq \lambda_{\min} \begin{bmatrix} \sum_0 \left(\left\{ \dfrac{\phi(x_t'\alpha)}{1 - \Phi(x_t'\alpha)} - x_t'\alpha \right\} \phi(x_t'\alpha) + \Phi(x_t'\alpha) \right) x_t x_t' & 0 \\ 0 & \dfrac{T_1}{h^2} \end{bmatrix}.$$

Therefore $\lambda_{\min} F_t \to \infty$ if and only if $\lambda_{\min} \sum_{t=1}^{T} x_t x_t' \to \infty$ because

$$\lambda_{\min} \sum_{t=1}^{T} x_t x_t' \to \infty \; iff \; \lambda_{\min} \sum_0 x_t x_t' \to \infty.$$

.

3. Growing Regressors

However, there are situations where growing regressors are of interest, e.g., time trend models (e.g., Judge et al. 1985:791) . Fahrmeir and Kaufmann (1986:187) have given gives a sharp upper bound for admissible growth of regressors for the Probit. Note that Tobit like-likelihood in (2) is expressible as the sum of the probit log-likelihood and the truncated likelihood and the Probit MLE is asymptotically normal and $\sqrt{T}$ —consistent for the original Tobit parameter vector divided by the standard deviation of the Tobit error term, one may think that the sufficient conditions on the growing regressors in the Tobit model and in the Probit model coincide.

It is true that $\Phi(x_t'\beta, \sigma^2)$ in (2) will tend to one or zero if some regressors are growing monotonically to +∞ or -∞. However, the statement by Fahrmeir and Kaufmann (1986:189) about the Probit: "Thus for large T nearly all response y_t will fall into one category and there will be too less information to draw inference about the relevant parameters" is only one-half right for the Tobit. Specifically, if the variable x_{it} is growing monotonically with t, and if its associated coefficient β_1 is negative, then for large T nearly all responses y_t will be zero, so that additional observations will be indeed add too little information for asymptotic theory to work. That is the case that looks like Probit. But if the coefficient β_1 is positive, then nearly all $y_t > 0$ for large T, meaning new observations will add as much information as they would in the classical linear model. Thus, the upper bound on the growth rates for regressors in the censored normal model, is true as stated in (5) and (6), but only for half of the parameter space to assure that enough information is available and the asymptotic theory works.

The following theorem gives a sharp upper bound for admissible growth of regressors for half of the parameter space.

Theorem 2: If $\|x_t\|^2 = o(\log t)$ and $\lambda_{\min}\sum_{t=1}^{T} x_t x_t' \geq cT^{\alpha}$, for some $\alpha > 0$ and $c > 0$. Then

$\lambda_{\min} F_t \to \infty$.

Proof: The information matrix $F_t(\beta)$ can also be written as $F_t(\beta) = \sum_{t=1}^{T} x_t x_t' a_t^2$, with

$$a_t^2 = \left(\left\{ \frac{\phi(x_t'\beta)}{(1-\Phi(x_t'\beta))} - x_t'\beta \right\} \phi(x_t'\beta) + \Phi(x_t'\beta) \right) \tag{15}$$

From Magnus and Neudecker (1988:204), we have

$$\lambda_{\min} F_t(\beta) = \lambda_{\min} \sum_{t=1}^{T} x_t x_t' a_t^2 \geq \lambda_{\min} \sum_{t=1}^{T} \left[\Phi(x_t'\beta) \right] x_t x_t'. \tag{16}$$

We also note that

$$\Phi(x_t'\beta) \geq c \exp(-\{x_t'\beta\}^2) \tag{17}$$

for some c > 0. The admissible growth rate of x_t, $\|x_t\|^2 = o(\log t)$, in Theorem 2 is equivalent to

$$\exp\left(-\|x_t\|^2 \|\beta\|^2\right) \geq t^{-\delta} \text{ for all } \beta, \delta > 0, t \geq t_1(\beta, \delta). \tag{18}$$

In combination with (16) and (17),

$$\lambda_{\min} F_t(\beta) \geq \lambda_{\min} \sum_{t=1}^{T} x_t x_t' t^{-\delta}.$$

Since $t^{-\delta} \geq T^{-\delta}$ for $t \leq T$, $\delta > 0$. Under $\lambda_{\min}\sum_{t=1}^{T} x_t x_t' \geq cT^{\alpha}$, therefore, we have

$$\lambda_{\min} F_t(\beta) \geq cT^{\alpha-\delta}, \; T \geq T_1$$

With $\delta = \alpha/2$, this implies that $\lambda_{\min} F_t \to \infty$.

Remark 8: If $\left|x_t^T \beta\right|^2 \geq c \log t$, c > 1, then $\lambda_{\min} F_t$ converges.

Example: Let $x_t = (1,\ w_t)'$ with $w_t^2 = (\log t)^a$ for the Tobit Model in (1). $\|x_t\|^2 = o(\log t)$ holds if $0 < \alpha < 1$. If $\alpha \geq 1$, then $\lambda_{\min} F_t$ converges for some β.

Remark 9: It implies that $x_t = (1,\ t)'$ in Judge et al. (1985:791) for the Tobit is not admissible.

Remark 10: It also implies that when $x_t = (1,\ w_t)$ has a unit root with the drift, i.e., $w_t = \mu + w_{t-1} + \omega_t$ for the Tobit is not admissible. Since w_t can be written as $w_t = w_0 + ut + \sum_{i=1}^{t} \omega_i$.

4. Summary

Regression models for censored data have found numerous applications. Statistical analysis of these models relies heavily on large sample theory, i.e., asymptotic properties of the MLE. However, previously published conditions assuring these properties may be too strong. Consistency and Asymptotic normality of the MLE are shown under weak and easily verifiable requirements. This paper gives a sharp upper bound on the admissible growth of regressors.

References

Amemiya, T. 1973. "Regression Analysis when the Dependent Variable is Truncated Normal," *Econometrica*, 41: 997-1016.

Amemiya, T. 1985. *Advanced Econometrics*. Cambridge, MA: Harvard University Press.

Billingsley, P. 1986. *Probability and Measure*. New York: John Wiley.

Drygas, R.A. 1976. "Weak and Strong Consistency of the Least Squares Estimators in Regression Models," *Zeitschrift Fur Washrscheinlichkeitstheorie und Verwandte Gebiete*, 34: 119-127.

Fahrmeir, L. and H. Kaufmann. 1985. "Consistency and Asymptotic Normality of the Maximum Likelihood Estimator in Generalized Linear Model," *Annals of Statistics*, 13: 342-368; Correction in *Annals of Statistics*, 14: 1643.

Fahrmeir, L. and H. Kaufmann. 1986. "Asymptotic Inference in Discrete Response Models," *Statistische Hefte*, 27: 179-205.

Fahrmeir, L. 1987. "Asymptotic Likelihood Inference for Nonhomogeneous Observations," *Statistische Hefte*, 28: 81-116.

Gourieroux, C. and A. Monfort. 1981. "Asymptotic Properties of the Maximum Likelihood Estimator in Dichotomous Logit Models," *Journal of Econometrics*, 17: 83-97.

James, I.R. and P.J. Smith. 1984. "Consistency Results for Linear Regression with Censored Data," *Annals of Statistics*, 12: 590-600.

Judge, G., W.E. Griffiths, R.C. Hill, H. Lutkepohl, and T.-C. Lee. 1985. *The Theory and Practice and Econometrics*. New York: John Wiley.

Lai, T.L., H. Robbins, and C.Z. Wei. 1979. "Strong Consistency of Least Squares Estimates in Multiple Regression II," *Journal of Multivariate Analysis*, 9: 343-361.

Magnus, J.R. and H. Neudecker. 1988. *Matrix Differential Calculus with Applications in Statistics and Econometrics*. New York: John Wiley.

Olsen, R.J. 1978. "Note on the Uniqueness of the Maximum Likelihood Estimator for the Tobit Model," *Econometrica*, 46: 1211-1215.