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The time at which a Lévy process creeps

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Abstract

We show that if a Lévy process creeps then, as a function of u, the renewal function $V(t,u)$ of the bivariate ascending ladder process (L^{-1},H) is absolutely continuous on $[0,\infty)$ and left differentiable on $(0,\infty)$, and the left derivative at u is proportional to the (improper) distribution function of the time at which the process creeps over level u, where the constant of proportionality is d_H^{-1}, the reciprocal of the (positive) drift of H. This yields the (missing) term due to creeping in the recent quintuple law of Doney and Kyprianou (2006). As an application, we derive a Laplace transform identity which generalises the second factorization identity. We also relate Doney and Kyprianou's extension of Vigon's équation amicale inversée to creeping. Some results concerning the ladder process of X, including the second factorization identity, continue to hold for a general bivariate subordinator, and are given in this generality.

Keywords: Lévy process, quintuple law, creeping by time t, second factorization identity, bivariate subordinator
AMS 2010 Subject Classifications: 60G51; 60K05; 60G50.

1 Introduction

Let $X = \{X_t : t \geq 0\}$, $X_0 = 0$, be a real-valued Lévy process with characteristic triplet $(\gamma, \sigma^2, \Pi_X)$, thus the characteristic function of X is given by the Lévy-Khintchine representation, $Ee^{i\theta X_t} = e^{t\Psi_X(\theta)}$, where

$$\Psi_X(\theta) = \mathrm{i}\theta\gamma - \sigma^2\theta^2/2 + \int_{\mathbb{R}} (e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x|<1\}})\Pi_X(\mathrm{d}x), \text{ for } \theta \in \mathbb{R},\ t \geq 0. \tag{1.1}$$

X is said to *creep across a level* $u > 0$ if $P(\tau_u < \infty, X_{\tau_u} = u) > 0$ where

$$\tau_u = \inf\{t \geq 0 : X_t > u\}.$$

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Our initial interest is in the *time at which X creeps*. Thus we introduce the (improper) distribution function

$$p(t,u) = P(\tau_u \le t, X_{\tau_u} = u),\ u > 0, t \ge 0. \tag{1.2}$$

We prove certain regularity properties of $p(t,u)$ which allow us to relate it to the renewal function of the bivariate ascending ladder process of X. This yields the missing term, due to creeping, in Doney and Kyprianou's (2006) *quintuple law*. Using the quintuple law, we derive a Laplace transform identity which generalises the *second factorization identity* due to Percheskii and Rogozin (1969). We also relate creeping to Doney and Kyprianou's extension of the équation amicale inversée of Vigon (2002). Some of these results extend from the bivariate ladder process to general bivariate subordinators, and we develop several of the results in this setting. In particular, it appears to have gone previously unnoticed, that the second factorization identity is a special case of a general transform result for bivariate subordinators. The results in the fluctuation setting are stated in Section 3, with their proofs given in Sections 5. The general bivariate subordinator case is developed in Section 4.

By a compound Poisson process we will mean a Lévy process with finite Lévy measure, no Brownian component and zero drift. The indicator of an event A will be denoted by $\mathbf{1}_A$, or sometimes by $\mathbf{1}(A)$, and we adopt the convention that the inf of the empty set is $+\infty$.

2 Fluctuation Setup

We need some notation, which is very standard in the area. Let $(L_s)_{s\ge 0}$ be the local time at the maximum, and $(L_s^{-1}, H_s)_{s\ge 0}$ the weakly[1] ascending bivariate ladder process of X. Bertoin (1996), Chapter VI, and Kyprianou (2006), Chapter 6, give detailed discussions of these processes and their properties; see also Doney (2005). When $X_t \to -\infty$ a.s., (L^{-1}, H) is defective and may be obtained from a nondefective bivariate subordinator $(\mathcal{L}^{-1}, \mathcal{H})$ by exponential killing at some rate $q > 0$, say. Thus

$$(L_s^{-1}, H_s) = \begin{cases} (\mathcal{L}_s^{-1}, \mathcal{H}_s) & \text{if } s < e(q), \\ (\infty, \infty) & \text{if } s \ge e(q), \end{cases} \tag{2.1}$$

where $e(q)$ is independent of $(\mathcal{L}^{-1}, \mathcal{H})$ and has exponential distribution with parameter q. In the case that (L^{-1}, H) is nondefective there is no need to introduce exponential killing and we set $(L^{-1}, H) = (\mathcal{L}^{-1}, \mathcal{H})$.

We denote the bivariate Lévy measure of $(\mathcal{L}^{-1}, \mathcal{H})$ by $\Pi_{L^{-1},H}(\cdot,\cdot)$, and its marginals by $\Pi_{L^{-1}}(\cdot)$ and $\Pi_H(\cdot)$. The Laplace exponent $\kappa(a,b)$ of (L^{-1}, H) is given by

$$e^{-\kappa(a,b)} = E(e^{-aL_1^{-1} - bH_1}; 1 < L_\infty) = e^{-q} E e^{-a\mathcal{L}_1^{-1} - b\mathcal{H}_1} \tag{2.2}$$

[1]The distinction between weak and strict only makes a difference when X is compound Poisson.

for values of $a, b \in \mathbb{R}$ for which the expectation is finite. It may be written

$$\kappa(a,b) \;=\; q + \mathrm{d}_{L^{-1}} a + \mathrm{d}_H b + \int_{t\geq 0}\int_{x\geq 0} \left(1 - e^{-at-bx}\right) \Pi_{L^{-1},H}(\mathrm{d}t, \mathrm{d}x), \tag{2.3}$$

where $\mathrm{d}_{L^{-1}} \geq 0$ and $\mathrm{d}_H \geq 0$ are drift constants. The bivariate renewal function of (L^{-1}, H) is[2]

$$V(t,x) = \int_0^\infty P(L_s^{-1} \leq t, H_s \leq x)\mathrm{d}s = \int_0^\infty e^{-qs} P(\mathcal{L}_s^{-1} \leq t, \mathcal{H}_s \leq x)\mathrm{d}s. \tag{2.4}$$

It has Laplace transform

$$\int_{t\geq 0}\int_{x\geq 0} e^{-at-bx} V(\mathrm{d}t, \mathrm{d}x) = \frac{1}{\kappa(a,b)} \tag{2.5}$$

for all a, b such that $\kappa(a,b) > 0$. The positivity condition on κ clearly holds when $a, b \geq 0$ and either $a \vee b > 0$ or $q > 0$.

Let $\widehat{X}_t = -X_t$, $t \geq 0$ denote the dual process, and $(\widehat{L}^{-1}, \widehat{H})$ the corresponding *strictly* ascending bivariate ladder processes of $\widehat{X}$. This is the same as the weakly ascending process if $\widehat{X}$ is not compound Poisson. The definition of $(\widehat{L}^{-1}, \widehat{H})$ when $\widehat{X}$ is compound Poisson is as the limit of the ascending bivariate ladder process of $\widehat{X}_t - \varepsilon t$ as $\varepsilon \downarrow 0$. All quantities relating to $\widehat{X}$ will be denoted in the obvious way; for example $\Pi_{\widehat{L}^{-1},\widehat{H}}(\cdot,\cdot)$, $\widehat{\kappa}(\cdot,\cdot)$ and $\widehat{V}(\cdot,\cdot)$. We choose the normalisation of the local times L and $\widehat{L}$ so that the Weiner-Hopf factorisation takes the form

$$\kappa(a,0)\widehat{\kappa}(a,0) = a, \quad a \geq 0. \tag{2.6}$$

This would not be possible in the compound Poisson case if $(\widehat{L}^{-1}, \widehat{H})$ were the weak bivariate ladder process; see Section 6.4 of Kyprianou (2006).

3 Creeping Time

It is well known that X creeps across some $u > 0$ iff X creeps across all $u > 0$, in which case we say that X creeps. A necessary and sufficient condition for creeping is that $\mathrm{d}_H > 0$; see Theorem VI.19 of Bertoin (1996). Our first result describes the (improper) distribution function of the time at which X creeps across level u.

Theorem 3.1 (Creeping Time)
(i) The following are equivalent:

$$p(t,u) > 0 \textit{ for some } t > 0,\ u > 0; \tag{3.1}$$

[2]Throughout, we will write $\Pi_{L^{-1},H}(\cdot,\cdot)$ and $V(\cdot,\cdot)$ with the time variable in first position, followed by the space variable. This is at variance with some established literature, but seems desirable for consistency.

$$p(t,u) > 0 \textit{ for all } u > 0 \textit{ and all } t \textit{ sufficiently large (depending on } u\textit{);} \tag{3.2}$$

$$p(t,u) > 0 \textit{ for all } t > 0 \textit{ and all } u \textit{ sufficiently small (depending on } t\textit{);} \tag{3.3}$$

$$\mathrm{d}_H > 0. \tag{3.4}$$

(ii) If $\mathrm{d}_H > 0$*, then for every* $t \geq 0$*,* $V(t,0) = 0$*,* $V(t,\cdot)$ *is absolutely continuous on* $[0,\infty)$ *with a left continuous left derivative on* $(0,\infty)$*, and for each* $u \in (0,\infty)$ *satisfies*

$$p(t,u) = \mathrm{d}_H \frac{\partial_-}{\partial_- u} V(t,u) \tag{3.5}$$

where $\partial_-/\partial_- u$ *denotes the left partial derivative in* u*.*
(iii) If $\mathrm{d}_H > 0$ *and* X *is not compound Poisson with positive drift, then* $V(t,\cdot)$ *is differentiable and* $p(t,\cdot)$ *is continuous on* $(0,\infty)$ *for each* $t \geq 0$*, and* $p(\cdot,u)$ *is continuous on* $[0,\infty)$ *for each* $u > 0$*.*

Remark 3.1 (i) Unlike the creeping case, it is possible that X creeps over some u but not all u by a fixed time $t > 0$. This is illustrated in Examples 5.1 and 5.2. Nevertheless, from Theorem 3.1, X creeps over some $u > 0$ by some time $t > 0$ iff X creeps, and we obtain the generalisation (3.5) of Kesten and Neveu's formula for the probability of eventually creeping over u; see pp. 119–121 of Kesten (1969).

(ii) It follows immediately from Theorem 3.1 that

$$\mathrm{d}_H V(\mathrm{d}t, \mathrm{d}u) = P(\tau_u \in \mathrm{d}t, X_{\tau_u} = u)\mathrm{d}u. \tag{3.6}$$

This formula has already been noted by Savov and Winkel (2010), p.8, and attributed to Andreas Kyprianou. Conversely, from Savov and Winkel's observation (3.6), it follows that $\mathrm{d}_H V(t, \mathrm{d}u)$ has a density given by $p(t,u)$ for a.e. u. This however gives no information about $p(t,u)$ for a given level u. One of the main points of Theorem 3.1 is that $p(t,u)$ *is the left derivative of* $\mathrm{d}_H V(t,u)$ *for every* $u > 0$, $t \geq 0$. This is particularly relevant in the quintuple law below.

(iii) In the case that X is a subordinator which creeps, the Laplace transform of the time at which it creeps over u is given by

$$E(e^{-\alpha\tau_u}; X_{\tau_u} = u) = \mathrm{d}_X v^\alpha(u) \tag{3.7}$$

where v^α is the bounded continuous density of the resolvent kernel

$$V^\alpha(\mathrm{d}u) = \int_0^\infty e^{-\alpha t} P(X_t \in \mathrm{d}u)\mathrm{d}t;$$

see page 80 of Bertoin (1996). This in principle gives the distribution of the time at which X creeps over u. Indeed

$$v^\alpha(u) = \frac{\mathrm{d}}{\mathrm{d}u}\int_0^\infty e^{-\alpha t} P(X_t \leq u)\mathrm{d}t \;\; = \frac{\mathrm{d}}{\mathrm{d}u}\int_0^\infty \alpha e^{-\alpha t}\mathrm{d}t \int_0^t P(X_s \leq u)\mathrm{d}s.$$

Hence if the derivative could be moved inside the integral, from (3.7) we would obtain

$$P(\tau_u \le t, X_{\tau_u} = u) = \mathrm{d}_X \frac{\partial}{\partial u} \int_0^t P(X_s \le u)\mathrm{d}s. \tag{3.8}$$

Since

$$\mathrm{d}_H V(t,u) = \mathrm{d}_X \int_0^t P(X_s \le u)\mathrm{d}s$$

when X is a subordinator, it follows from (3.5) that

$$P(\tau_u \le t, X_{\tau_u} = u) = \mathrm{d}_X \frac{\partial_-}{\partial_- u} \int_0^t P(X_s \le u)\mathrm{d}s.$$

Thus (3.8) is correct provided $\partial/\partial u$ is replaced by $\partial_-/\partial_- u$. Conversely one can use (3.5) to give an alternative proof of (3.7).

(iv) Theorem 3.1 is concerned with regularity of $V(t,\cdot)$. Some information about regularity of $V(\cdot,u)$ may be gleaned from Theorem 5 of Alili and Chaumont (2001), from which it follows that $V(\cdot,u)$ is absolutely continuous for each $u>0$ provided 0 is regular for both $(-\infty,0)$ and $(0,\infty)$, for X.

The quintuple law is a fluctuation identity, due to Doney and Kyprianou (2006), describing the joint distribution of five random variables associated with the first passage of X over a fixed level $u>0$ when $X_{\tau_u} > u$. Using Theorem 3.1, we are able to account for the contribution due to creeping, that is the term when $X_{\tau_u} = u$. To give the result set

$$\overline{X}_t = \sup_{0\le s\le t} X_s \quad \text{and} \quad G_t = \sup\{0 \le s \le t : X_s = \overline{X}_s\}.$$

The quintuple law concerns the following quantities:

- First Passage Time Above Level u: $\tau_u = \inf\{t \ge 0 : X_t > u\}$;
- Time of Last Maximum Before Passage: $G_{\tau_u -}$;
- Overshoot Above Level u: $X_{\tau_u} - u$;
- Undershoot of Level u: $u - X_{\tau_u -}$;
- Undershoot of the Last Maximum Before Passage: $u - \overline{X}_{\tau_u -}$.

Theorem 3.2 (Quintuple Law with Creeping) *Fix $u>0$; then for $x \ge 0$, $v \ge 0$, $0 \le y \le u \wedge v$, $s \ge 0$ and $t \ge 0$*

$$\begin{aligned} &P\left(X_{\tau_u} - u \in \mathrm{d}x, u - X_{\tau_u -} \in \mathrm{d}v, u - \overline{X}_{\tau_u -} \in \mathrm{d}y, \tau_u - G_{\tau_u -} \in \mathrm{d}s, G_{\tau_u -} \in \mathrm{d}t\right) \\ &\quad = \mathbf{1}_{\{x>0\}} |V(\mathrm{d}t, u - \mathrm{d}y)| \widehat{V}(\mathrm{d}s, \mathrm{d}v - y) \Pi_X(\mathrm{d}x + v) + \mathrm{d}_H \frac{\partial_-}{\partial_- u} V(\mathrm{d}t, u) \delta_0(\mathrm{d}s, \mathrm{d}x, \mathrm{d}v, \mathrm{d}y), \end{aligned} \tag{3.9}$$

where δ_0 is a point mass at the origin, and with the convention that the term containing the differential $\partial_- V(\mathrm{d}t,u)/\partial_- u$ is absent when $\mathrm{d}_H = 0$ (in which case $\partial_- V(t,u)/\partial_- u$ need not be defined).

The contribution to (3.9) for $x > 0$ is Doney and Kyprianou's quintuple law. Theorem 3.2 then follows easily for a.e. u from Savov and Winkel's observation (3.6), but this is clearly unsatisfactory, since it says nothing about any given u. To get the result for every u, (3.5) is needed. As a simple consequence of Theorem 3.2, we record the joint distribution of the first passage time and overshoot of a level $u > 0$;

Corollary 3.1 *Fix $u > 0$. Then for $x, r \geq 0$*

$$P(X_{\tau_u} - u \in \mathrm{d}x, \tau_u \in \mathrm{d}r) = I(x > 0) \int_{0 \leq s \leq r} \int_{0 \leq y \leq u} |V(\mathrm{d}s, u - \mathrm{d}y)| \Pi_{L^{-1},H}(\mathrm{d}r - s, y + \mathrm{d}x) + \mathrm{d}_H \frac{\partial_-}{\partial_- u} V(\mathrm{d}r, u) \delta_{\{0\}}(\mathrm{d}x).$$

In particular the distribution of the first passage time is

$$P(\tau_u \in \mathrm{d}r) = \int_{0 \leq s \leq r} \int_{0 \leq y \leq u} |V(\mathrm{d}s, u - \mathrm{d}y)| \Pi_{L^{-1},H}(\mathrm{d}r - s, (y, \infty)) + \mathrm{d}_H \frac{\partial_-}{\partial_- u} V(\mathrm{d}r, u).$$

Using the quintuple law, Doney and Kyprianou (2006) (Corollary 6) obtain the following useful extension of the équation amicale inversée of Vigon (2002); for $s \geq 0$ and $x > 0$,

$$\Pi_{L^{-1},H}(\mathrm{d}s, \mathrm{d}x) = \int_{v \geq 0} \widehat{V}(\mathrm{d}s, \mathrm{d}v) \Pi_X(\mathrm{d}x + v). \tag{3.10}$$

They state this result for $s > 0, x > 0$, but their proof works equally well when $s = 0$. Observe that (3.10) gives no information about $\Pi_{L^{-1},H}$ on $\{(s, 0) : s > 0\}$. When X is not compound Poisson, $\Pi_{L^{-1},H}(\mathrm{d}s, \{0\})$ is seen to relate to creeping as the following result shows:

Theorem 3.3 *Assume X is not compound Poisson. Then X creeps iff $\Pi_{L^{-1},H}(\mathrm{d}s, \{0\})$ is not the zero measure.*

Despite the connection with creeping, it is easily seen that the jumps of (L^{-1}, H) for which $\Delta H = 0$ do not occur when X creeps over a fixed level; see (5.3) in Section 5. As a consequence of Theorem 3.3 we are able to characterise when (3.10) holds for all $s \geq 0$ and $x \geq 0$:

Theorem 3.4 (3.10) *holds for all $s \geq 0$, $x \geq 0$ iff X does not creep.*

When X is compound Poisson it does not creep, and we can deduce from Theorem 3.4 that

$$\Pi_{L^{-1},H}(\mathrm{d}s, \{0\}) = \int_{v \geq 0} \widehat{V}(\mathrm{d}s, \mathrm{d}v) \Pi_X(\{v\}), \ s \geq 0. \tag{3.11}$$

If Π_X is diffuse then $\Pi_{L^{-1},H}(\mathrm{d}s, \{0\})$ reduces to the zero measure, but in general it may have positive mass. Thus Theorem 3.3 cannot be extended to the compound Poisson case.

The next result is an application of Theorem 3.2 to computing a quadruple Laplace transform. The finiteness conditions on κ, below, clearly hold when $\mu, \rho, \lambda, \nu, \theta \geq 0$, and in that case $\kappa(\nu, \mu) > 0$ except when $\nu = \mu = 0$ and $q = 0$ in (2.3). More generally the conditions allow for distributions with exponential moments, which can arise quite frequently in applications.

Theorem 3.5 (A Laplace Transform Identity)
Fix $\mu, \rho, \lambda, \nu, \theta$ so that $\kappa(\theta, \mu+\lambda), \kappa(\theta, \rho)$ are finite and $\kappa(\nu, \mu) > 0$.
(i) If $\lambda \neq \rho - \mu$ then

$$\int_{u \geq 0} e^{-\mu u} E\left(e^{-\rho(X_{\tau_u}-u)-\lambda(u-\overline{X}_{\tau_u-})-\nu G_{\tau_u-}-\theta(\tau_u-G_{\tau_u-})}; \tau_u < \infty\right) \mathrm{d}u = \frac{\kappa(\theta, \mu+\lambda) - \kappa(\theta, \rho)}{(\mu+\lambda-\rho)\kappa(\nu, \mu)}. \tag{3.12}$$

(ii) If $\lambda = \rho - \mu$ then

$$\int_{u \geq 0} E\left(e^{-\rho\Delta\overline{X}_{\tau_u}-\mu\overline{X}_{\tau_u-}-\nu G_{\tau_u-}-\theta(\tau_u-G_{\tau_u-})}; \tau_u < \infty\right) \mathrm{d}u = \frac{1}{\kappa(\nu, \mu)} \frac{\partial_+ \kappa(\theta, \rho)}{\partial_+ \rho} \tag{3.13}$$

provided the right derivative exists.

The right derivative in (3.13) exists and equals the derivative if $\kappa(\theta, \rho - \varepsilon) < \infty$ for some $\varepsilon > 0$. When $\lambda = 0, \theta = \nu \geq 0, \rho > 0, \mu > 0$, (3.12) reduces to the *second factorization identity*, (3.2) of Percheskii and Rogozin (1969). [Alili and Kyprianou (2005) give a short and elegant proof of the second factorization identity using the strong Markov property.] Theorem 3.5 can be used in the computation of certain exponential Gerber-Shiu functionals from insurance risk, see Griffin and Maller (2010a). Another application, to stability of the exit time, can be found in Griffin and Maller (2010b).

It is natural to ask if there is a quintuple Laplace transform identity, analogous to the quintuple law. It is straightforward to follow calculations similar to those in the proof of Theorem 3.5 and derive a corresponding expression to (3.12), but because the component X_{τ_u-} cannot be expressed in terms of the ladder process, the resulting expression cannot be expressed simply in terms of the kappa functions.

4 Bivariate Subordinators

(L^{-1}, H) and $(\widehat{L}^{-1}, \widehat{H})$ are, possibly killed, bivariate subordinators, and some of our results require only this property. In this section we prove several theorems in this generality. These will then be applied in Section 5 to the fluctuation variables.

In the fluctuation setting, let

$$T_u = T_u^{\mathcal{H}} = \inf\{s \geq 0 : \mathcal{H}_s > u\}, \ u \geq 0. \tag{4.1}$$

Using $H_s = X_{L_s^{-1}}$ on $\{L_s^{-1} < \infty\}$, $s \geq 0$, and recalling the exponential killing described in (2.1), we have

$$X_{\tau_u} = \mathcal{H}_{T_u}, \ \overline{X}_{\tau_u-} = \mathcal{H}_{T_u-}, \ \tau_u = \mathcal{L}_{T_u}^{-1}, \text{ and } G_{\tau_u-} = \mathcal{L}_{T_u-}^{-1} \text{ on } \{T_u < e(q)\}. \tag{4.2}$$

Thus, via (1.2),

$$p(t,u) = P(\mathcal{L}_{T_u}^{-1} \leq t, \mathcal{H}_{T_u} = u, T_u < e(q)). \tag{4.3}$$

This suggests the following setup. Let (Z,Y) be any two dimensional subordinator obtained from a true subordinator $(\mathcal{Z},\mathcal{Y})$ by exponential killing at rate $q \geq 0$, say. Corresponding to (2.3), (Z,Y) has Laplace exponent $\kappa_{Z,Y}(a,b) = q - \log Ee^{-a\mathcal{Z}_1 - b\mathcal{Y}_1}$ where

$$\kappa_{Z,Y}(a,b) = q + \mathrm{d}_Z a + \mathrm{d}_Y b + \int_{t\geq 0}\int_{x\geq 0} \left(1 - e^{-at-bx}\right) \Pi_{Z,Y}(\mathrm{d}t, \mathrm{d}x), \tag{4.4}$$

for values of $a, b \in \mathbb{R}$ for which the expression is finite. Analogous to the fluctuation variables, we define

$$T_u^{\mathcal{Y}} = \inf\{s \geq 0 : \mathcal{Y}_s > u\},\ u \geq 0,$$

and

$$p_{Z,Y}(t,u) = P(\mathcal{Z}_{T_u^{\mathcal{Y}}} \leq t, \mathcal{Y}_{T_u^{\mathcal{Y}}} = u, T_u^{\mathcal{Y}} < e(q)), \tag{4.5}$$

where $e(q)$ is an independent exponential random variable with parameter q. Also set

$$V_{Z,Y}(t,u) = \int_0^\infty e^{-qs} P(\mathcal{Z}_s \leq t, \mathcal{Y}_s \leq u)\mathrm{d}s.$$

So $V_{Z,Y}(\cdot,\cdot)$ has Laplace transform

$$\int_{t\geq 0}\int_{x\geq 0} e^{-at-bx} V_{Z,Y}(\mathrm{d}t, \mathrm{d}x) = \frac{1}{\kappa_{Z,Y}(a,b)} \quad \text{if } \kappa_{Z,Y}(a,b) > 0. \tag{4.6}$$

We begin by investigating aspects of the regularity of $p_{Z,Y}$ defined in (4.5).

Lemma 4.1 *The function $p_{Z,Y}(\cdot,\cdot)$ has the following properties:*

(a) $p_{Z,Y}(\cdot,u)$ is right continuous and non-decreasing on $[0,\infty)$ for every $u > 0$;

(b) $p_{Z,Y}(t,\cdot)$ is left continuous on $(0,\infty)$ for every $t \geq 0$;

(c) If $p_{Z,Y}(\cdot,u)$ is continuous on $(0,\infty)$ for every $u > 0$, then $p_{Z,Y}(t,\cdot)$ is continuous on $(0,\infty)$ for every $t > 0$.

Proof of Lemma 4.1: First observe that the results trivially hold if $\mathrm{d}_Y = 0$, since then $\mathcal{Y}$ does not creep and so $p_{Z,Y}(t,u) \leq P(\mathcal{Y}_{T_u^{\mathcal{Y}}} = u) = 0$ for all $t \geq 0, u > 0$. Thus for the remainder of the proof we assume $\mathrm{d}_Y > 0$.

Part (a) follows immediately from the definition of $p_{Z,Y}$. To prove Parts (b) and (c) we will use the following two equations which are simple consequences of the strong Markov property (cf. Andrew (2006)): for any $x > 0, y > 0, r \geq 0, s \geq 0$, we have

$$p_{Z,Y}(r+s, x+y) \geq p_{Z,Y}(r,x)p_{Z,Y}(s,y) \tag{4.7}$$

and

$$p_{Z,Y}(s, x+y) \leq p_{Z,Y}(r,x)p_{Z,Y}(s,y) + 1 - p_{Z,Y}(r,x). \tag{4.8}$$

By Theorem III.5 of Bertoin (1996), which applies to nondefective subordinators with $\mathrm{d}_Y > 0$, we have $\lim_{\varepsilon\downarrow 0} P(\mathcal{Y}_{T_\varepsilon^{\mathcal{Y}}} = \varepsilon) = 1$. Since $\mathrm{d}_Y > 0$, $\mathcal{Y}$ is strictly increasing, and so $T_\varepsilon^{\mathcal{Y}} \downarrow 0$ a.s. as $\varepsilon \downarrow 0$. Thus for every $\delta > 0$

$$\lim_{\varepsilon\downarrow 0} p_{Z,Y}(\delta,\varepsilon) = \lim_{\varepsilon\downarrow 0} P(\mathcal{Z}_{T_\varepsilon^{\mathcal{Y}}} \le \delta, \mathcal{Y}_{T_\varepsilon^{\mathcal{Y}}} = \varepsilon, T_\varepsilon^{\mathcal{Y}} < e(q)) = 1. \tag{4.9}$$

Now fix $u > 0$ and $t \ge 0$. Then for any $0 < \varepsilon < u$ and $\delta > 0$ we have by (4.7) and (4.8)

$$p_{Z,Y}(t,u) - 1 + p_{Z,Y}(\delta,\varepsilon) \le p_{Z,Y}(\delta,\varepsilon)p_{Z,Y}(t,u-\varepsilon) \le p_{Z,Y}(t+\delta,u).$$

Letting $\varepsilon \downarrow 0$ then $\delta \downarrow 0$, and using Part (a) and (4.9), proves Part (b). Similarly if in addition, $t > 0$ and $\delta < t$, then

$$p_{Z,Y}(\delta,\varepsilon)p_{Z,Y}(t-\delta,u) \le p_{Z,Y}(t,u+\varepsilon) \le p_{Z,Y}(\delta,\varepsilon)p_{Z,Y}(t,u) + 1 - p_{Z,Y}(\delta,\varepsilon).$$

Letting $\varepsilon \downarrow 0$ then $\delta \downarrow 0$, we conclude that

$$p_{Z,Y}(t-,u) \le \liminf_{\varepsilon\downarrow 0} p_{Z,Y}(t,u+\varepsilon) \le \limsup_{\varepsilon\downarrow 0} p_{Z,Y}(t,u+\varepsilon) \le p_{Z,Y}(t,u).$$

Thus if $p_{Z,Y}(\cdot,u)$ is continuous on $(0,\infty)$ for every $u > 0$, then $p_{Z,Y}(t,\cdot)$ is right continuous on $(0,\infty)$ for every $t > 0$. Combining this with Part (b) proves $p_{Z,Y}(t,\cdot)$ is continuous on $(0,\infty)$ for every $t > 0$. □

Lemma 4.2 *For any $u \ge 0$ and $t \ge 0$,*

$$\int_0^u p_{Z,Y}(t,v)\mathrm{d}v = \mathrm{d}_Y V_{Z,Y}(t,u). \tag{4.10}$$

Proof of Lemma 4.2: If $d_Y = 0$ then $\mathcal{Y}$ does not creep, so both sides of (4.10) are zero. Thus we may assume $d_Y > 0$. First observe that for any s, $\{v : \mathcal{Y}_{T_v^{\mathcal{Y}}} = v, T_v^{\mathcal{Y}} = s\}$ is at most a singleton, and so

$$\int_0^\infty \mathbf{1}\{\mathcal{Y}_{T_v^{\mathcal{Y}}} = v, T_v^{\mathcal{Y}} = s\}\mathrm{d}v = 0. \tag{4.11}$$

Next, if $T_t^{\mathcal{Z}} = \inf\{s : \mathcal{Z}_s > t\}$, then

$$\{s < T_t^{\mathcal{Z}}\} \subset \{\mathcal{Z}_s \le t\} \subset \{s \le T_t^{\mathcal{Z}}\}. \tag{4.12}$$

Thus, using (4.11) and (4.12),

$$\begin{aligned}\int_0^u p_{Z,Y}(t,v)\mathrm{d}v &= \int_0^u P(\mathcal{Z}_{T_v^{\mathcal{Y}}} \le t, \mathcal{Y}_{T_v^{\mathcal{Y}}} = v, T_v^{\mathcal{Y}} < e(q))\mathrm{d}v \\ &= \int_0^u P(T_v^{\mathcal{Y}} \le T_t^{\mathcal{Z}}, \mathcal{Y}_{T_v^{\mathcal{Y}}} = v, T_v^{\mathcal{Y}} < e(q))\mathrm{d}v \\ &= \int_0^u P(\mathcal{Y}_{T_v^{\mathcal{Y}}} = v, T_v^{\mathcal{Y}} \le T_t^{\mathcal{Z}} \wedge e(q))\mathrm{d}v.\end{aligned}$$

Now since $d_Y > 0$, $\mathcal{Y}$ is strictly increasing. Thus if $\mathcal{Y}$ hits v then it does so at time $T_v^{\mathcal{Y}}$. Hence

$$\int_0^u \mathbf{1}\{\mathcal{Y}_{T_v^{\mathcal{Y}}} = v, T_v^{\mathcal{Y}} \le T_t^{\mathcal{Z}} \wedge e(q)\}\mathrm{d}v = \mathcal{Y}_{T_u^{\mathcal{Y}} \wedge T_t^{\mathcal{Z}} \wedge e(q)} - \sum_{s \le T_u^{\mathcal{Y}} \wedge T_t^{\mathcal{Z}} \wedge e(q)} \Delta\mathcal{Y}_s$$

as each quantity represents the Lebesgue measure of the set of points in $[0, u]$ hit by $\mathcal{Y}$ by time $T_t^{\mathcal{Z}} \wedge e(q)$. Since $\mathcal{Y}_r = \mathrm{d}_Y r + \sum_{s \le r} \Delta\mathcal{Y}_s$, this gives

$$\int_0^u p_{Z,Y}(t, v)\mathrm{d}v = \mathrm{d}_Y E(T_u^{\mathcal{Y}} \wedge T_t^{\mathcal{Z}} \wedge e(q)). \tag{4.13}$$

But by (4.12) (which also applies to $\mathcal{Y}$ and $T_u^{\mathcal{Y}}$)

$$\begin{aligned}\int_0^\infty \mathbf{1}\{\mathcal{Z}_s \le t, \mathcal{Y}_s \le u, s < e(q)\}\mathrm{d}s &\le \int_0^\infty \mathbf{1}\{s \le T_t^{\mathcal{Z}}, s \le T_u^{\mathcal{Y}}, s < e(q)\}\mathrm{d}s \\ &= \int_0^\infty \mathbf{1}\{s < T_t^{\mathcal{Z}}, s < T_u^{\mathcal{Y}}, s < e(q)\}\mathrm{d}s \\ &\le \int_0^\infty \mathbf{1}\{\mathcal{Z}_s \le t, \mathcal{Y}_s \le u, s < e(q)\}\mathrm{d}s.\end{aligned} \tag{4.14}$$

Thus by (4.13) and (4.14),

$$\int_0^u p_{Z,Y}(t, v)\mathrm{d}v = \mathrm{d}_Y \int_0^\infty P(\mathcal{Z}_s \le t, \mathcal{Y}_s \le u, s < e(q))\mathrm{d}s = \mathrm{d}_Y V_{Z,Y}(t, u).$$

□

Theorem 4.1 *Parts (i) and (ii) of Theorem 3.1 hold precisely as stated with $p_{Z,Y}$ in place of p, d_Y in place of d_H, and $V_{Z,Y}$ in place of V.*

Proof of Theorem 4.1: Since $V_{Z,Y}(t, u) > 0$ for all $t > 0$ and $u > 0$ by right continuity of (Z, Y), we have by (4.10) and Lemma 4.1 (b) that

$$\mathrm{d}_Y > 0 \text{ iff } \quad p_{Z,Y}(t, u) > 0 \text{ for some } t > 0, u > 0.$$

On the other hand

$$\mathrm{d}_Y > 0 \text{ iff } 0 < P(\mathcal{Y}_{T_u^{\mathcal{Y}}} = u, T_u^{\mathcal{Y}} < e(q)) = \lim_{t\to\infty} p_{Z,Y}(t, u) \text{ for every } u > 0.$$

Combined with monotonicity of $p(\cdot, u)$ for $u > 0$, these give the equivalence of the subordinator versions of (3.1), (3.2) and (3.4). To complete the proof of Part (i) observe that the subordinator version of (3.4) implying the subordinator version of (3.3) was proved in (4.9), while the subordinator version of (3.3) implying the subordinator version of (3.1) is trivial.

If $d_Y > 0$ then $\mathcal{Y}$ is not compound Poisson, so $V_{Z,Y}(t, 0) = 0$. The remainder of part (ii) follows immediately from (4.10) and Lemma 4.1(b). □

Lemma 4.3 *For every $u > 0$,*

$$P(\Delta\mathcal{Z}_{T_u^{\mathcal{Y}}} > 0, \Delta\mathcal{Y}_{T_u^{\mathcal{Y}}} = 0) = 0. \tag{4.15}$$

Proof of Lemma 4.3: If $\mathcal{Y}$ is compound Poisson, then $P(\Delta\mathcal{Y}_{T_u^{\mathcal{Y}}} = 0) = 0$, so the result is trivial. Thus we may assume that $\mathcal{Y}$ is not compound Poisson, in which case $\mathcal{Y}$ is strictly increasing. Thus by the compensation formula, p7 of Bertoin (1996), for every $\varepsilon > 0$

$$\begin{aligned} P(\Delta\mathcal{Y}_{T_u^{\mathcal{Y}}} = 0, \Delta\mathcal{Z}_{T_u^{\mathcal{Y}}} > \varepsilon) &= E\sum_{t>0} \mathbf{1}\{\mathcal{Y}_{t-} = u, \Delta\mathcal{Y}_t = 0, \Delta\mathcal{Z}_t > \varepsilon\} \\ &= \Pi_{Z,Y}((\varepsilon,\infty)\times\{0\})\int_0^\infty P(\mathcal{Y}_t = u)\mathrm{d}t. \end{aligned}$$

The last expression is 0 by Proposition I.15 of Bertoin (1996). □

The next result is a "quadruple law" for (Z, Y), in a similar spirit to the quintuple law.

Theorem 4.2 (Quadruple Law) *For $u > 0$, $x \geq 0$, $0 \leq y \leq u$, $s \geq 0$, $t \geq 0$, we have*

$$\begin{aligned} &P\left(\mathcal{Y}_{T_u^{\mathcal{Y}}} - u \in \mathrm{d}x, u - \mathcal{Y}_{T_u^{\mathcal{Y}}-} \in \mathrm{d}y, \Delta\mathcal{Z}_{T_u^{\mathcal{Y}}} \in \mathrm{d}s, \mathcal{Z}_{T_u^{\mathcal{Y}}-} \in \mathrm{d}t; T_u^{\mathcal{Y}} < e(q)\right) \\ &= \mathbf{1}_{\{x>0\}}|V_{Z,Y}(\mathrm{d}t, u-\mathrm{d}y)|\Pi_{Z,Y}(\mathrm{d}s, \mathrm{d}x+y) + \mathrm{d}_Y\frac{\partial_-}{\partial_- u}V_{Z,Y}(\mathrm{d}t, u)\delta_0(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y), \end{aligned} \tag{4.16}$$

with the convention that the term containing the differential $\partial_- V_{Z,Y}(\mathrm{d}t, u)/\partial_- u$ is absent when $\mathrm{d}_Y = 0$ (in which case $\partial_- V_{Z,Y}(t, u)/\partial_- u$ need not be defined).

Proof of Theorem 4.2: Fix $u > 0$. By the compensation formula, we get for $x > 0$, $0 \leq y \leq u$, $s \geq 0$, $t \geq 0$,

$$\begin{aligned} &P(\mathcal{Y}_{T_u^{\mathcal{Y}}} - u \in \mathrm{d}x, u - \mathcal{Y}_{T_u^{\mathcal{Y}}-} \in \mathrm{d}y, \Delta\mathcal{Z}_{T_u^{\mathcal{Y}}} \in \mathrm{d}s, \mathcal{Z}_{T_u^{\mathcal{Y}}-} \in \mathrm{d}t; T_u^{\mathcal{Y}} < e(q)) \\ &\quad = P(\Delta\mathcal{Y}_{T_u^{\mathcal{Y}}} \in \mathrm{d}x + y, u - \mathcal{Y}_{T_u^{\mathcal{Y}}-} \in \mathrm{d}y, \Delta\mathcal{Z}_{T_u^{\mathcal{Y}}} \in \mathrm{d}s, \mathcal{Z}_{T_u^{\mathcal{Y}}-} \in \mathrm{d}t; T_u^{\mathcal{Y}} < e(q)) \\ &\quad = E\sum_{r>0}\mathbf{1}\{\Delta\mathcal{Y}_r \in \mathrm{d}x + y, u - \mathcal{Y}_{r-} \in \mathrm{d}y, \Delta\mathcal{Z}_r \in \mathrm{d}s, \mathcal{Z}_{r-} \in \mathrm{d}t, r < e(q)\} \\ &\quad = E\int_0^\infty \mathbf{1}\{u - \mathcal{Y}_{r-} \in \mathrm{d}y, \mathcal{Z}_{r-} \in \mathrm{d}t, r < e(q)\}\mathrm{d}r\,\Pi_{Z,H}(\mathrm{d}s, \mathrm{d}x+y) \\ &\quad = |V_{Z,H}(\mathrm{d}t, u-\mathrm{d}y)|\Pi_{Z,H}(\mathrm{d}s, \mathrm{d}x+y). \end{aligned} \tag{4.17}$$

Thus we have left to consider the case $x = 0$.

First observe that from Proposition III.2 of Bertoin (1996), it follows that

$$P(\mathcal{Y}_{T_u^{\mathcal{Y}}-} < u = \mathcal{Y}_{T_u^{\mathcal{Y}}}) = 0, \text{ for all } u > 0. \tag{4.18}$$

Now suppose $\mathrm{d}_Y > 0$. By part (ii) of Theorem 4.1

$$p_{Z,Y}(t,u) = \mathrm{d}_Y \frac{\partial_-}{\partial_- u} V_{Z,Y}(t,u). \tag{4.19}$$

This, together with (4.15) and (4.18), shows

$$\begin{aligned} P(\mathcal{Y}_{T_u^{\mathcal{Y}}} = u, u - \mathcal{Y}_{T_u^{\mathcal{Y}}-} \in \mathrm{d}y, \Delta\mathcal{Z}_{T_u^{\mathcal{Y}}} \in \mathrm{d}s, \mathcal{Z}_{T_u^{\mathcal{Y}}-} \in \mathrm{d}t; T_u^{\mathcal{Y}} < e(q)) \\ = P(\mathcal{Y}_{T_u^{\mathcal{Y}}} = u, \mathcal{Z}_{T_u^{\mathcal{Y}}} \in \mathrm{d}t; T_u^{\mathcal{Y}} < e(q))\delta_0(\mathrm{d}s, \mathrm{d}y) \\ = \mathrm{d}_Y \frac{\partial_-}{\partial_- u} V_{Z,Y}(\mathrm{d}t, u)\delta_0(\mathrm{d}s, \mathrm{d}y) \end{aligned} \tag{4.20}$$

for $0 \le y \le u$, $s \ge 0$ and $t \ge 0$. When $\mathrm{d}_Y = 0$, (4.20) continues to hold since the lefthand side of (4.20) is 0 because $\mathcal{Y}$ does not creep. Adding (4.20) to (4.17) then gives (4.16). □

The next result is a generalisation of Theorem 3.5 to the subordinator setup. The conditions on $\kappa_{Z,Y}$ are analogous to those on κ in Theorem 3.5.

Theorem 4.3 (A Laplace Transform Identity for Subordinators)
Fix $\mu, \rho, \lambda, \nu, \theta$ so that $\kappa_{Z,Y}(\theta, \mu+\lambda), \kappa_{Z,Y}(\theta, \rho)$ are finite and $\kappa_{Z,Y}(\nu, \mu) > 0$.
(i) If $\lambda \neq \rho - \mu$ then

$$\begin{aligned} \int_{u\ge 0} e^{-\mu u} E\left(e^{-\rho(\mathcal{Y}_{T_u^{\mathcal{Y}}}-u)-\lambda(u-\mathcal{Y}_{T_u^{\mathcal{Y}}-})-\nu\mathcal{Z}_{T_u^{\mathcal{Y}}-}-\theta\Delta\mathcal{Z}_{T_u^{\mathcal{Y}}}}; T_u^{\mathcal{Y}} < e(q)\right) \mathrm{d}u \\ = \frac{\kappa_{Z,Y}(\theta, \mu+\lambda) - \kappa_{Z,Y}(\theta, \rho)}{(\mu+\lambda-\rho)\kappa_{Z,Y}(\nu, \mu)}. \end{aligned} \tag{4.21}$$

(ii) If $\lambda = \rho - \mu$ then

$$\int_{u\ge 0} E\left(e^{-\rho\Delta\mathcal{Y}_{T_u^{\mathcal{Y}}}-\mu\mathcal{Y}_{T_u^{\mathcal{Y}}-}-\nu\mathcal{Z}_{T_u^{\mathcal{Y}}-}-\theta\Delta\mathcal{Z}_{T_u^{\mathcal{Y}}}}; T_u^{\mathcal{Y}} < e(q)\right) \mathrm{d}u = \frac{1}{\kappa_{Z,Y}(\nu, \mu)} \frac{\partial_+ \kappa_{Z,Y}(\theta, \rho)}{\partial_+ \rho} \tag{4.22}$$

provided the right derivative exists.

Proof of Theorem 4.3: Taking the expectation over the set $\{\mathcal{Y}_{T_u^{\mathcal{Y}}} > u\}$, from (4.16) we find

$$\begin{aligned} E\left(e^{-\rho(\mathcal{Y}_{T_u^{\mathcal{Y}}}-u)-\lambda(u-\mathcal{Y}_{T_u^{\mathcal{Y}}-})-\nu\mathcal{Z}_{T_u^{\mathcal{Y}}-}-\theta(\Delta\mathcal{Z}_{T_u^{\mathcal{Y}}})}; T_u^{\mathcal{Y}} < e(q), \mathcal{Y}_{T_u^{\mathcal{Y}}} > u\right) \\ = \int_{0\le y\le u} \int_{x>0} \int_{s\ge 0} \int_{t\ge 0} e^{-\rho x-\lambda y-\nu t-\theta s} |V_{Z,Y}(\mathrm{d}t, u-\mathrm{d}y)| \Pi_{Z,Y}(\mathrm{d}s, \mathrm{d}x+y) \\ = \int_{0\le w\le u} \int_{x>u-w} \int_{s\ge 0} \int_{t\ge 0} e^{-\rho(x-u+w)-\lambda(u-w)-\nu t-\theta s} V_{Z,Y}(\mathrm{d}t, \mathrm{d}w) \Pi_{Z,Y}(\mathrm{d}s, \mathrm{d}x). \end{aligned} \tag{4.23}$$

Now take the Laplace transform of both sides of (4.23). For $\lambda \neq \rho - \mu$, we obtain

$$\begin{aligned}\int_{u\geq 0} e^{-\mu u}\mathrm{d}u \int_{0\leq w\leq u}\int_{x>u-w} & e^{-\rho(x-u+w)-\lambda(u-w)} \\ &= \int_{w\geq 0}\int_{x>0}\int_{w\leq u<w+x} e^{-(\mu+\lambda-\rho)u}e^{-(\rho-\lambda)w}e^{-\rho x}\mathrm{d}u \\ &= \frac{1}{\rho-\mu-\lambda}\int_{w\geq 0} e^{-\mu w}\int_{x>0}(e^{-(\mu+\lambda)x}-e^{-\rho x}).\end{aligned} \tag{4.24}$$

Since we may clearly also include $x = 0$ in the last integral, we then have

$$\begin{aligned}&\int_{u\geq 0} e^{-\mu u}E\left(e^{-\rho(\mathcal{Y}_{T_u^{\mathcal{Y}}}-u)-\lambda(u-\mathcal{Y}_{T_u^{\mathcal{Y}}-})-\nu\mathcal{Z}_{T_u^{\mathcal{Y}}-}-\theta(\Delta\mathcal{Z}_{T_u^{\mathcal{Y}}})}; T_u^{\mathcal{Y}} < e(q), \mathcal{Y}_{T_u^{\mathcal{Y}}} > u\right)\mathrm{d}u \\ &= \frac{1}{\rho-\mu-\lambda}\int_{w\geq 0}\int_{t\geq 0} e^{-\nu t-\mu w}V_{Z,Y}(\mathrm{d}t,\mathrm{d}w) \\ &\qquad\qquad \times\int_{s\geq 0}\int_{x\geq 0} e^{-\theta s}(e^{-(\mu+\lambda)x}-e^{-\rho x})\Pi_{Z,Y}(\mathrm{d}s,\mathrm{d}x) \\ &= \frac{\kappa_{Z,Y}(\theta,\mu+\lambda)-\kappa_{Z,Y}(\theta,\rho)-(\mu+\lambda-\rho)\mathrm{d}_Y}{(\mu+\lambda-\rho)\kappa_{Z,Y}(\nu,\mu)} \\ &= \frac{\kappa_{Z,Y}(\theta,\mu+\lambda)-\kappa_{Z,Y}(\theta,\rho)}{(\mu+\lambda-\rho)\kappa_{Z,Y}(\nu,\mu)} - \frac{\mathrm{d}_Y}{\kappa_{Z,Y}(\nu,\mu)}\end{aligned} \tag{4.25}$$

by (4.4) and (4.6).

If $\mathrm{d}_Y > 0$ then we need to add in the second term in (4.16) due to creeping. From (4.16) and part (ii) of Theorem 4.1 we have

$$\begin{aligned}&\int_{u\geq 0} e^{-\mu u}E\left(e^{-\rho(\mathcal{Y}_{T_u^{\mathcal{Y}}}-u)-\lambda(u-\mathcal{Y}_{T_u^{\mathcal{Y}}-})-\nu\mathcal{Z}_{T_u^{\mathcal{Y}}-}-\theta(\Delta\mathcal{Z}_{T_u^{\mathcal{Y}}})}; T_u^{\mathcal{Y}} < e(q), \mathcal{Y}_{T_u^{\mathcal{Y}}} = u\right)\mathrm{d}u \\ &\qquad = \mathrm{d}_Y\int_{u\geq 0} e^{-\mu u}\int_{t\geq 0} e^{-\nu t}\frac{\partial_-}{\partial_- u}V_{Z,Y}(\mathrm{d}t,u)\mathrm{d}u \\ &\qquad = \mathrm{d}_Y\int_{u\geq 0} e^{-\mu u}\int_{t\geq 0} e^{-\nu t}V_{Z,Y}(\mathrm{d}t,\mathrm{d}u) \\ &\qquad = \frac{\mathrm{d}_Y}{\kappa_{Z,Y}(\nu,\mu)}.\end{aligned}$$

Added to (4.25), this gives (4.21).

Now consider the case where $\lambda = \rho - \mu$. Let $\varepsilon > 0$ and set $\lambda' = \rho - \mu + \varepsilon$. Then $\kappa_{Z,Y}(\theta, \mu + \lambda')$ is finite since $\kappa_{Z,Y}(\theta, \rho)$ is finite. Thus by (4.21)

$$\begin{aligned}&\int_{u\geq 0} E\left(e^{-\rho\Delta\mathcal{Y}_{T_u^{\mathcal{Y}}}-\mu\mathcal{Y}_{T_u^{\mathcal{Y}}-}-\varepsilon(u-\mathcal{Y}_{T_u^{\mathcal{Y}}-})-\nu\mathcal{Z}_{T_u^{\mathcal{Y}}}-\theta\Delta\mathcal{Z}_{T_u^{\mathcal{Y}}}}; T_u^{\mathcal{Y}} < e(q)\right)\mathrm{d}u \\ &\qquad\qquad = \frac{\kappa_{Z,Y}(\theta,\rho+\varepsilon)-\kappa_{Z,Y}(\theta,\rho)}{\varepsilon\kappa_{Z,Y}(\nu,\mu)}.\end{aligned}$$

Letting $\varepsilon \downarrow 0$ and using monotone convergence completes the proof of (4.22). $\square$

Results similar to (4.21) can be found in Winkel (2005). Winkel's interest in bivariate subordinators is in modelling electronic foreign exchange markets and he does not make the connection with the ladder height process. As mentioned in the introduction, it appears to have gone previously unnoticed, that the second factorization identity is a special case of a general transform result for bivariate subordinators.

5 Proofs for Section 3

We now turn to the proofs of the fluctuation results from Section 3. Recall the definitions of $p(t,u)$ and T_u in (1.2) and (4.1) respectively, and from (4.3) that

$$p(t,u) = P(\mathcal{L}_{T_u}^{-1} \leq t, \mathcal{H}_{T_u} = u, T_u < e(q)).$$

In view of the correspondences $(L^{-1}, H) \leftrightarrow (Z, Y)$, $p(t,u) \leftrightarrow p_{Z,Y}(t,u)$, and $V(t,u) \leftrightarrow V_{Z,Y}(t,u)$, we can carry a number of results directly across from Section 4. In particular, the results of Lemma 4.1 and Lemma 4.2 hold with p in place of $p_{Z,Y}$ and V in place of $V_{Z,Y}$.

Proof of Theorem 3.1: Parts (i) and (ii) follow immediately from Theorem 4.1. For Part (iii), supposing X is not compound Poisson with positive drift, by Theorem 27.4 of Sato (1999),

$$P(\tau_u = t, X_{\tau_u} = u) \leq P(X_t = u) = 0$$

for all $u > 0$. Consequently $p(\cdot, u)$ is continuous on $[0, \infty)$ for every $u > 0$, and hence by Lemma 4.1, $p(t, \cdot)$ is continuous on $(0, \infty)$ for every $t > 0$. Since $p(0, \cdot) \equiv 0$ on $(0, \infty)$ this continues to hold for $t = 0$. By (4.10), this then implies differentiability of $V(t, \cdot)$. $\square$

Example 5.1 Let $X_t = t + Y_t$ where Y is compound Poisson with Lévy measure $\Pi_Y(\mathrm{d}x) = \delta_{\{1\}}(\mathrm{d}x) + \delta_{\{-1\}}(\mathrm{d}x)$. Since $X_t = t + \sum_{s \leq t} \Delta Y_s$ and $\sum_{s \leq t} \Delta Y_s$ is an integer, we have that for any $t \in (0,1)$, $u > 0$

$$p(t,u) > 0 \text{ iff } u \in \cup_{n=0}^{\infty} (n, n+t].$$

Furthermore $p(t,u) \geq e^{-2tu}$ for $u \in (0,t]$. Thus neither $p(t,\cdot)$ nor $p(\cdot,u)$ is continuous, and, using (4.10), $V(t,\cdot)$ is not differentiable. Finally, in contrast to monotonicity of $p(\cdot,u)$, $p(t,\cdot)$ is not monotone.

Example 5.2 Set $X_t = t - Y_t$, where Y is a subordinator and $\Pi_Y(\mathbb{R}) = \infty$. Then clearly $p(t,u) = 0$ for $u > t$. Thus there is no hope of proving $p(t,u) > 0$ for all $t, u > 0$ even in the situation of (iii) of Theorem 3.1.

Before turning to the proof of Theorem 3.2, we need a preliminary result, generalising (4.18), which is surely well known, but for which we can not find an exact reference.

Lemma 5.1 *Fix $u > 0$.*
(i) If X is not compound Poisson, then

$$P(X_{t-} \neq u, X_t = u \text{ for some } t > 0) = P(X_{t-} = u, X_t \neq u \text{ for some } t > 0) = 0; \quad (5.1)$$

(ii) For any Lévy process X and $u > 0$

$$P(X_{\tau_u -} < u, X_{\tau_u} = u, \tau_u < \infty) = 0. \quad (5.2)$$

Proof of Lemma 5.1: (i) Use the compensation formula to write for $u > 0$

$$\begin{aligned} P(X_{t-} \neq u, X_t = u \text{ for some } t > 0) &\leq E \sum_{t>0} \mathbf{1}\{X_{t-} \neq u, X_{t-} + \Delta X_t = u\} \\ &= E \int_0^\infty \mathrm{d}t \int_{\xi \neq 0} \mathbf{1}\{X_{t-} \neq u, X_{t-} + \xi = u\} \Pi_X(d\xi) \\ &= \int_{\xi \neq 0} \int_0^\infty P(X_{t-} = u - \xi) \mathrm{d}t \, \Pi_X(d\xi). \end{aligned}$$

The last expression is 0 when X is not compound Poisson, since the potential measure of X is diffuse by Proposition I.15 of Bertoin (1996). Similarly for any $\varepsilon > 0$

$$\begin{aligned} P(X_{t-} = u, |X_t - u| > \varepsilon \text{ for some } t > 0) &\leq E \sum_{t>0} \mathbf{1}\{X_{t-} = u, |\Delta X_t| > \varepsilon\} \\ &= \Pi_X([-\varepsilon, \varepsilon]^c) \int_0^\infty P(X_{t-} = u) \, \mathrm{d}t = 0. \end{aligned}$$

Letting $\varepsilon \to 0$ completes the proof.

(ii) Clearly we may assume $P(X_{\tau_u} = u) > 0$ else there is nothing to prove. In that case X is not compound Poisson. Since

$$\{X_{\tau_u -} < u, X_{\tau_u} = u, \tau_u < \infty\} \subseteq \{X_{t-} \neq u, X_t = u \text{ for some } t > 0\},$$

(5.2) follows from (5.1). □

Proof of Theorem 3.2: Fix $v \geq 0$, $0 \leq y \leq u \wedge v$, $s \geq 0$ and $t \geq 0$. For $x > 0$ this is Doney and Kyprianou's quintuple law. If $x = 0$, then from (4.2), (4.16) and (5.2)

$$\begin{aligned} &P\left(X_{\tau_u} - u \in \mathrm{d}x, u - X_{\tau_u -} \in \mathrm{d}v, u - \overline{X}_{\tau_u -} \in \mathrm{d}y, \tau_u - G_{\tau_u -} \in \mathrm{d}s, G_{\tau_u -} \in \mathrm{d}t\right) \\ &\quad = P\left(\mathcal{H}_{T_u} - u \in \mathrm{d}x, u - \mathcal{H}_{T_u -} \in \mathrm{d}y, \Delta \mathcal{L}^{-1}_{T_u} \in \mathrm{d}s, \mathcal{L}^{-1}_{T_u -} \in \mathrm{d}t; T_u < e(q)\right) \delta_0(\mathrm{d}v) \\ &\quad = \mathrm{d}_H \frac{\partial_-}{\partial_- u} V(\mathrm{d}t, u) \delta_0(\mathrm{d}s, \mathrm{d}x, \mathrm{d}v, \mathrm{d}y). \end{aligned}$$

□

Proof of Corollary 3.1: This follows easily from Theorem 3.2 and (3.10). Alternatively use Theorem 4.2. □

Proof of Theorem 3.3: Assume X is not compound Poisson. We can first easily dispense with the case that 0 is irregular for $(0,\infty)$; because then, 0 is irregular for $[0,\infty)$, so by construction, L^{-1} and H jump at the same times (see p.24 of Doney (2005)). Hence $\Pi_{L^{-1},H}(\mathrm{d}s,\{0\})$ is the zero measure. On the other hand, by the strong Markov property at time τ_u, X does not creep over any $u \geq 0$. Thus the result holds in this case.

We now assume that 0 is regular for $(0,\infty)$, in which case the closure of the zero set of $\overline{X}-X$ is a perfect nowhere dense set with probability one; see for example the discussion on p.104 of Bertoin (1996). Fix $r \in \mathbb{Q}, r > 0$ and set $Y_t^r = X_{r+t} - X_r$. Let

$$\tau^r = \inf\{t > 0 : Y_t^r > \overline{X}_r - X_r\}$$

and

$$A_r = \{Y_{\tau^r}^r = \overline{X}_r - X_r,\ \overline{X}_r - X_r > 0\}.$$

By independence

$$P(A_r) = \int_{u>0} P(X_{\tau_u} = u) P(\overline{X}_r - X_r \in \mathrm{d}u).$$

Thus $P(A_r) > 0$ iff X creeps. On the other hand, $\Pi_{L^{-1},H}(\mathrm{d}s,\{0\})$ is the zero measure iff $P(\Delta L_s^{-1} > 0, \Delta H_s = 0 \text{ for some } s > 0) = 0$. The result then follows from the key observation that

$$\{\Delta L_s^{-1} > 0, \Delta H_s = 0 \text{ for some } s > 0\} \overset{a.s.}{=} \bigcup_{r\in\mathbb{Q}} A_r.$$

To see this, assume s is such that $\Delta H_s = 0$ and $\Delta L_s^{-1} > 0$. Let $r \in (L_{s-}^{-1}, L_s^{-1}) \cap \mathbb{Q}$. Then $\overline{X}_r - X_r > 0$, $X_{L_s^{-1}} = \overline{X}_r$ and $X_t < \overline{X}_r$ for $L_{s-}^{-1} < t < L_s^{-1}$. Since a.s. the zero set of $\overline{X}-X$ contains no isolated points, it follows that off this P-null set $Y_{\tau^r}^r = \overline{X}_r - X_r$. Thus A_r occurs. Conversely fix $r \in \mathbb{Q}$ and assume that $Y_{\tau^r}^r = \overline{X}_r - X_r > 0$. Choose $s > 0$ so that $r \in (L_{s-}^{-1}, L_s^{-1})$. Then on A_r, $H_s = X_{L_s^{-1}} = \overline{X}_r = H_{s-}$. Hence $\Delta H_s = 0$ and $\Delta L_s^{-1} > 0$. □

As remarked earlier, the jumps of (L^{-1}, H) for which $\Delta H = 0$, do not occur when X creeps over a fixed level. This follows from an application of Lemma 4.3, which gives

$$P(\Delta \mathcal{L}_{T_u}^{-1} > 0, \Delta \mathcal{H}_{T_u} = 0) = 0,\ u > 0. \tag{5.3}$$

Let

$$V_H(\mathrm{d}v) = \int_{t\geq 0} V(\mathrm{d}t, \mathrm{d}v) = \int_0^\infty P(H_s \in \mathrm{d}v)\mathrm{d}s$$

be the potential measure of H, and similarly for $\widehat{V}_{\widehat{H}}$.

Proof of Theorem 3.4 Assume X is not compound Poisson. Then $\widehat{V}_{\widehat{H}}$ is diffuse on $(0,\infty)$ by Lemma 1 of Chaumont and Doney (2010). Then since $\Pi_X(\{0\}) = 0$ we have

$$\int_{s\geq 0}\int_{v\geq 0} \widehat{V}(\mathrm{d}s, \mathrm{d}v)\Pi_X(\{v\}) = \int_{v\geq 0} \widehat{V}_{\widehat{H}}(\mathrm{d}v)\Pi_X(\{v\}) = 0. \tag{5.4}$$

Now by (3.10),

$$\Pi_{L^{-1},H}(\mathrm{d}s,\mathrm{d}x) = \int_{v\geq 0} \widehat{V}(\mathrm{d}s,\mathrm{d}v)\Pi_X(\mathrm{d}x+v) \text{ for all } s\geq 0, x\geq 0 \tag{5.5}$$

is equivalent to

$$\Pi_{L^{-1},H}(\mathrm{d}s,\{0\}) = \int_{v\geq 0} \widehat{V}(\mathrm{d}s,\mathrm{d}v)\Pi_X(\{v\}) \text{ for all } s\geq 0 \tag{5.6}$$

which in turn, by (5.4), is equivalent to $\Pi_{L^{-1},H}(\mathrm{d}s,\{0\})$ being the zero measure. This is equivalent to X not creeping by Theorem 3.3.

If X is compound Poisson, then recalling the definition of $(\widehat{L}^{-1},\widehat{H})$ prior to (2.6) in this case, a natural approach is to apply the result in the non compound Poisson case to the approximating process $X_t^\varepsilon = X_t + \varepsilon t$ and take the limit as $\varepsilon \downarrow 0$. Unfortunately this cannot work since X^ε creeps and so the non compound Poisson result does not apply. Consequently we are forced to appeal to a direct construction of $(\widehat{L}^{-1},\widehat{H})$ in this case. We defer the details of this to the appendix. □

Example 5.3 Assume that X is a spectrally negative compound Poisson process with positive drift d_X. For simplicity, to avoid killing, also assume $EX_1 \geq 0$. In this example X creeps, and it is a simple matter to find $\Pi_{L^{-1},H}(\mathrm{d}s,\{0\})$. Choose the normalisation of local time so that

$$L_t = \int_0^t \mathbf{1}(X_r = \overline{X}_r)\mathrm{d}r.$$

Let σ_1 be the time of the first jump of X and

$$\alpha = \inf\{t > \sigma_1 : X_t \geq \overline{X}_{\sigma_1}\}. \tag{5.7}$$

Then it is easy to see that (draw a picture)

$$L_s^{-1} = s + \sum_{i=1}^{N_s} R_i, \quad H_s = d_X s,$$

where N is a Poisson process of rate $\lambda = \Pi_X(\mathbb{R})$ independent of the i.i.d. sequence $R_i, i\geq 1$, where $R_1 \stackrel{d}{=} \alpha - \sigma_1$. From this we conclude

$$\Pi_{L^{-1},H}(\mathrm{d}s,\{0\}) = \lambda P(R_1 \in \mathrm{d}s) = \Pi_{L^{-1}}(\mathrm{d}s), s > 0.$$

On the other hand, since X is not compound Poisson, (5.4) shows that the lefthand side of (3.10) is the zero measure when $x = 0$.

Proof of Theorem 3.5: Upon using (4.2), this is an immediate application of Theorem 4.3. □

6 Appendix; Proof of Theorem 3.4 in the Compound Poisson Case

We want to prove Theorem 3.4 in the compound Poisson case. As mentioned earlier the natural approach using the approximating process $X_t^\varepsilon = X_t + \varepsilon t$ does not work. Thus we proceed by using the random walk embedded in X to give a direct construction of the bivariate ladder processes.

Throughout this section we assume that X is compound Poisson. We write it in the form $X_t = \sum_{k=1}^{N_t} Y_k$, where N is a Poisson process of rate $\lambda > 0$, Y_k are i.i.d. rvs independent of N with distribution function F, and $P(Y_k = 0) = 0$. Thus the Lévy measure of X is $\Pi_X(\mathrm{d}x) = \lambda F(\mathrm{d}x)$.

Let $S_0 = 0$, and $S_n = \sum_{k=1}^n Y_k$, $n \geq 1$. Let $\sigma_0 = 0$ and σ_k, $k \geq 1$, denote the successive jump times of X_t. Then $\{S_n, n \geq 0\}$ is independent of $\{\sigma_k, k \geq 0\}$ and $X_t = S_n$ for $\sigma_n \leq t < \sigma_{n+1}$, $n \geq 0$. Let $t_0 = 0$,

$$t_{n+1} = \min\{m > t_n : S_m \geq S_{t_n}\}, \quad n = 0, 1, 2, \ldots, \tag{6.1}$$

be the *weak ascending* ladder times of S_n, $h_n = S_{t_n}$ the corresponding ladder height sequence, and

$$U(k, x) = \sum_{n \geq 0} P(t_n = k, h_n \leq x), \ k \geq 0, x \geq 0,$$

the corresponding bivariate renewal measure. The *strict ascending* ladder process is obtained by replacing the inequality in (6.1) with a strict inequality. The analogous quantities for the *strict descending* ladder process will be denoted with a hat; thus

$$\widehat{U}(k, x) = \sum_{n \geq 0} P(\widehat{t}_n = k, \widehat{h}_n \leq x), \ k \geq 0, x \geq 0.$$

We choose the normalisation of the local time L so that

$$L_t = \int_0^t \mathbf{1}(X_r = \overline{X}_r)\mathrm{d}r = \int_0^t \sum_{n=0}^\infty \mathbf{1}_{[\sigma_{t_n}, \sigma_{t_n+1})}(r)\mathrm{d}r, \ t \geq 0. \tag{6.2}$$

As remarked earlier this gives rise to the *weak bivariate ladder process* (L^{-1}, H) of X. For $\widehat{X}$, we require $(\widehat{L}^{-1}, \widehat{H})$ to be the *strict* bivariate ladder process, which may be viewed as the limit of the ascending bivariate ladder process of $\widehat{X}_t - \varepsilon t$ as $\varepsilon \downarrow 0$. For a direct construction of $(\widehat{L}^{-1}, \widehat{H})$, let M_s be an auxiliary Poisson process of rate 1, independent of X, and set

$$(\widehat{L}_s^{-1}, \widehat{H}_s) = \begin{cases} (\sigma_{\widehat{t}_n}, \widehat{h}_n) & \text{if } M_s = n, \widehat{t}_n < \infty, n \geq 0, \\ (\infty, \infty) & \text{if } M_s = n, \widehat{t}_n = \infty, n \geq 0. \end{cases} \tag{6.3}$$

This is analogous to the definition of the ascending ladder processes of $\widehat{X}$ in the non-compound Poisson case when 0 is irregular for $[0, \infty)$ for $\widehat{X}$; see for example p.24 of Doney (2005). To emphasize, $\widehat{V}(\cdot, \cdot)$ and $\widehat{\kappa}(\cdot, \cdot)$ are defined in terms of $(\widehat{L}^{-1}, \widehat{H})$ given by

(6.3). We should also remark that with these normalisations of the local times, one can check that (2.6) holds.

The connection between the bivariate ladder processes of X and S is given by

Lemma 6.1 *For any $t \geq 0, x \geq 0$,*

$$V(t,x) = \lambda^{-1} \sum_{k \geq 0} P(\sigma_{k+1} \leq t) U(k,x), \tag{6.4}$$

and

$$\widehat{V}(t,x) = \sum_{k \geq 0} P(\sigma_k \leq t) \widehat{U}(k,x). \tag{6.5}$$

Proof of Lemma 6.1: Since (6.4) will not be used in the sequel, we only prove (6.5). We have

$$\begin{aligned} P(\widehat{L}_s^{-1} \leq t, \widehat{H}_s \leq x) &= \sum_{n=0}^{\infty} P(\sigma_{\widehat{t}_n} \leq t, \widehat{h}_n \leq x, M_s = n) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P(\widehat{t}_n = k, \widehat{h}_n \leq x, \sigma_k \leq t, M_s = n) \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} P(\widehat{t}_n = k, \widehat{h}_n \leq x) P(\sigma_k \leq t) P(M_s = n). \end{aligned} \tag{6.6}$$

One easily checks that $\int_0^\infty P(M_s = n)\, \mathrm{d}s = 1$ for every $n \geq 0$, hence integrating (6.6) we obtain

$$\begin{aligned} \widehat{V}(t,x) &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} P(\widehat{t}_n = k, \widehat{h}_n \leq x) P(\sigma_k \leq t) \\ &= \sum_{k \geq 0} P(\sigma_k \leq t) \widehat{U}(k,x), \end{aligned}$$

as required. □

Recall the definition of α in (5.7), which in the present case reduces to

$$\alpha = \inf\{t > \sigma_1 : X_t \geq 0\}.$$

Also introduce

$$\beta = \inf\{n > 0 : S_n \geq 0\}.$$

Lemma 6.2 *For $s > 0, x \geq 0$ and $v \geq 0$*

$$\widehat{V}(\mathrm{d}s, \mathrm{d}v) F(\mathrm{d}x + v) = P(X_{\alpha-} \in -\mathrm{d}v, X_\alpha \in \mathrm{d}x, \alpha - \sigma_1 \in \mathrm{d}s). \tag{6.7}$$

Proof of Lemma 6.2: Using $P(\sigma_0 \in \mathrm{d}s) = 0$ when $s > 0$ in the fourth equality below, duality in the fifth, and (6.5) in the sixth, we have

$$\begin{aligned}
P(X_{\alpha-} \in -\mathrm{d}v, & X_\alpha \in \mathrm{d}x, \alpha - \sigma_1 \in \mathrm{d}s) \\
&= \mathbf{1}_{\{v>0\}} P(X_{\alpha-} \in -\mathrm{d}v, X_\alpha \in \mathrm{d}x, \alpha - \sigma_1 \in \mathrm{d}s) \\
&= \mathbf{1}_{\{v>0\}} \sum_{i\geq 2} P(\beta = i, S_{i-1} \in -\mathrm{d}v, S_i \in \mathrm{d}x, \sigma_i - \sigma_1 \in \mathrm{d}s) \\
&= \mathbf{1}_{\{v>0\}} \sum_{i\geq 2} P(\beta \geq i-1, S_{i-1} \in -\mathrm{d}v) F(\mathrm{d}x + v) P(\sigma_{i-1} \in \mathrm{d}s) \\
&= \mathbf{1}_{\{v>0\}} \sum_{i\geq 0} P(\beta \geq i, S_i \in -\mathrm{d}v) P(\sigma_i \in \mathrm{d}s) F(\mathrm{d}x + v) \\
&= \mathbf{1}_{\{v>0\}} \sum_{i\geq 0} \widehat{U}(i, \mathrm{d}v) P(\sigma_i \in \mathrm{d}s) F(\mathrm{d}x + v) \\
&= \mathbf{1}_{\{v>0\}} \widehat{V}(\mathrm{d}s, \mathrm{d}v) F(\mathrm{d}x + v) \\
&= \widehat{V}(\mathrm{d}s, \mathrm{d}v) F(\mathrm{d}x + v)
\end{aligned}$$

since $s > 0$ and $\widehat{V}(\mathrm{d}s, \{0\}) = \delta_0(\mathrm{d}s)$ by (6.5) (or (6.3)). □

Proof of Theorem 3.4, Compound Poisson case: Since X does not creep in this case, we need to prove (3.10) for all $s \geq 0, x \geq 0$. When $s = x = 0$, (3.10) reduces to showing

$$\int_{v\geq 0} \widehat{V}(\{0\}, \mathrm{d}v) \Pi_X(\{v\}) = 0. \tag{6.8}$$

But $\widehat{V}(\{0\}, \mathrm{d}v) = \delta_0(\mathrm{d}v)$ by (6.5), which proves (6.8). Thus we may assume $s \vee x > 0$.

If $s > 0$ and $x \geq 0$, then integrating out v in (6.7) gives

$$\int_{v\geq 0} \widehat{V}(\mathrm{d}s, \mathrm{d}v) F(\mathrm{d}x + v) = P(\Delta L_{\sigma_1}^{-1} \in \mathrm{d}s, \Delta H_{\sigma_1} \in \mathrm{d}x). \tag{6.9}$$

This continues to hold when $s = 0$ and $x > 0$ since

$$\int_{v\geq 0} \widehat{V}(\{0\}, \mathrm{d}v) F(\mathrm{d}x + v) = \int_{v\geq 0} \delta_0(\mathrm{d}v) F(\mathrm{d}x + v) = F(\mathrm{d}x),$$

while

$$P(\Delta L_{\sigma_1}^{-1} = 0, \Delta H_{\sigma_1} \in \mathrm{d}x) = P(X_{\sigma_1} \in \mathrm{d}x) = F(\mathrm{d}x).$$

Thus (6.9) holds whenever $s \vee x > 0$. Now by the compensation formula (which requires

$s \vee x > 0$) we have

$$\begin{aligned} P(\Delta L_{\sigma_1}^{-1} \in \mathrm{d}s, \Delta H_{\sigma_1} \in \mathrm{d}x) &= E \sum_{t \geq 0} \mathbf{1}(L_{t-}^{-1} = t, H_{t-} = 0, \Delta L_t^{-1} \in \mathrm{d}s, \Delta H_t \in \mathrm{d}x) \\ &= \int_{t \geq 0} P(L_t^{-1} = t, H_t = 0) \mathrm{d}t \; \Pi_{L^{-1},H}(\mathrm{d}s, \mathrm{d}x) \\ &= \int_{t \geq 0} P(\sigma_1 > t) \; \mathrm{d}t \; \Pi_{L^{-1},H}(\mathrm{d}s, \mathrm{d}x) \\ &= \lambda^{-1} \Pi_{L^{-1},H}(\mathrm{d}s, \mathrm{d}x), \end{aligned}$$

which together with $\Pi_X(\mathrm{d}x) = \lambda F(\mathrm{d}x)$ completes the proof of (3.10). □

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