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Optimal Solvability for the Dirichlet and Neumann Problem in Dimension Two

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OPTIMAL SOLVABILITY FOR THE DIRICHLET AND NEUMANN PROBLEMS IN DIMENSION TWO

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ABSTRACT. We show existence and uniqueness for the solutions of the regularity and the Neumann problems for harmonic functions on Lipschitz domains with data in the Hardy spaces $H^p_1(\partial D)(H^p(\partial D))$, $p > \frac{2}{3} - \varepsilon$, where $D \subset \mathbb{R}^2$ and ε is a (small) number depending on the Lipschitz nature of D. This in turn implies that solutions to the Dirichlet problem with data in the Hölder class $C^{1/2+\varepsilon}(\partial D)$ are themselves in $C^{1/2+\varepsilon}(\bar{D})$. Both of these results are sharp. In fact, we prove a more general statement regarding the H^p solvability for divergence form elliptic equations with bounded measurable coefficients.

We also provide $H^{2/3-\varepsilon}$ and $C^{1/2+\varepsilon}$ solvability result for the regularity and Dirichlet problem for the biharmonic equation on Lipschitz domains.

1. Introduction and Main Results

In this paper we study the Dirichlet and Neumann problems for harmonic functions on Lipschitz domains and their biharmonic counterparts. More precisely let X, Y, Z be function spaces on the boundary ∂D of D. Then

$$(D_X)\left|\begin{array}{ll} \Delta u & = 0 \\ u|_{\partial D} & = f \\ M(u) & \in X \end{array}\right.$$

is the Dirichlet problem with underlying space X, and

$$(N_Y)\left|\begin{array}{ll} \Delta u & = 0 \\ \left.\frac{\partial u}{\partial N}\right|_{\partial D} & = g \\ M(\nabla u) & \in Y \end{array}\right.$$

$$(R_Z)\left|\begin{array}{ll} \Delta u & = 0 \\ u|_{\partial D} & = h \\ M(\nabla u) & \in Z \end{array}\right.$$

are the Neumann and regularity problems. Here N is the outer normal vector to D, $\frac{\partial u}{\partial N} = \langle N, \nabla u\rangle$ and $M(u)$ is the usual non-tangential maximal function of u.

In this setting the canonical choices are $X = Y = L^p(\partial D)$, $Z = L^p_1(\partial D)$- the space of functions with one distributional derivative in $L^p(\partial D)$. In the sequel, we will slightly abuse notations by using D_p instead of D_{L^p}, N_p instead of N_{L^p} etc.

1.1. **Harmonic functions.** We state now the classical results related to the L^p theory.

Theorem [3, 12, 4, 5] Let $D \subset \mathbb{R}^n$ be a connected Lipschitz domain. Then

1. there exists an $\varepsilon = \varepsilon(D) > 0$ such that for $2 - \varepsilon < p \leq \infty$ and $f \in L^p(\partial D)$ there is an unique solution to D_p. Moreover, there is the *apriori* estimate $\|M(u)\|_{L^p(\partial D)} \leq C\|f\|_{L^p(\partial D)}$.
2. there exists an $\varepsilon = \varepsilon(D) > 0$ such that for $1 < p < 2 + \varepsilon$ and $g \in L^p(\partial D)$, $\int_{\partial D} g d\sigma = 0$ there is an unique (up to a constant) solution to N_p. There is the *apriori* estimate $\|M(\nabla u)\|_{L^p(\partial D)} \leq C\|g\|_{L^p(\partial D)}$.
3. there exists an $\varepsilon = \varepsilon(D) > 0$ such that for $1 < p < 2 + \varepsilon$ and $h \in L^p_1(\partial D)$ there is an unique solution to R_p. There is the *apriori* estimate $\|M(\nabla u)\|_{L^p(\partial D)} \leq C\|h\|_{L^p_1(\partial D)}$.

This theorem summarizes the results in [3], but some earlier version and ideas originated in [12]. Actually,the L^p theory described above is a consequence of the duality between the Dirichlet and regularuty problems, the L^2 solvability for all three problems and the following endpoint result due to Dahlberg and Kenig.

Theorem 1 (Dahlberg-Kenig). *Let $D \subset \mathbb{R}^n$ be a connected star-like Lipschitz domain. Then*

1.
$$(N_1)\left|\begin{array}{ll} \Delta u & = 0 \\ \left.\dfrac{\partial u}{\partial N}\right|_{\partial D} & = f \\ M(\nabla u) & \in L^1 \end{array}\right.$$
is uniquely solvable provided $f \in H^1(\partial D)$ and $\|M(\nabla u)\|_{L^1(\partial D)} \leq C\|f\|_{H^1(\partial D)}$.
2. *Given $f \in H^1_1(\partial D)$ there exists an unique solution to*
$$(R_1)\left|\begin{array}{ll} \Delta u & = 0 \\ u|_{\partial D} & = f \\ M(\nabla u) & \in L^1 \end{array}\right.$$
Moreover $\|M(\nabla u)\|_{L^1(\partial D)} \leq C\|f\|_{H^1_1(\partial D)}$.

As a corollary to this result, one proves (weak) maximum principle, solvability for BMO data etc. We refer to [3] for excellent treatise of these questions.

Recently, Brown [1] was able to extend Theorem 1 to show that there exists $\varepsilon = \varepsilon(D)$, such that for $1 - \varepsilon < p < 1$ the Neumann problem N_p is still uniquely solvable with the usual estimates $\|M(\nabla u)\|_p \leq C\|f\|_{H^p}$. This result has the interesting corollary that the double layer potential is invertible operator on the Hölder space $C^\alpha(\partial D)$ for α close to zero and thus we have a representation formula for the solutions of the Dirichlet problem with C^α data. This raises the following natural question, see Question 3.2.10 in [5].

Question 1. *Are R_p and N_p solvable for p significantly below one? What does that imply for solutions of the Dirichlet problem with C^α data for α significantly above zero?*

The purpose of this paper is to establish the optimal p range for solvability of both R_p and N_p in dimension two. That is our Theorem 2 below. Let us remark only that known counterexamples in dimensions bigger than two imply that the Neumann problem N_p may not be solvable for $p < 1 - \varepsilon(D)$, i.e. for fixed $p < 1$, there exists a Lipschitz domain D such that $N_p(D)$ is not uniquely solvable.

Theorem 2. *Let $D \subset \mathbb{R}^2$ be a star-like Lipschitz domain with connected boundary. There exists $\varepsilon = \varepsilon(D)$, such that for $2/3 - \varepsilon < p < 1$ and $0 < \alpha < 1/2 + \varepsilon$*

1. *The Neumann problem*

$$(N_p)\left|\begin{array}{c}\Delta u = 0\\ \dfrac{\partial u}{\partial N}\,|_{\partial D} = f \in H^p(\partial D)\end{array}\right.$$

is uniquely solvable and $\|M(\nabla u)\|_{L^p(\partial D)} \leq C\|f\|_{H^p(\partial D)}$.

2. *The regularity problem*

$$(R_p)\left|\begin{array}{c}\Delta u = 0\\ u|_{\partial D} = f \in H^p_1(\partial D)\end{array}\right.$$

has unique solution and $\|M(\nabla u)\|_{L^p(\partial D)} \leq C\|f\|_{H^p_1(\partial D)}$.

3. *The Dirichlet problem*

$$(D_\alpha)\left|\begin{array}{c}\Delta u = 0\\ u|_{\partial D} = f \in C^\alpha(\partial D)\end{array}\right.$$

has unique solution and $\|u\|_{C^\alpha(\bar{D})} \leq C\|f\|_{C^\alpha(\partial D)}$.

Moreover, the ranges $2/3 - \varepsilon < p$ and $\alpha < 1/2 + \varepsilon$ are sharp.

In fact, we consider more general divergence form elliptic equations in the form $\operatorname{div}(A(\nabla u)) = 0$, where A is a symmetric, elliptic matrix with real-valued bounded measurable coefficients. We prove the following theorem.

Theorem 3. *Let $A(x,t) = A(x) = \begin{pmatrix} a(x) & b(x)\\ b(x) & c(x)\end{pmatrix}$ be a real, symmetric, uniformly elliptic matrix with bounded and measurable coefficients, independent of the time variable. Then there exists $\varepsilon = \varepsilon(D) > 0$ such that for $2/3 - \varepsilon < p < 2 + \varepsilon$*

1. *the Neumann problem in the upper half-space $\mathbb{R}^2_+ = \{(x,t) : t > 0\}$*

$$(N_p)\left|\begin{array}{c}\operatorname{div}(A\nabla u) = 0 \quad \text{for } t > 0,\\ A\nabla u(x,0)\cdot(0,-1)| = f \in H^p(\mathbb{R}^1)\end{array}\right.$$

has unique solution and $\|M(\nabla u)\|_{L^p} \leq C\|f\|_{H^p}$.

2. *The regularity problem in the upper half-space $\mathbb{R}^2_+$*

$$(R_p)\left|\begin{array}{c}\operatorname{div}(A\nabla u) = 0 \quad \text{for } t > 0,\\ u(x,0) = h \in H^p_1(\mathbb{R}^1)\end{array}\right.$$

has unique solution and $\|M(\nabla u)\|_{L^p} \leq C\|h\|_{H^p_1}$.

Moreover, the range $2/3 - \varepsilon < p$ is sharp.

Remarks

- For divergence form equations $\operatorname{div}(A\nabla u) = 0$, $A = A(x,t)$ one cannot expect solvability even for D_2 or N_2. Indeed, counterexamples show that unless we require *radial independence* for such a problem in the unit ball, we may encounter non-uniqueness for N_2, see [6] and [5], p. 63.
  We will however work in the upper half-space instead of the unit ball. These two problems are not so much different. In fact, the appropriate assumption in the upper-half space is *time independence* (see [5], p.68 for a relevant discussion) and so our theorem 3

is formulated in that fashion. Let us remark only, that the problems D_2, R_2 and N_2 are all solvable for matrices $A = A(x)$ with time independent coefficients ([6]). We make use of these facts later on in our proofs.

- The restriction to the upper-half space in theorem 3 is just for technical reasons. In fact, one can state the theorem for a general Lipschitz domain in $\mathbb{R}^2$. The following argument shows that for the Dirichlet (regularity) problem.

Let D be the domain above the Lipschitz graph $t = \varphi(x)$ and u solves the Dirichlet problem $\mathrm{div}(A(x)\nabla u) = 0$, $u(x, \varphi(x)) = f(x)$. Define $\Phi(x,t) = (x, t - \varphi(x))$ and set $\widetilde{u}(\Phi(x,t)) = u(x,t)$. It is not difficult to check that $\widetilde{u} : \mathbb{R}^2_+ \to \mathbb{R}^1$ is a solution to $\mathrm{div}(\widetilde{A}\nabla\widetilde{u}) = 0$, $\widetilde{u}(x,0) = f$, where

$$\widetilde{A}(x) = \begin{pmatrix} 1 & 0 \\ -\varphi'(x) & 1 \end{pmatrix} A(x) \begin{pmatrix} 1 & -\varphi'(x) \\ 0 & 1 \end{pmatrix}.$$

In particular, we have shown that Theorem 3 implies parts one and two of Theorem 2.

Unfortunately, at this moment we cannot claim part three of our Theorem 2 for general divergence form elliptic equations with time independent coefficients. Our proof for harmonic functions is based on Brown's duality technique for the double-layer potential, which does not seem to generalize in the setting of Theorem 3. Thus we pose the following:

Question 2. *Assume that $A = A(x)$ is a real, symmetric elliptic matrix. Prove that the Dirichlet problem in the upper half-space $\mathbb{R}^2_+$*

$$(D_\alpha)\left|\begin{array}{c} \mathrm{div}(A\nabla u) = 0 \quad \text{for } t > 0, \\ u|_{\partial D} = f \in C^\alpha(\mathbb{R}^1) \end{array}\right.$$

is solvable with $\|u\|_{C^\alpha(\mathbb{R}^2_+)} \leq C\|f\|_{C^\alpha(\mathbb{R}^1)}$ for $\alpha < 1/2 + \varepsilon(A)$.

We now state the following result, which gives a connection between the Neumann Hardy spaces and the usual atomic Hardy spaces. This is an extension of Theorem 2.3.18 in [5], for the case $p < 1$.

Theorem 4. *Let $D \subset \mathbb{R}^2$ is a domain above Lipschitz graph and u is a harmonic function on D that satisfies $u|_{\partial D} \in H^p_1(\partial D)$, $2/3 - \varepsilon < p \leq 1$. Then*

$$\left\|\frac{\partial u}{\partial N}\right\|_{H^p(\partial D)} \lesssim \|u\|_{H^1_p(\partial D)}.$$

Conversely, given $f \in H^p(\partial D)$, there exists a harmonic function u, such that $\dfrac{\partial u}{\partial N} = f$ and

$$\|u\|_{H^p_1(\partial D)} \lesssim \|f\|_{H^p(\partial D)}.$$

1.2. **Biharmonic functions.** For the biharmonic equation, we consider the Dirichlet problem

$$BD_p\left|\begin{array}{cc} \Delta^2 u & = 0 \\ u|_{\partial D} & = f_0 \\ \dfrac{\partial u}{\partial N}|_{\partial D} & = \sum\limits_{j=1}^n f_j N_j, \\ \|M(\nabla u)\|_{L^p(\partial D)} & < \infty, \end{array}\right.$$

where $N_1, N_2, \dots, N_n$ are the components of the normal vector and $f_0, f_1, f_2, \dots, f_n$ satisfy the compatibility condition $(f_0, f_1, f_2, \dots, f_n) \in WA_2(\partial D)$ (cf. [10]). The regularity problem is

$$BR_p \left| \begin{array}{ll} \Delta^2 u & = 0 \\ D_2 u|_{\partial D} & = f \\ \sum_{j=1}^{n-1} \langle \nabla_{T_j}, \nabla D_j u \rangle|_{\partial D} & = g \\ \|M(\nabla\nabla u)\|_{L^p(\partial D)} & < \infty \end{array} \right.$$

The L^2 theory for these problems (with the necessary adjustments for the order of the derivatives) is very similar to the harmonic case and we present it in Section 2. We have the following results in two dimensions.

Theorem 5. *There exists an $\varepsilon = \varepsilon(D) > 0$, such that if $0 < \alpha < 1/2 + \varepsilon$, $f_1, f_2 \in C^\alpha(\partial D) \cap L^2(\partial D)$, then the unique L^2 solution to BD_α satisfies $\nabla u \in C^\alpha(D)$. In fact,*

$$\|\nabla u\|_{C^\alpha(D)} + \sup_{X \in D} \operatorname{dist}(X, \partial D)^{-1-\alpha} |u(X) - u(X^*) - \langle X - X^*, \nabla u(X^*) \rangle| \leq C \sum_{j=1}^{2} \|f_j\|_{C^\alpha(D)},$$

where C is a constant depending only on the Lipschitz nature of D and X^* is the projection of X along the "time" axis onto ∂D. Moreover the range $\alpha < 1/2 + \varepsilon$ is sharp.

Theorem 6. *There exists an $\varepsilon = \varepsilon(D) > 0$, such that the regularity problem BR_p with $2/3 - \varepsilon < p < 2 + \varepsilon$, $(f, g) \in H_1^p(\partial D) \times H^p(\partial D)$ has unique (up to a constant) solution. Moreover the estimate*

$$\|M(\nabla\nabla u)\|_{L^p(\partial D)} \leq C \|\nabla_{T_1} f\|_{H^p(\partial D)} + \|g\|_{H^p(\partial D)},$$

holds with a constant C depending only on the Lipschitz nature of D. The range $2/3 - \varepsilon < p$ is sharp.

2. Preliminaries

We separate this section into two parts - about harmonic and biharmonic functions respectively. The corresponding equations exhibit some common features like the L^2 theory, but there are some dissimilarities as well. We try to present the similarities in the technically simpler harmonic context and we briefly outline some specifics for the biharmonic operator. We constantly refer in the text to the papers [9], [10] for the necessary background results. Since the two dimensional case is of utmost interest to us, we sometimes avoid the explicit formulas (with the inevitable technicalities that arise) for dimensions higher than two.

2.1. Harmonic functions.

Let $D \subseteq \mathbb{R}^n$ be a Lipschitz domain, such that D, D^c are connected. For technical reasons, we restrict our attention to the case of domains above Lipschitz graphs, i.e.

$$D = \{(x,t) : t > \varphi(x)\}, \quad \varphi : \mathbb{R}^{n-1} \to \mathbb{R}^1,$$
$$|\varphi(x) - \varphi(y)| \leq M|x - y|.$$

The surface measure on ∂D is defined via the usual $d\sigma = \sqrt{1 + |\nabla\varphi|^2} dx$.

Following [3], we introduce the atomic Hardy spaces $H^p(\partial D)$ for $1 \geq p > (n-1)/n$. First an $H^p(\partial D)$ atom is a function $a : \partial D \to \mathrm{R}$, such that

$$\begin{aligned}
&\mathrm{supp}(a) \subseteq B(Q,d) = \{P \in \partial D : |P-Q| < d\}, \\
&\int a d\sigma = 0, \\
&\|a\|_{L^2(\partial D)} \leq C d^{(n-1)(1/2-1/p)}.
\end{aligned}$$

Then,

$$\begin{aligned}
&H^p(\partial D) = \{\sum \lambda_i a_i : \sum |\lambda_i|^p < \infty\} \\
&\|f\|_{H^p(\partial D)} = \inf_{f=\sum \lambda_i a_i} (\sum |\lambda_i|^p)^{1/p}.
\end{aligned}$$

where a_i are $H^p(\partial D)$ atoms.

We also define $H^p_1(\partial D)$ atoms by requiring that

$$\begin{aligned}
&\mathrm{supp}(a) \subseteq B(Q,d) = \{P \in \partial D : |P-Q| < d\}, \\
&\|\nabla_T a\|_{L^2(\partial D)} \leq C d^{(n-1)(1/2-1/p)},
\end{aligned}$$

where $\nabla_{T_j} u = \langle T_j, \nabla u \rangle = \left(\frac{\partial}{\partial x_j} + \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial x_n} \right) u$. The space $H^p_1(\partial D)$ of distributions with one derivative in $H^p(\partial D)$ may be defined as the l^p span of such atoms. It is well known that $H^1(\partial D) \subset L^1(\partial D)$, while the spaces $H^p(\partial D)$, $p < 1$ contain non-integrable distributions. Sometimes, we will abuse notations by writing $a(x)$, instead of $a(x, \varphi(x))$. Observe that $\nabla_{T_j} a = \frac{\partial}{\partial x_j} a(x, \varphi(x))$.

We also define the (homogeneous) Hölder spaces $C^\alpha(\partial D)$, $0 < \alpha \leq 1$ by

$$C^\alpha(\partial D) = \left\{ f : \partial D \to \mathbb{R}^1 : \|f\|_{C^\alpha(\partial D)} = \sup_{Q \neq P} \frac{|f(P) - f(Q)|}{|Q-P|^\alpha} \right\}.$$

We remark that the Hölder spaces C^α and the Hardy spaces H^p, $p = \frac{n-1}{n-1+\alpha}$ can be paired in the sense that every element in one of them defines via integration a continuous linear functional on the other (cf. [11] p. 130). Let

$$\Gamma(x) = \begin{cases} \frac{|x|^{2-n}}{(n-2)\omega_n} & n > 2 \\ \frac{1}{2\pi} \ln |x| & n = 2 \end{cases}$$

be the fundamental solution for the Laplace's equation in $\mathbb{R}^n$. Define the single and double layer potentials $\mathcal{S}$ and $\mathcal{K}$ by

$$\begin{aligned}
\mathcal{S}(f)(X) &= \text{p.v.} \int_{\partial D} \Gamma(X-Q) f(Q) d\sigma(Q), \quad x \in \mathbb{R}^n \setminus \partial D \\
\mathcal{K}(f)(X) &= \text{p.v.} \int_{\partial D} \frac{\partial \Gamma}{\partial N_Q}(X-Q) f(Q) d\sigma(Q), \quad x \in \mathbb{R}^n \setminus \partial D
\end{aligned}$$

We also define the formal adjoint of $\mathcal{K}$

$$K^*(f)(X) = \text{p.v.} \int_{\partial D} \frac{\partial \Gamma}{\partial N_X}(X-Q) f(Q) d\sigma(Q).$$

2.2. **Biharmonic functions.** We start with the L^2 theory for the biharmonic equation, due to Kenig and Verchota [7] (see also Theorem 3.7 in [9]).

Proposition 1. *Let $D \subset \mathbb{R}^n$ be a Lipschitz domain. Then there exists an $\varepsilon > 0$ depending only on the Lipschitz nature of D, such that for $p: \quad 2-\varepsilon < p < 2+\varepsilon$ the equation*

$$\left| \begin{array}{ll} \Delta^2 u & = 0 \\ u|_{\partial D} & = f \\ \langle N, \nabla u \rangle|_{\partial D} & = g \\ \|M(\nabla u)\|_{L^p(\partial D)} & < \infty \end{array} \right.$$

is uniquely solvable. In addition, there are the estimates

- $|\nabla u(X)| \lesssim \text{dist}(X, \partial D)^{-(n-1)/p}$,
- $\|M(\nabla u)\|_{L^p \partial D)} \lesssim \|\nabla u\|_{\partial D}$.

There is a also the regularity result, which we now state(cf. Theorem 4.6, [9]).

Proposition 2. *Let $D \subset \mathbb{R}^n$ be a Lipschitz domain. Then there exists an $\varepsilon > 0$ depending only on the Lipschitz nature of D, such that for $p: \quad 2-\varepsilon < p < 2+\varepsilon$ the equation*

$$\left| \begin{array}{ll} \Delta^2 u & = 0 \\ D_n u|_{\partial D} & = f \\ \sum_{j=1}^{n-1} \langle \nabla_{T_j}, \nabla D_j u \rangle|_{\partial D} & = g \\ \|M(\nabla\nabla u)\|_{L^p(\partial D)} & < \infty \end{array} \right.$$

is uniquely solvable. In addition, there are the estimates

- $|\nabla\nabla u(X)| \lesssim \text{dist}(X, \partial D)^{-(n-1)/p}$,
- $\|M(\nabla\nabla u)\|_{L^p \partial D)} \lesssim \sum_j (\|\nabla_{T_j} f\|_{L^p(\partial D)} + \|g\|_{L^p(\partial D)})$.

The fundamental solution of the biharmonic equation in two dimensions is

$$\Sigma(X) = \frac{1}{8\pi} |X|^2 \ln |X|.$$

Based on the L^2 theory, one is able to define the Green's function as follows. Let X be inside the domain and fix a point $X_0 \notin \partial D$. Let

$$\begin{aligned} f_0(Q) &= \Sigma(X-Q) - \Sigma(X_0 - Q) = \frac{1}{8\pi}(|X-Q|^2 \ln |X-Q| - |X_0 - Q|^2 \ln |X_0 - Q|), \\ f_j(Q) &= D_j(f_0(Q)), \quad j \in \{1,2\}. \end{aligned}$$

Consider then the unique solution $\gamma_X(Y)$ to the problem

$$\left| \begin{array}{ll} \Delta^2 u & = 0, \\ u|_{\partial D} & = f_0, \\ \nabla u|_{\partial D} & = \{f_1, f_2\}. \end{array} \right.$$

Since $\nabla_{T_j} f_j \sim |X-Q|^{-1}$, we have by the L^2 regularity result that $\|M(\nabla\nabla\gamma_X)\|_{L^2(\partial D)} \lesssim 1$. The Green's function can be defined as $G(X,Y) = \Sigma(X-Y) - \Sigma(X_0 - Y) - \gamma_X(Y)$. Observe that since two tangential derivatives of the explicit quantity $\Sigma(X-Y)-\Sigma(X_0-Y)$ also belong to $L^2(\partial D)$, we can conclude

$$\|M(\nabla\nabla G(X,\cdot))\|_{L^2(\partial D)} \lesssim 1. \tag{1}$$

Later on, we will be able to show a much stronger estimate than (1) when the integration is over dyadic pieces away from the origin.

Integration by parts shows that one has the following representation formula for biharmonic functions (cf. [10]):

$$u(x) = \int_{\partial D} u(Q)\frac{\partial}{\partial N_Q}\Delta_Q G(X,Q) + \int_{\partial D} \frac{\partial u}{\partial N_Q}(Q)\Delta_Q G(X,Q). \tag{2}$$

3. Existence and Uniqueness for harmonic functions

In this section, we prove existence and uniqueness for R_p and N_p, and Dirichlet problem with Hölder data in Theorem 3. Our plan is as follows. First, we show that the regularity and Neumann problems are equivalent, i.e. if one can solve the regularity problem uniquely in $H^p_1(\partial D)$, then one can solve uniquely the Neumann problem in $H^p(\partial D)$ and vice versa. Secondly, we show that the R_p is solvable in $H^p_1(\partial D)$, $p>2/3-\varepsilon$ and finally we prove uniqueness for R_p. The uniqueness result will be almost automatic in view of Lemma 2.2 in [1] and the usual L^q uniqueness result for the Dirichlet problem, $q\geq 2-\varepsilon$. Finally, we show the existence and uniqueness result for the Dirichlet problem with $C^\alpha(\partial D)$ data in Theorem 2.

3.1. **Equivalence for the regularity and Neumann problem.**

Let u satisfy the Neumann problem N_p with data $f\in H^p(\partial D)$. By the properties of $H^p(\partial D)$, we consider without loss of generality only smooth compactly supported data f. Let $u_{-1}(x,t)=\int_0^t u(x,z)dz$. Define

$$v = a(x)(u_{-1})_x + b(x)(u_{-1})_t.$$

It is not difficult to check that

$$v_x(x,0) = (a(u_{-1})_x + b(u_{-1})_t)_x = -(b(u_{-1})_{xt} + c(u_{-1})_{tt}) = -bu_x - cu_t = f(x).$$

We observe that v satisfies

$$(R_p)\left|\begin{array}{ll} \operatorname{div}(\tilde{A}\nabla v) & = 0 \\ v(x,0) & = g(x) \end{array}\right. ,$$

where $\tilde{A} = \dfrac{A}{ac-b^2}$ and g is the (unique) function with $g'(x)=f(x), \int g = 0$.

Since A is an uniformly elliptic matrix with time independent coefficients, so is $\tilde{A}$. Note that starting with a solution of a Neumann problem, we have produced an L^2 solution to an associated regularity problem. Moreover, since $\nabla v = \begin{pmatrix} -b & -c \\ a & b \end{pmatrix}\nabla u$, one obtains pointwise equivalence $M(\nabla v)\sim M(\nabla u)$. Hence, if one can prove estimates for R_p

$$\|M(\nabla v)\|_{L^p(\partial D)} \lesssim \|g\|_{H^p_1(\partial D)},$$

they would imply the corresponding estimates for N_p

$$\|M(\nabla u)\|_{L^p(\partial D)} \lesssim \|f\|_{H^p(\partial D)}.$$

Thus, we have showed that if one can solve the regularity problem in $H^p_1(\partial D)$, then one can also solve the Neumann problem. The reverse implication can be proved by retracing back the argument above, so we omit the details.

We note that the equivalence of the regularity and Neumann problems in the sense described above is purely two dimensional phenomena. Actually, in the important case of the Laplace's equation, it is not difficult to check that u and v above are in fact conjugate harmonic functions and thus one cannot expect the equivalence to persist in higher dimensions. Actually, by the equivalence of the regularity and Neumann problem and the existence results of Theorem 2 (to be proved below), we establish Theorem 4.

3.2. **Solvability for the regularity problem in $H^p_1(\partial D)$.** By well known approximation techniques (see for example [5], section 1.10), it will suffice to prove the estimate

$$\|M(\nabla u)\|_{L^p} \leq C\|g\|_{H^p_1} \tag{3}$$

for solutions u of R_p corresponding to smooth matrices A and smooth data g, which are known to exist, as long as the constant C is independent of everything, but the ellipticity constant of A.

Thanks to the "atomic" nature of $H^p_1(\mathbb{R}^1)$, one can take g to be a H^p_1-atom. Indeed, if we assume (3) for atoms and take into account the p-subaditivity of the L^p quasi-norm ($p < 1$) we get for $g = \sum \lambda_i g_i$

$$\|M(\nabla u_g)\|_p^p \lesssim \sum |\lambda_i|^p \|M(\nabla u_{g_i})\|_p^p \lesssim \sum |\lambda_i|^p \lesssim \|g\|^p_{H^p_1}.$$

Simple dilation and translation argument allows us to reduce to the case of an unit atom, i.e.

1. supp $\quad g \subset [-1, 1]$,
2. $\|g\|_\infty, \|g'\|_\infty \lesssim 1$.

For $\tau \in (1/2, 1)$, consider the intervals $R^\tau_j = (2^j\tau, 2^{j+1}/\tau)$ and $R_j := R^1_j$. Let

$$q^\tau_j(x) = \begin{cases} 100(2^j\tau - x) & x < 2^j\tau, \\ 0 & 2^j\tau \leq x \leq 2^{j+1}/\tau, \\ 100(x - 2^{j+1}/\tau), & x \geq 2^{j+1}/\tau, \end{cases}$$

and $\Omega^\tau_j = \{(x,t) : t > q^\tau_j(x)\}$. Observe that since Ω^τ_j is a domain above Lipschitz graph, the L^2 theory for divergence form equations with time independent coefficients applies to it (see the discussion after Theorem 3). We have the following lemma.

Lemma 1. *Let u be the unique L^2 solution to the problem*

$$\left| \begin{array}{ll} \operatorname{div}(A\nabla u) & = 0 \\ u(x,0) & = g(x), \end{array} \right.$$

where g is an unit atom in $H^p_1(\mathbb{R}^1)$. Then there exists $\varepsilon > 0$, such that

$$\int\limits_{R_j} M(\nabla u)^2 \leq C_\varepsilon 2^{(-\varepsilon-2)j}.$$

Let us show that Lemma 1 implies (3). By Hölder's inequality,

$$\int_{R_j} M(\nabla u)^p \lesssim |R_j|^{1-p/2} \left(\int_{R_j} M(\nabla u)^2 \right)^{p/2} \lesssim 2^{j(1-p/2+p(-\varepsilon-2)/2)},$$

and for every $p > 2/3 - O(\varepsilon)$, the series $\sum_j \int_{R_j} M(\nabla u)^p$ converges.

Thus, it remains to prove Lemma 1.

Proof. (Lemma 1) We use the standard Cacciopoli type argument. By the L^2 theory for Ω_j^τ and since $R_j \subset \partial\Omega_j^\tau$, we derive

$$\int_{R_j} M(\nabla u)^2 \lesssim \int_{1/2}^{1} \int_{\partial\Omega_j^\tau} M(\nabla u)^2 d\tau \lesssim 2^{-j} \int_{\Omega_j} |\nabla u|^2, \tag{4}$$

where $\Omega_j = \Omega_j^{1/2}$. Break Ω_j into "good" and "bad" part, so that

$$G_j = \Omega_j \cap \{t \geq 2^j\}$$
$$B_j = \Omega_j \setminus G_j.$$

On the good part, we further decompose $G_j = \bigcup_{k=1}^{\infty} G_j^k$, so that $G_j^k = G_j \cap \{t \sim 2^{j+k}\}$. For each G_j^k, one applies the usual interior estimates for the solution (cf. [5]), to get

$$\int_{G_j^k} |\nabla u|^2 \leq 2^{-2(j+k)} \int_{G_j^k} |u|^2. \tag{5}$$

By the D_2 solvability, we can always estimate $\|u\|_{L^2(G_j^k)} \lesssim 2^{(j+k)/2}$, which would give the desired estimate, except fot the extra decay factor $2^{-\varepsilon j}$.

For the "bad" part, select an even function $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, so that supp $\psi \subset (1/4, 4) \times (0, 2)$ and $\psi(x,t) = 1$ for $(|x|, |t|) \in (1/2, 2) \times (0, 1)$. Denote $\psi_j := \psi(2^{-j} \cdot)$. Observe that since $u(x, 0) = 0$ on R_j, we may extend $u(x,t)$ for $t < 0$ across R_j as an *even* function. By the ellipticity of A and the divergence theorem, one then derives

$$\int_{B_j} |\nabla u|^2 \leq \int_{\mathbb{R}^2} \langle A\nabla(u\psi_j), \nabla(u\psi_j)\rangle \lesssim 2^{-j} \int_{\mathbb{R}^2} |\nabla u||u|\psi_j \lesssim 2^{-j} \|\nabla u\|_{L^2(C_j)} \|u\|_{L^2(C_j)}, \tag{6}$$

where $C_j \supset B_j$ is again a box with sides $\sim 2^j$.

From (6), R_2, D_2 solvability and by iterating (6) (we will get back to this point later on), we easily get the bound $\int_{\Omega_j} |\nabla u|^2 \leq C_\delta 2^{-j} 2^{\delta j}$ for all positive δ. This estimate, together with (4) imply Lemma 1 without the crucial term $2^{-\varepsilon j}$.

The usual approach to get the improvement $2^{-\varepsilon j}$ is to use Sobolev embedding $H^1(\mathbb{R}^n) \subset L^{2n/(n-2)}(\mathbb{R}^n)$, which unfortunately *fails* for dimension two. We use instead the following multiplicative variant of Sobolev embedding

$$\|u\|_{L^4(\mathbb{R}^2)} \leq \|u\|_{L^2(\mathbb{R}^2)}^{1/2} \|\nabla u\|_{L^2(\mathbb{R}^2)}^{1/2}. \tag{7}$$

We have the following proposition.

Proposition 3. *Suppose*

$$\|\nabla u\|^2_{L^2(Q_j)} \lesssim 2^{-j}\|u\|_{L^2(P_j)}\|\nabla u\|_{L^2(P_j)}, \tag{8}$$

$$\|M(u)\|_{L^{2-\delta}(\mathbb{R}^1)} \lesssim 1, \tag{9}$$

$$\|M(\nabla u)\|_{L^2(\mathbb{R}^1)} \lesssim 1. \tag{10}$$

where $Q_j \subset P_j$ are boxes with sides $\sim 2^j$. Then there exists $\varepsilon = O(\delta) > 0$, such that

$$\|\nabla u\|_{L^2(Q_j)} \lesssim 2^{(-1/2-\varepsilon)j}. \tag{11}$$

It is clear that a direct application of (11) gives the estimate for the "bad" part, while for the good part one applies (11) for G^k_j and summation in $k > 0$ then gives (5). Hence, to complete the proof of Lemma 1, it remains to prove Proposition 3. □

Proof. (Proposition 3)

Apply (7) for $u(x,t)\psi_j(x-x_0,t-t_0)$, where (x_0,t_0) are suitably chosen so that $\psi_j(x-x_0,t-t_0) = 1$ on P_j and supp $\psi_j(\cdot - x_0, \cdot - t_0) \subset 4P_j$. Cauchy-Schwartz and (9) yield

$$\int_{P_j} |u|^2 \lesssim \|u\|_{L^{2-\delta}(P_j)}\|u\|_{L^{(2-\delta)'}(P_j)} \lesssim 2^{j/(2-\delta)-j/2)}2^{2j/(2-\delta)'}\|u\|_{L^4(P_j)} \lesssim \text{by } (7)$$

$$\lesssim 2^{j(1/2+1/(2-\delta)')}(\|u\|^{1/2}_{L^2(4P_j)}\|\nabla u\|^{1/2}_{L^2(4P_j)} + 2^{-j/2}\|u\|_{L^2(4P_j)}) \lesssim$$

$$\lesssim 2^{j(1/2+1/(2-\delta)')}(2^{j/4}\|\nabla u\|^{1/2}_{L^2(4P_j)} + 1) \lesssim 2^{(5/4-O(\delta))j}\|\nabla u\|^{1/2}_{L^2(4P_j)} + 2^j 2^{-O(\delta)j},$$

where $O(\delta)$ is a positive number of the order of δ. From the preceding estimate and (8), we get

$$\int_{Q_j} |\nabla u|^2 \lesssim 2^{-(3/8+O(\delta))j}\|\nabla u\|^{5/4}_{L^2(4P_j)} + 2^{-j/2-O(\delta)j}\|\nabla u\|_{L^2(4P_j)} \tag{12}$$

We can perform now the following iteration procedure. Call $\lambda_j = \|\nabla u\|_{L^2(Q_j)}$ and $\mu_j = \|\nabla u\|_{L^2(4P_j)}$. Clearly (12) reads as

$$\lambda_j^2 \lesssim 2^{-3/8j}2^{-O(\delta)j}\mu_j^{5/4} + 2^{-j/2}2^{-O(\delta)j}\mu_j. \tag{13}$$

Since (10) allows us to bound $\mu_j \le C2^{j/2}$, one gets from (13) improvement for the bounds for λ_j, μ_j. We continue in that fashion and use the improved bounds back at (13). That way one gets an improvement at every step. One has

$$\lambda_j, \mu_j \lesssim 2^{-j/2-\varepsilon j}$$

for some $\varepsilon = O(\delta)$ and the proof is complete. □

3.3. **Uniqueness for the regularity problem in H^1_p.** The uniqueness result is almost automatic in two dimension due to the following lemma of Brown [1], which we state verbatim.

Lemma 2. *Let $D \subset \mathbb{R}^n$ be a connected Lipschitz domain and suppose that u is harmonic in D. Let X^* be a fixed point in D and suppose $u(X^*) = 0$. For $p < n-1$ and $p^* = (n-1)p/(n-1-p)$ we have*

$$\|M(u)\|_{L^{p^*}(\partial D)} \le C\|M(\nabla u)\|_{L^p(\partial D)}, \tag{14}$$

where the constant C depends only on the distance of X^* to the boundary, p and the Lipschitz character of ∂D.

Carefull inspection of the proof shows that one can relax the harmonicity assumptions on u, by requiring that u satisfies a divergence form equation. Indeed in the proof of (14), Brown uses interior estimates and the equivalence of the square function with the non-tangential maximal function in L^2, which are of course available for solutions of divergence form equations as well.

In contrast with the higher dimensionional case, where one needs to have an additional argument to prove uniqueness for N_p, $p > 1-\varepsilon$ (cf. [1]), the two dimensional uniqueness result follows from the Brown's lemma for the range $1 > p > 2/3-\varepsilon$ and uniqueness for D_q, $q > 2-\varepsilon$. To this end, assume that u solves R_p with zero data, such that $\|M(\nabla u)\|_{L^p(\mathbb{R}^1)} < \infty$. From (14) we get

$$\|M(u)\|_{L^{p^*}(\mathbb{R}^1)} \leq C\|M(\nabla u)\|_{L^p(\mathbb{R}^1)}.$$

Observe that since $1 > p > 2/3-\varepsilon$, we have $2 - O(\varepsilon) < p^* = p/(1-p) < \infty$ and therefore u solves a Dirichlet problem (with zero data), with $\|M(u)\|_{L^{p^*}(\mathbb{R}^1)} < \infty$. Thus $u = 0$ by the uniqueness for D_{p^*}.

3.4. **The Dirichlet problem with Hölder data.** We start with a lemma in the spirit of Theorem 3.4 in [1].

Lemma 3. *Let $D \subset \mathbb{R}^2$ be a star-like Lipschitz domain. There exist $\varepsilon = \varepsilon(D) > 0$, so that for $2/3 - \varepsilon < p < 1$ the maps*

$$\begin{aligned} \frac{1}{2}I \quad &+ \quad K^* : H^p(\partial D) \to H^p(\partial D) \\ \frac{1}{2}I \quad &- \quad K^* : H^p(\partial D) \to H^p(\partial D) \end{aligned}$$

are invertible.

Assuming the validity of Lemma 3, we can easily show part three of Theorem 2. Indeed, observe that $\frac{1}{2}I+K : C^\alpha(\partial D) \to C^\alpha(\partial D)$ is the adjoint map to $\frac{1}{2}I+K^* : H^p(\partial D) \to H^p(\partial D)$ and $-\frac{1}{2}I + K : C^\alpha(\partial D) \to C^\alpha(\partial D)$ is the adjoint map to $-\frac{1}{2}I + K^* : H^p(\partial D) \to H^p(\partial D)$, where $\alpha = 1/p - 1$. Thus

$$\begin{aligned} \frac{1}{2}I \quad &+ \quad K : C^\alpha(\partial D) \to C^\alpha(\partial D) \\ \frac{1}{2}I \quad &- \quad K : C^\alpha(\partial D) \to C^\alpha(\partial D) \end{aligned}$$

are invertible operators for $\alpha < \left(\frac{1}{2/3-\varepsilon} - 1\right) = 1/2 + O(\varepsilon)$. Thus, the solution to

$$(D_\alpha)\left| \begin{array}{c} \Delta u = 0 \\ u|_{\partial D} = f \in C^\alpha(\partial D) \end{array} \right.$$

is in $C^\alpha(\bar{D})$.

Remark There exists more direct arguments towards proving the C^α estimates of Theorem 2 similar to the one employed for the system of elastostatic ([2]), and later on for biharmonic functions ([10]).

The tools provided by Lemma 3 however allow for unified treatment of the problem at hand. More specifically, one reduces the question for solvability of the regularity and Neumann problems in $H^p \quad 2/3 - \varepsilon < p < 2 + \varepsilon$ to the invertibility of unitary perturbations of the adjoint of the double-layer potential in the same spaces.

Lemma 3 follows from the existence and uniqueness statements for R_p and N_p, $2/3 - \varepsilon < p < 1$ combined with the usual duality argument. The proof of Lemma 3 is essentially contained in [1] (cf. Proposition 3.1–3.5). One can easily adapt the argument there to the two dimensional case and the extended range of p's, thus we omit the details.

4. Existence and Uniqueness for biharmonic functions

In this section, we briefly sketch the proofs of Theorems 5, 6. We follow closely the ad-hoc approach of [10], which originated in [2]. As we have mentioned earlier, a more systematic way of studying the problem would be the method of Lemma 3, i.e. one could build an operator T, whose invertibility is equivalent to the solvability of BR_p and by duality to the Dirichlet problem with α-Hölder data in the sense of Theorem 5. This program was implicitely carried out in [9]. We choose however the direct method for sake of clarity of the exposition.

4.1. **Uniqueness for biharmonic functions.**

An easy adaptation of Lemma 2 gives the following.

Lemma 4. *Suppose that for a given* **biharmonic** *function u, there is $X^* \in D \subset \mathbb{R}^n$, such that $|\nabla u(X^*)| = 0$. For $p < n-1$ and $p^* = (n-1)p/(n-1-p)$ there is*

$$\|M(\nabla u)\|_{L^{p^*}(\partial D)} \leq C \|M(\nabla\nabla u)\|_{L^p(\partial D)}, \tag{15}$$

where the constant C depends only on the distance of X^* to the boundary, p and the Lipschitz character of ∂D.

Indeed, one uses the equivalence of the area integral and the non-tangential maximal function for biharmonic functions as in the proof for the harmonic case to show (15). Since the maximum principle of [10] is valid for dimensions two and three (but not for dimensions bigger than three), we argue as follows. Assume that a biharmonic function solves BR_p for $2/3 - \varepsilon < p < 1$, such that $D_2 u|_{\partial D} = 0$ and $\nabla_{T_1} D_1 u|_{\partial D} = 0$ and $\|M(\nabla\nabla u)\|_{L^p(\partial D)} < \infty$. Thus, after an eventual correction with a linear term, we may assume that $\nabla u|_{\partial D} = 0$ and $u|_{\partial D} = 0$. By Lemma 4, we conclude

$$\|M(\nabla u)\|_{L^{p^*}(\partial D)} \lesssim \|M(\nabla\nabla u)\|_{L^p(\partial D)} < \infty,$$

where $2 - \varepsilon < p^* < \infty$. By the maximum principle, we have uniqueness for the Dirichlet problem in L^{p^*}, hence $|\nabla u| = 0$.

4.2. **Existence for the biharmonic regularuty problem.**

For the existence part, we will use the following Cacciopolli type inequality.

Lemma 5. *Let $D \subset \mathbb{R}^n$ be a domain above Lipschitz graph. Let $\Omega_1 \subset \Omega_2 \subset D$ be bounded Lipschitz domains. Let $\Delta^2 u = 0$ in D with $M(\nabla^2 u) \in L^2(\partial D)$. Let ε is a small number as in Proposition 1. Let also $2 - \varepsilon < p < 2 + \varepsilon$ and $d = \mathrm{dist}(\Omega_1, D \setminus \Omega_2)$. Then there is a* *constant C, depending only on the Lipschitz constant and p, so that*

$$\begin{aligned}\int\limits_{\Omega_1} |\nabla^2 u|^2 dX \quad &\leq \quad C(\|\nabla u\|_{L^{p'}(\partial D\cap\partial\Omega_2)}\big\|M(\nabla^2 u)\big\|_{L^p(\partial D)} + \\ &+ \quad d^{-1}\|u\|_{L^{p'}(\partial D\cap\partial\Omega_2)}\big\|M(\nabla^2 u)\big\|_{L^p(\partial D)} + \\ &+ \quad d^{-1}\|\nabla u\|_{L^2(\Omega_2)}\big\|\nabla^2 u\big\|_{L^2(\Omega_2)} + d^{-2}\|u\|_{L^2(\Omega_2)}\big\|\nabla^2 u\big\|_{L^2(\Omega_2)}).\end{aligned}$$

Lemma 5 appears as Lemma 5.6 in [9] for dimension three. The proof though can be easily adapted to this generality. We state now our main estimate for "atomic" solutions.

Lemma 6. *Let a be an unit atom in $H^p(\partial D)$, $D \subset \mathbb{R}^2$, with $\int a(z)z = 0$. There exists $\varepsilon = \varepsilon(D) > 0$, such that the unique solution to the regularity problem*

$$BR_p \left| \begin{array}{ll} \Delta^2 u & = 0 \\ D_2 u|_{\partial D} & = 0 \\ \langle \nabla_{T_1}, \nabla D_1 u\rangle|_{\partial D} & = a \\ \|M(\nabla\nabla u)\|_{L^p(\partial D)} & < \infty \end{array}\right.$$

satisfies

$$\int\limits_{\partial D} M(\nabla^2 u)^p \lesssim 1, \tag{16}$$

for $2/3 - \varepsilon < p < 1$.

Remark The extra cancellation condition $\int a(z)zdz = 0$ is technical and it is possible to remove. Simple translation and dilation argument yields (16) for arbitrary atoms a in $H^p(\partial D)$ with the special cancelation property $\int a(z)zdz = 0$. Since such atoms suffice to span $H^p(\partial D)$, we get

$$\int\limits_{\partial D} M(\nabla^2 u_a)^p \lesssim \|a\|^p_{H^p(\partial D)}$$

In particular, we get (16) for atoms without the extra cancellation $\int a(z)zdz = 0$.

Proof. (Lemma 6) We make the standard assumption that the boundary is smooth, so that smooth solutions exist according to the classical theory. As usual, our estimates will not involve the smoothness constants and after one proves the result in that fashion, a standard approximation technique yields (16) for general Lipschitz domains.

Next, observe that the boundary conditions for BR_p imply that $\frac{d^2}{dx^2}u(x,\varphi(x)) = a(x)$ and therefore

$$u(x,\varphi(x)) = \int\limits_{-\infty}^{x}\int\limits_{-\infty}^{y} a(z)dzdy.$$

By support considerations and since $\int a = 0$, $\int a(z)zdz = 0$, we get $u(x,\varphi(x)) = 0$, for $x > 2$ and $u(x,\varphi(x)) = 0$, for $x < -2$. Thus, u also satisfies a *Dirichlet* type boundary conditions

$$\left| \begin{array}{ll} \Delta^2 u & = 0 \\ D_2 u|_{\partial D} & = 0 \\ u(x,\varphi(x)) & = \int_{-\infty}^{x}\int_{-\infty}^{y} a(z)dzdy. \end{array}\right.$$

The advantage of casting u as a solution to both Dirichlet and regularity type problems will be seen later on in the proof.

We first estimate (16) for x-small. By Hölder and L^2 regularity

$$\left(\int_{\partial D\cap\{(x,\varphi(x)):|x|<100\}} M(\nabla^2 u)^p\right)^{1/p} \lesssim \left(\int_{\partial D} M(\nabla^2 u)^2\right)^{1/2} \lesssim 1.$$

For every point $X \in \partial D$ fix a right cone $\Gamma(X)$ opening upward with axis along the "time" axis and sides having slopes $100\|\varphi'\|_\infty$. Define the auxilliary maximal functions

$$\begin{aligned} M_1(\nabla^2 u)(X) &= \sup_{\Gamma(X)\cap\Gamma} |\nabla^2 u|, \\ M_2(\nabla^2 u)(X) &= \sup_{\Gamma(X)\setminus\Gamma} |\nabla^2 u|. \end{aligned}$$

M_1 measures the behavior of the solution away from the boundary and is somewhat easier to control, while M_2 captures the behavior of the solution close to the boundary. It is clear that $M \lesssim M_1 + M_2$.

Since $D_2 u = 0$ and $\nabla u|_{\partial D} \in L^{2-\varepsilon}(\partial D)$, we deduce from the L^2 Dirichlet theory

$$\|M(\nabla u)\|_{L^{2-\varepsilon}(\partial D)} \lesssim 1.$$

Thus, interior estimates imply

$$\begin{aligned} |\nabla u(X)| &\lesssim \operatorname{dist}(X,\partial D)^{-1/(2-\varepsilon)}, \\ |\nabla\nabla u(X)| &\lesssim \operatorname{dist}(X,\partial D)^{-1-1/(2-\varepsilon)}. \end{aligned}$$

Therefore for $|Q| > 100$, we infer

$$M_1(\nabla^2 u)(Q) \lesssim |Q|^{-1-1/(2-\varepsilon)} \in L^{2/3+O(\varepsilon)}(\partial D). \tag{17}$$

For $R > 10$ and $1 \le \tau \le 2$ define the Carleson region Ω_τ^R above $Z_R = \{(x,\varphi(x)) : |x| \sim R\}$

$$\Omega_\tau = \Omega_\tau^R = \{(x,t) : R/\tau \le |x| \le R\tau, \varphi(x) < t < 100\tau\|\varphi'\|_\infty\}.$$

By Hölder's inequality, we have

$$\int_{\partial D\cap\partial\Omega_1} M(\nabla^2 u)^p \lesssim R^{1-p/2}\left(\int_{\partial D\cap\partial\Omega_1} M(\nabla^2 u)^2\right)^{p/2}.$$

Hence to show (16) it will be enough to prove

$$\int_{\partial D\cap\partial\Omega_1} M(\nabla^2 u)^2 \lesssim R^{-2-\varepsilon},$$

for some positive $\varepsilon > 0$.

From the L^2 regularity result in Ω_τ^R, we have

$$\int_{\partial\Omega_\tau\cap\partial D} M_2(\nabla\nabla u)^2 \lesssim \int_{\partial\Omega_\tau\setminus\partial D} |\nabla\nabla u|^2 + \int_{\partial\Omega_\tau\cap\partial D} |\nabla_{T_1}\nabla u|^2.$$

Averaging in τ and $\nabla_{T_1}\nabla u|_{\partial\Omega_t\cap\partial D}=0$ yield

$$\int\limits_{\partial\Omega_1\cap\partial D} M_2(\nabla\nabla u)^2 \lesssim R^{-1}\int\limits_{\Omega_2}|\nabla\nabla u|^2. \tag{18}$$

The boundary conditions $u|_{\partial\Omega_2\cap\partial D}=0$, $\nabla u|_{\partial\Omega_2\cap\partial D}=0$ and Lemma 5 imply

$$\int\limits_{\Omega_2}|\nabla^2 u|^2 \lesssim R^{-1}\|\nabla u\|_{L^2(\Omega_3)}\left\|\nabla^2 u\right\|_{L^2(\Omega_3)} + R^{-2}\|u\|_{L^2(\Omega_3)}\left\|\nabla^2 u\right\|_{L^2(\Omega_3)}, \tag{19}$$

for some eventually bigger box $\Omega_3\subset D$ still having diameter $\sim R$. Since $u|_{\partial\Omega_3\cap\partial D}=0$, one estimates

$$\|u\|_{L^2(\Omega_3)}\lesssim R\|\nabla u\|_{L^2(\Omega_3)}. \tag{20}$$

Combining (18),(19), (20) with the obvious $\|\nabla^2 u\|_{L^2(\Omega_3)}\lesssim R^{1/2}\|M_2(\nabla^2 u)\|_{L^2(\partial\Omega_3\cap\partial D)}$ yield

$$\int\limits_{\Omega_2}|\nabla^2 u|^2 \lesssim R^{-1/2}\|\nabla u\|_{L^2(\Omega_3)}\left(\int\limits_{\partial\Omega_3\cap\partial D} M_2(\nabla^2 u)^2\right)^{1/2} \tag{21}$$

$$\int\limits_{\partial\Omega_1\cap\partial D} M_2(\nabla^2 u)^2 \lesssim R^{-3/2}\|\nabla u\|_{L^2(\Omega_3)}\left(\int\limits_{\partial\Omega_3\cap\partial D} M_2(\nabla^2 u)^2\right)^{1/2} \tag{22}$$

As usual, one has $\|\nabla u\|_{L^2(\Omega_3)}\lesssim R^{1/2}\|M(\nabla u)\|_{L^2(\partial\Omega_3\cap\partial D)}\lesssim R^{1/2}$ and by iterating (22), one gets for every $\delta>0$

$$\int\limits_{\partial\Omega_1\cap\partial D} M_2(\nabla^2 u)^2 \le C_\delta R^{-2+\delta}, \tag{23}$$

which barely fails to make $\int\limits_{\partial D} M_2(\nabla^2 u)^p$ convergent. One also has

$$\left\|\nabla^2 u\right\|_{L^2(\Omega_1)}\lesssim R^{1/2}\left\|M_2(\nabla^2 u)\right\|_{L^2(\Omega_1)}\le C_\delta R^{-1/2+\delta/2}. \tag{24}$$

Thus in order to get a better estimate we resort to (7). Write

$$\begin{aligned} \int\limits_{\Omega_3}|\nabla u|^2 &\lesssim \left(\int_{\Omega_3}|\nabla u|^{2-\varepsilon}\right)^{1/(2-\varepsilon)}\left(\int_{\Omega_3}|\nabla u|^{(2-\varepsilon)'}\right)^{1/(2-\varepsilon)'} \\ &\lesssim R^{1/2+1/(2-\varepsilon)'}\|M(\nabla u)\|_{L^{2-\varepsilon}(\partial D)}\left(\int\limits_{\Omega_3}|\nabla u|^4\right)^{1/4} \quad \text{by} \quad (7) \\ &\lesssim R^{1-O(\varepsilon)}\|\nabla u\|^{1/2}_{L^2(\Omega_4)}\|\nabla\nabla u\|^{1/2}_{L^2(\Omega_4)} + R^{1/2-O(\varepsilon)}\|\nabla u\|_{L^2(\Omega_4)}, \end{aligned} \tag{25}$$

where $\Omega_3\subset\Omega_4$ is still a domain with diameter $\sim R$. Since one can clearly derive (23), (24) for Ω_4 instead of Ω_1 (with eventually bigger constants), we use those with $\delta=\varepsilon/100$ to estimate the right hand side of (25). We get

$$\int_{\Omega_3} |\nabla u|^2 \lesssim R^{3/4-O(\varepsilon)} \|\nabla u\|^{1/2}_{L^2(\Omega_4)} + R^{1/2-O(\varepsilon)} \|\nabla u\|_{L^2(\Omega_4)}. \tag{26}$$

Iterate (26) to get $\|\nabla u\|_{L^2(\Omega_3)} \lesssim R^{1/2-O(\varepsilon)}$. Going back to (22) and using the newly obtained bound for $\|\nabla u\|_{L^2(\Omega_3)}$, we have

$$\int_{\partial\Omega_1\cap\partial D} M_2(\nabla^2 u)^2 \lesssim R^{-1-O(\varepsilon)} \left(\int_{\partial\Omega_3\cap\partial D} M_2(\nabla^2 u)^2 \right)^{1/2}. \tag{27}$$

Iterating (27) now gives the desired estimate

$$\int_{\partial\Omega_1\cap\partial D} M_2(\nabla^2 u)^2 \lesssim R^{-2-O(\varepsilon)}, \tag{28}$$

which completes the proof of the Lemma. □

We finish the proof of Theorem 6, based on Lemma 6.

Proof. (Theorem 6) Start with data $(f,g) \in H^p_1(\partial D) \times H^p(\partial D)$ in BR_p. Select a harmonic function h with Dirichlet data f. Define $H = h_{-1}$ to be the primitive of h. Stein's lemma and the regularity estimates in Theorem 2 imply

$$\left\| M(\nabla^2 H) \right\|_{L^p(\partial D)} \lesssim \|M(\nabla D_2 H)\|_{L^p(\partial D)} \lesssim \|M(\nabla h)\|_{L^p(\partial D)} \lesssim \|f\|_{H^p_1(\partial D)}. \tag{29}$$

By Theorem 4

$$\left\| \frac{\partial h}{\partial N} \right\|_{H^p(\partial D)} \lesssim \|f\|_{H^p_1(\partial D)}.$$

Lemma 6 then provides a biharmonic function v, with data $g - \dfrac{\partial h}{\partial N}$ such that

$$\left\| M(\nabla^2 v) \right\|_{L^p(\partial D)} \lesssim \left\| g - \frac{\partial h}{\partial N} \right\|_{H^p(\partial D)}. \tag{30}$$

Set $u = H + v$ to get a biharmonic function satisfying the desired boundary conditions. Combining (29), (30) yields the estimate

$$\left\| M(\nabla^2 u) \right\|_{L^p(\partial D)} \lesssim \|(f,g)\|_{H^p_1(\partial D)\times H^p(\partial D)}.$$

□

4.3. **Hölder continuity of solutions.** We will show Theorem 5 based on the estimates for solutions of the regularity problem, in particular for the Green's function.

Proof. (Theorem 5) Recall the representation formula for biharmonic solutions (2)

$$u(X) = \int_{\partial D} f_0(Q) \frac{\partial}{\partial N_Q} \Delta_Q G(X,Q) d\sigma(Q) + \int_{\partial D} \sum_{j=1}^{2} f_j(Q) N_j(Q) \Delta_Q G(X,Q) d\sigma(Q). \tag{31}$$

By rescaling and dilation, it will suffice to show

$$|u(X) - u(X^*) - \langle X - X^*, \nabla u(X^*)\rangle| \lesssim \sum_{j=1}^{2} \|f_j\|_{C^\alpha(\partial D)},$$

when $dist(X, \partial D) = 1$. To achieve that one obviously needs estimates for $\Delta G|_{\partial D}$.

Observe that by (28), (11) and the construction of the solution for the regularity problem (see the proof of Theorem 6), we can establish the following estimate for the Green's function of Δ^2 in D

$$\int\limits_{\partial D \cap |Q - X^*| \sim 2^k} M(\nabla^2 G(X,Q))dQ \lesssim 2^{-2k} 2^{-O(\varepsilon)k}, \tag{32}$$

when $dist(X, \partial D) = 1$ (cf. (3.5) in [10]).

For the first term in (31), the normal derivatives of the harmonic function $\Delta_Q G(X, \cdot)$ are converted into tangential derivatives for f_0 and one eventually replaces the harmonic function $\Delta_Q G(X, \cdot)$ by its conjugate harmonic functions (cf. [10], p. 395). The estimates that one needs are then the same ones as for the second term. We estimate the second term in (31) below.

Note that by considering $u(X) - u(X^*) - \langle X - X^*, \nabla u(X^*)\rangle$ instead of $u(X)$, we may assume without loss of generality that $f_0(X^*) = 0$ and $f_j(X^*) = 0$. Thus

$$\begin{aligned}
&\left|\int\limits_{\partial D} \sum_j f_j(Q) N_j(Q) \Delta_Q G(X,Q) d\sigma Q\right| = \\
&= \left|\int\limits_{\partial D} \sum_j (f_j(Q) - f_j(X^*)) N_j(Q) \Delta_Q G(X,Q) d\sigma Q\right| \\
&\lesssim \sum_j \|f_j\|_{C^\alpha(\partial D)} \int\limits_{\partial D} |Q - X^*|^\alpha |\Delta_Q G(X,Q)| d\sigma(Q) \lesssim \\
&\lesssim \sum_j \|f_j\|_{C^\alpha(\partial D)} \sum_{k=0}^{\infty} 2^{k/2 + \alpha k} \left(\int\limits_{|Q - X^*| \sim 2^k} |\Delta_Q G(X,Q)|^2 d\sigma(Q) \right)^{1/2} \lesssim \text{by} \quad (32) \\
&\lesssim \sum_j \|f_j\|_{C^\alpha(\partial D)} \sum_{k>0} 2^{(\alpha - 1/2 - O(\varepsilon))k} \lesssim \sum_j \|f_j\|_{C^\alpha(\partial D)},
\end{aligned}$$

provided $\alpha < 1/2 + O(\varepsilon)$. □

5. Counterexamples

In this section, we provide counterexamples to show that the statement of Theorem 2 (and consequently Theorem 3) is sharp. Since Theorems 5 and 6 provide an extension and essentially contain the results of Theorem 2, the harmonic functions considered below should be viewed also as biharmonic counterexamples showing the sharpness of the statements of Theorem 6 and Theorem 5.

Consider the domain

$$\Omega_\delta = \{z \in \mathcal{C} : |z| < 1, \quad 0 < \arg z < 2\pi/(1+\delta)\},$$

for some small $\delta > 0$. Observe that the domain has Lipschitz constant $O(1/\delta)$ and is very "non-convex" as $\delta \to 0$. We will show that the Dirichlet problem with Hölder data in $C^\alpha(\partial\Omega_\delta)$, $\alpha > (1+\delta)/2$ is not uniquely solvable.

Define the harmonic function

$$u(z) = \mathrm{Im} \quad z^{(1+\delta)/2},$$

with the obvious identification of the complex plane $\mathcal{C}$ with $\mathbb{R}^2$. It is easy to check that $\partial\Omega_\delta$ consist of two segments and an arc, so that u vanishes on the segments and $u \in C^1$ on the arc. Altogether, we have that u solves a Dirichlet problem on Ω_δ with C^1 data, while one obviously cannot control more than $\|u\|_{C^{(1+\delta)/2}(\Omega_\delta)}$. Thus, we have shown that part three of Theorem 2 is sharp.

Next, we invoke the equivalence results of Section 3 to conclude that since one cannot solve the Dirichlet problem with Hölder data, then the regularity and Neumann problem must fail to be solvable as well.

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