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Factoring the Adjoint and Maximal Cohen-Macaulay Modules over the Generic Determinant

Ragnar-Olaf Buchweitz
University of Toronto

Graham J. Leuschke
Syracuse University

FACTORING THE ADJOINT AND MAXIMAL COHEN–MACAULAY MODULES OVER THE GENERIC DETERMINANT

RAGNAR-OLAF BUCHWEITZ AND GRAHAM J. LEUSCHKE

Abstract. A question of Bergman [3] asks whether the adjoint of the generic square matrix over a field can be factored nontrivially as a product of square matrices. We show that such factorizations indeed exist over any coefficient ring when the matrix has even size. Establishing a correspondence between such factorizations and extensions of maximal Cohen–Macaulay modules over the generic determinant, we exhibit all factorizations where one of the factors has determinant equal to the generic determinant. The classification shows not only that the Cohen–Macaulay representation theory of the generic determinant is wild in the tame-wild dichotomy, but that it is quite wild: even in rank two, the isomorphism classes cannot be parametrized by a finite-dimensional variety over the coefficients. We further relate the factorization problem to the multiplicative structure of the Ext–algebra of the two nontrivial rank-one maximal Cohen–Macaulay modules and determine it completely.

1. Introduction

Let K be a field, $X = (x_{ij})$ the generic $(n \times n)$–matrix, whose entries thus form a family of n^2 indeterminates, and $S = K[x_{ij}]$, the polynomial ring over K in those variables. The determinant $\det X$ of X is a homogeneous polynomial of degree n with coefficients ± 1, and the hypersurface ring $R = S/(\det X)$ is a normal domain of dimension $n^2 - 1$.

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The classical adjoint, or adjugate, $\operatorname{adj}(X)$ of X is uniquely determined through either of the two matrix equations

$$\operatorname{adj}(X)X = (\det X)\cdot \mathrm{id}_n \quad \text{and} \quad X\operatorname{adj}(X) = (\det X)\cdot \mathrm{id}_n\,, \tag{1.0.1}$$

where id_n denotes the $(n\times n)$ identity matrix.

G.M. Bergman asks [3] whether the factorizations (1.0.1) and those arising from the transposes $X^T,\ \operatorname{adj}(X)^T$ are the only nontrivial matrix factorizations of $(\det X)\cdot \mathrm{id}_n$. More specifically, he inquires about possible refinements of the factorization (1.0.1) obtained by writing $\operatorname{adj}(X) = YZ$ for noninvertible $(n\times n)$–matrices Y and Z. He shows, for K an algebraically closed field of characteristic zero, that there are no such refinements when n is odd and that for n even the only possible refinements have either $\det Y = \det X$ or $\det Z = \det X$, up to multiplication by units in S. The proofs of [3] use a recent theorem by C. De Concini and Z. Reichstein [9] about maps between Grassmannians, generalizing the well-known topological theorem that the hairy sphere cannot be combed.

Here, in Section 2, we show that when n is even, the adjoint can in fact be factored nontrivially (over any commutative ring K). We give explicit matrix factorizations for each invertible *alternating* matrix A over S, based on the following key result:

Theorem 2.8. *Let U, A be $(n\times n)$–matrices over a commutative ring K with A alternating and $\det U$ a nonzerodivisor in K. There exist then unique alternating $(n\times n)$–matrices B_A and ${}_AB$ satisfying*

$$A\operatorname{adj}(U) = U^T B_A \qquad \textit{and} \qquad \operatorname{adj}(U)A = {}_ABU^T\,.$$

When n is even, there exist *invertible* alternating matrices A, so that $Y = A^{-1}U^T$, $Z = B_A$ gives one of the factorizations of $\operatorname{adj}(U)$ allowed by Bergman's result.

Corollary 2.14. *Let X be the generic square matrix of even size over the commutative ring K. Then the adjoint $\operatorname{adj}(X)$ factors nontrivially.*

The remainder of the paper has two main purposes: to show that the factorizations arising from this Corollary are the only factorizations possible with $\det Y = \det X$ or $\det Z = \det X$ up to units, and to cover, in reverse, the path by which we found them. To this end, we observe (Proposition 4.2) that a factorization of $\operatorname{adj}(X)$ into a product of square matrices Y and Z exhibits the cokernel of $\operatorname{adj}(X)$ as the middle term in a short exact sequence of R-modules, with ends the modules presented by Z and Y. Each of the three modules in this extension is a maximal Cohen–Macaulay R-module, and so is given by a matrix factorization of $\det X$. In Section 3 we briefly discuss the essential features that we will need from the theory of matrix factorizations.

Bergman's question can thus be rephrased in terms of extensions: Is it possible to write the cokernel of the adjoint as an extension of two maximal Cohen–Macaulay R-modules? When K is a unique factorization domain, W. Bruns has shown [4] (see also [6]) that up to isomorphism there are only three MCM R-modules of rank one, namely the cokernel of X, the cokernel of the transpose X^T, and R itself. This observation, together with a calculation in the divisor class group of R, already allows us to give a negative answer to the $n = 3$ case of Bergman's question over any UFD.

Theorem 4.5. *Let $X = (x_{ij})$ be the generic (3×3)–matrix over a unique factorization domain K. Then there are no nontrivial factorizations of* $\operatorname{adj}(X)$.

The general question of identifying whether and under what conditions a given module can be the middle term of a nonsplit short exact sequence is interesting and rarely addressed. We avoid it here as well. Looking instead for inspiration to Bergman's theorem we observe that the condition $\det Y = u \det X$, with u a unit, is equivalent to the module presented by Y, $\operatorname{cok} Y$, having rank one as an R-module. Given the classification of rank-one MCM R-modules, we obtain an explicit correspondence between nontrivial factorizations $\operatorname{adj}(X) = YZ$ with $\det Y = u \det X$ and short exact sequences

$$0 \longrightarrow \operatorname{cok} Y \longrightarrow Q \longrightarrow \operatorname{cok} X \longrightarrow 0$$

such that Q is a homomorphic image of R^n (see Lemma 4.10 and Proposition 4.7).

The classification of factorizations $\operatorname{adj}(X) = YZ$ with $\det Y = u \det X$ thus naturally leads to the calculation of Ext^1 for the rank-one MCM R-modules. In Section 5 we show that $\operatorname{Ext}^1_R(\operatorname{cok} X, \operatorname{cok} X) = 0$. This follows from a theorem of

R. Ile [18]; we reprove Ile's result, simplifying the proof slightly. We compute the minimal graded free resolution of $\operatorname{Ext}^1_R(\operatorname{cok} X, \operatorname{cok} X^T)$ in Theorem 7.4.

Sections 6 and 7 classify the nontrivial factorizations $\operatorname{adj}(X) = YZ$ with $\det Y = u \det X$ and the associated extensions. We show

Theorem 6.5. *Let* $\operatorname{adj}(X) = YZ$ *be a factorization of* $\operatorname{adj}(X)$ *with* $\det Y = u \det X$ *for some unit* u. *Then* $\operatorname{cok} Y \cong \operatorname{cok} X^T$ *and* $Y = JX^TZ$ *for an invertible* $(n \times n)$*–matrix* J. *Moreover, there exist then a unique invertible alternating* $(n \times n)$*–matrix* A *and a matrix* U *of the same size such that*

$$J^{-1} = A + X^T U \qquad \text{and} \qquad Z = B_A + U \operatorname{adj}(X)\,.$$

Two such factorizations $\operatorname{adj}(X) = JX^TZ$ *and* $\operatorname{adj}(X) = J'X^TZ'$ *give the same extension if and only if* $J^{-1} - J'^{-1} = X^TV$ *for some* $(n \times n)$*–matrix* V, *and in that case* $Z - Z' = V \operatorname{adj}(X)$.

As explained above, this result depends upon the structure of $\operatorname{Ext}^1_R(\operatorname{cok} X, \operatorname{cok} X^T)$ determined in Section 7.

Given the classification of the maximal Cohen–Macaulay R-modules of rank 1, one may ask for a description of the maximal Cohen–Macaulay modules of small rank in general. From the results of earlier sections, in Section 8 we make a first step in this direction by classifying all extensions of the rank-one maximal Cohen–Macaulay modules. In representation-theoretic terms, the class of such extension modules is (very) *wild*:

Corollary 8.9. *Let* $X = (x_{ij})$ *be the generic* $(n \times n)$*–matrix over a field* K, $n \geq 3$. *Let* $R = K[x_{ij}]/(\det X)$ *be the generic determinantal hypersurface ring. Then the rank-two orientable MCM* R*-modules cannot be parametrized by the points of any finite-dimensional algebraic variety over* K.

Finally, we construct a graded ring $\mathcal{E}$, the *stable* Ext-*algebra* of the rank-one maximal Cohen–Macaulay modules and describe its multiplication, given by the Yoneda product, explicitly. This algebra controls the higher-order extension theory of the rank-one MCM R-modules.

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2. The Adjoint of even size factors

In this section we give the promised factorizations of $\operatorname{adj}(X)$, after some background on determinants and derivations. Throughout, K denotes a commutative ring, $X = (x_{ij})$ the generic $(n \times n)$–matrix over K, and $S = K[x_{ij}]$.

2.1. We will use the following notation for minors of the generic matrix X: Let $[i_1 i_2 \cdots i_k \mid j_1 j_2 \cdots j_k]$ denote the (unsigned) determinant of the $(k \times k)$–submatrix of X that consists of the rows indexed $1 \le i_1 < \cdots < i_k \le n$, and of the columns indexed $1 \le j_1 < \cdots < j_k \le n$.

The symbol $[i_1 i_2 \cdots i_k \,\widehat{|}\, j_1 j_2 \cdots j_k]$ will denote the complementary minor, thus, the determinant of the $(n-k) \times (n-k)$–submatrix of X obtained by removing the rows indexed i_ν and the columns indexed j_ν. For consistency, the empty determinant $[\ \mid\ ]$, for $k = 0$, has value 1, whereas the empty complementary minor $[\ \widehat{|}\ ]$ equals $\det X$.

We extend the symbols $[? \mid ?]$ and $[? \,\widehat{|}\, ?]$ to not necessarily strictly increasing index sets by requiring them to be alternating in both the left and right arguments. In particular, each symbol vanishes if there is repetition of indices either before or after the vertical bar.

2.2. If U is any $(n \times n)$–matrix over some K-algebra R, then there exists a unique K-algebra homomorphism $\operatorname{ev}_U : S \longrightarrow R, x_{ij} \mapsto u_{ij}$, that transforms the entries of X to those of U. The evaluation homomorphism ev_U is compatible with the formation of minors, thus $[\cdots](U) = \operatorname{ev}_U([\cdots])$ represents the corresponding minor of the matrix U. Of particular interest is the case $U = X^T$. The corresponding evaluation homomorphism $\tau := \operatorname{ev}_{X^T}$ is then a K-algebra involution of S, given by $\tau(x_{ij}) = x_{ji}$, that fixes the determinant as $\det \tau X = \tau(\det X) = \det X$.

We write $I_t(U) \subseteq R$ for the ideal generated by all the $(t \times t)$–minors of U. The transpose of a matrix U will be denoted U^T. We also sometimes write $|U| := \det U$ to abbreviate.

Example 2.3. The $(i,j)^{\text{th}}$ entry of the adjoint matrix can be written as

$$\mathrm{adj}(X)_{ij} = (-1)^{i+j}[j \,\widehat{|}\, i] = (-1)^{i+j}[1\cdots\widehat{j}\cdots n \mid 1\cdots\widehat{i}\cdots n]\,.$$

2.4. Recall that a map $D : R \longrightarrow R$, on a not necessarily commutative ring R, is a *derivation* if it satisfies the Leibnitz rule $D(ab) = D(a)b + aD(b)$ for all elements $a, b \in R$.

For example, the *partial derivative* $\partial_{ij} = \frac{\partial}{\partial x_{ij}}$ with respect to the variable x_{ij} defines a derivation on S that is furthermore K-linear. These partial derivations form indeed a basis of the free S-module $\mathrm{Der}_K(S)$ of all K-linear derivations on S,

$$\mathrm{Der}_K(S) \cong \bigoplus_{1\le i,j\le n} S\partial_{ij}\,.$$

Now we state the facts on derivations and minors that we will use.

Lemma 2.5. *If R is a commutative ring, $D : R \longrightarrow R$ a derivation, and U an $(n\times n)$–matrix over R, then $D(\det U)$ can be written as a sum of determinants,*

$$D(\det U) = \sum_{i=1}^{n} \begin{vmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & & \vdots \\ D(u_{i1}) & \cdots & D(u_{in}) \\ \vdots & & \vdots \\ u_{n1} & \cdots & u_{nn} \end{vmatrix} = \sum_{i=1}^{n} \begin{vmatrix} u_{11} & \cdots & D(u_{1j}) & \cdots & u_{1n} \\ \vdots & & \vdots & & \vdots \\ u_{n1} & \cdots & D(u_{nj}) & \cdots & u_{nn} \end{vmatrix}\,.$$

Proof. This follows immediately from the Leibnitz rule for D applied to the complete expansion of the determinant. □

Lemma 2.6. *Let X be again the generic matrix and S the associated polynomial ring over K.*

(1) *For any pair of indices $1 \le i, j \le n$,*

$$\partial_{ij}(\det X) = \mathrm{adj}(X)_{ji}\,,$$

equivalently,

$$\mathrm{adj}(X)^T = (\partial_{ij}(\det X))_{ij}\,.$$

(2) *For any pair of indices $1 \le i, j \le n$,*

$$\sum_{\nu=1}^{n} x_{i\nu}\partial_{j\nu}(\det X) = \delta_{ij}\det X = \sum_{\nu=1}^{n} x_{\nu i}\partial_{\nu j}(\det X)\,,$$

where δ_{ij} is the Kronecker symbol*.*

(3) *For any indices* $1 \le i_1, i_2, \ldots, i_k \le n$ *and* $1 \le j_1, j_2, \ldots, j_k \le n$,

$$\partial_{i_1 j_1} \cdots \partial_{i_k j_k}(\det X) = (-1)^{i_1+\cdots+i_k+j_1+\cdots j_k}[i_1 \cdots i_k \,\widehat{|}\, j_1, \cdots, j_k]\,;$$

in particular, these terms vanish whenever there is a repetition among the i's or the j's.

Proof. Claim (1) follows from Lemma 2.5 with $D = \partial_{ij}$ and $U = X$. In view of (1), claim (2) is simply a reformulation of the equation (1.0.1) above. To see (3), apply first Lemma 2.5 or (1) to the generic matrix using the derivation $\partial_{i_k j_k}$, and then use induction on $k \ge 1$. □

2.7. We now use the "differential calculus" from above to establish two factorization results about products of the adjoint matrix with *alternating* matrices on one or both sides. Recall that an $(n \times n)$–matrix $A = (a_{kl})$ is alternating if $A^T = -A$ and the diagonal elements vanish, $a_{kk} = 0$ for each $k = 1, \ldots, n$. The latter condition is of course a consequence of the first as soon as 2 is a nonzerodivisor in K.

Theorem 2.8. *Let* U, A *be* $(n \times n)$*–matrices over a commutative ring* K*, with* A *alternating. The* $(n \times n)$*–matrix* $B_A = (b_{rs})$ *with entries from* $I_1(A) \cdot I_{n-2}(U) \subseteq K$*, given by*

$$b_{rs} = \sum_{k<l} a_{kl}(-1)^{r+s+k+l}[rs \,\widehat{|}\, kl](U)\,,$$

is then alternating as well and satisfies the matrix equation

$$A \operatorname{adj}(U) = U^T B_A\,. \tag{2.8.1}$$

If $\det U$ *is a nonzerodivisor in* K*, then* B_A *is the* unique *solution to this equation.*

Proof. As $[sr \,\widehat{|}\, kl] = -[rs \,\widehat{|}\, kl]$ and $[rr \,\widehat{|}\, kl] = 0$, the matrix $B = B_A$ is alternating. To verify that B satisfies (2.8.1), it suffices to establish the generic case, in which we replace K by S and U by X. Let E_{ij} denote the *elementary* $(n \times n)$–matrix with 1 at position (i, j) as its only nonzero entry. Recall that $E_{ab}E_{cd} = \delta_{bc}E_{ad}$ for any indices $1 \le a, b, c, d \le n$. As $\partial_{rk}\partial_{sl}(\det X) = (-1)^{r+s+k+l}[rs \,\widehat{|}\, kl]$ by Lemma 2.6(3), the right-hand side of (2.8.1) expands first as

$$
\begin{aligned}
X^T B &= \Big(\sum_{i,\nu} x_{\nu i} E_{i\nu}\Big)\Big(\sum_{\mu,j}\sum_{k<l} a_{kl}\partial_{\mu k}\partial_{jl}(\det X) E_{\mu j}\Big)\\
&= \sum_{k<l} a_{kl} \sum_{i,j}\Big(\sum_{\nu} x_{\nu i}\partial_{\nu k}\partial_{jl}(\det X)\Big) E_{ij}\,.
\end{aligned}
$$

The innermost sum can be simplified using first that partial derivatives commute, then applying the product rule, and finally invoking Lemma 2.6(2) together with the fact that $\partial_{jl}(x_{\nu i}) = \delta_{j\nu}\delta_{li}$. In detail, these steps yield the following equalities:

$$
\begin{aligned}
\sum_{\nu} x_{\nu i}\partial_{\nu k}\partial_{jl}(\det X) &= \sum_{\nu} x_{\nu i}\partial_{jl}\partial_{\nu k}(\det X)\\
&= \sum_{\nu}\partial_{jl}\big(x_{\nu i}\partial_{\nu k}(\det X)\big) - \sum_{\nu}\partial_{jl}(x_{\nu i})\partial_{\nu k}(\det X)\\
&= \partial_{jl}\Big(\sum_{\nu} x_{\nu i}\partial_{\nu k}(\det X)\Big) - \delta_{li}\sum_{\nu}\delta_{j\nu}\partial_{\nu k}(\det X)\\
&= \delta_{ik}\partial_{jl}(\det X) - \delta_{li}\partial_{jk}(\det X)\,.
\end{aligned}
$$

In light of this simplification, we may expand $X^T B$ further as follows:

$$
\begin{aligned}
X^T B &= \sum_{k<l} a_{kl}\sum_{i,j}\Big(\sum_{\nu} x_{\nu i}\partial_{\nu k}\partial_{jl}(\det X)\Big)E_{ij}\\
&= \sum_{k<l} a_{kl}\sum_{i,j}\Big(\delta_{ik}\partial_{jl}(\det X) - \delta_{li}\partial_{jk}(\det X)\Big)E_{ij}\\
&= \sum_{k<l} a_{kl}\sum_{j}\Big(\partial_{jl}(\det X)E_{kj} - \partial_{jk}(\det X)E_{lj}\Big)\\
&= \sum_{k<l} a_{kl}\Big(E_{kl}\sum_{j}\partial_{jl}(\det X)E_{lj} - E_{lk}\sum_{j}\partial_{jk}(\det X)E_{kj}\Big)\\
&= \sum_{k<l} a_{kl}\Big(E_{kl}\sum_{i,j}\partial_{ji}(\det X)E_{ij} - E_{lk}\sum_{i,j}\partial_{ji}(\det X)E_{ij}\Big)\\
&= \sum_{k<l} a_{kl}\big(E_{kl} - E_{lk}\big)\sum_{i,j}\partial_{ji}(\det X)E_{ij}\\
&= A\,\mathrm{adj}(X)
\end{aligned}
$$

with the last equality using that A is alternating, thus $A = \sum_{k<l} a_{kl}(E_{kl} - E_{lk})$, and that $\mathrm{adj}(X) = \sum_{i,j}\partial_{ji}(\det X)E_{ij}$, in view of Lemma 2.6(1).

The final assertion about uniqueness follows from (2.8.1) by multiplying from the left with $\operatorname{adj}(U)^T$ and using equation (1.0.1) to obtain

$$\operatorname{adj}(U)^T A \operatorname{adj}(U) = (\det U) \cdot B\,.$$

□

Corollary 2.9. *For U, A as in Theorem 2.8, there exists also an alternating $(n \times n)$–matrix ${}_AB$ so that*

$$\operatorname{adj}(U)A = {}_ABU^T\,.$$

Proof. Let B_A be the $(n \times n)$–matrix over $S = K[x_{ij}]$ given by Theorem 2.8, so that $A \operatorname{adj}(X) = X^T B_A$, and let $\tau = \operatorname{ev}_{X^T}$ be the involution introduced in 2.2. Clearly τ exchanges X and its transpose, and moreover, $\tau(\operatorname{adj}(X)) = \operatorname{adj}(X)^T$, in view of equation (1.0.1). Now

$$\begin{aligned}
A \operatorname{adj}(X) &= X^T B_A && \text{if, and only if,}\\
\tau(A)\tau(\operatorname{adj}(X)) &= \tau(X^T)\tau(B_A) && \text{if, and only if,}\\
\tau(A) \operatorname{adj}(X)^T &= X\tau(B_A) && \text{if, and only if,}\\
\operatorname{adj}(X)\tau(A)^T &= \tau(B_A)^T X^T\,.
\end{aligned}$$

As A and B_A are both alternating, so are $\tau(A)$ and $\tau(B_A)$, and the last equation is equivalent to

$$\operatorname{adj}(X)\tau(A) = \tau(B_A)X^T\,.$$

Interchanging the roles of A and $\tau(A)$, we have

$$\operatorname{adj}(X)A = \tau(B_{\tau(A)})X^T\,.$$

Put ${}_AB = \tau(B_{\tau(A)}$. □

We now investigate what happens when multiplying simultaneously from both left and right.

Proposition 2.10. *Let U, A be again $(n \times n)$–matrices over a commutative ring K, and let B_A be the matrix introduced in Theorem 2.8. For another alternating*

$(n\times n)$–matrix $A'=(a'_{uv})$, the $(n\times n)$–matrix $C=C_{A,A'}=(c_{wm})$ with entries from $I_1(A)\cdot I_{n-3}(U)\cdot I_1(A')\subseteq R$ given by

$$c_{wm}=\sum_{k<l,u<v}(-1)^{u+v+w+k+l+m}a_{kl}[uvw\widehat{\restriction}\,klm](U)a'_{uv}$$

satisfies

$$B_AA'=r\cdot\mathrm{id}_n+CU^T\qquad\text{and}\qquad A_{A'}B=r\cdot\mathrm{id}_n+U^TC\,,$$

where

$$r=-\sum_{k<l,u<v}(-1)^{u+v+k+l}a_{kl}[uv\widehat{\restriction}\,kl](U)a'_{uv}\;\in K\,.$$

Proof. It suffices again to verify the result for the generic matrix $U=X$, in which case we can employ once more the description of minors as given in Lemma 2.6(3). The straightforward calculation proceeds then as follows:

$$\begin{aligned}(B_AA'-r\cdot\mathrm{id}_n)_{ij}&=\sum_m\sum_{k<l}(-1)^{i+m+k+l}a_{kl}[im\widehat{\restriction}\,kl]a'_{mj}\\&\qquad+\delta_{ij}\sum_{k<l}\sum_{u<v}(-1)^{u+v+k+l}a_{kl}[uv\widehat{\restriction}\,kl]a'_{uv}\\&=\sum_{k<l}a_{kl}\left(\sum_m\partial_{ik}\partial_{ml}(\det X)a'_{mj}+\sum_{u<v}\partial_{uk}\partial_{vl}(\det X)a'_{uv}\delta_{ij}\right)\\&=\sum_{k<l,m}a_{kl}\left(\partial_{ik}\partial_{ml}(\det X)a'_{mj}+\sum_{u<v}\partial_{uk}\partial_{vl}\left(\partial_{im}(\det X)x_{jm}\right)a'_{uv}\right)\end{aligned}$$

where we have used 2.6(2) in the last step. Using the product rule twice together with $\partial_{ab}(x_{cd})=\delta_{ac}\delta_{bd}$, we find next

$$\begin{aligned}\partial_{uk}\partial_{vl}\left(\partial_{im}(\det X)x_{jm}\right)&=\partial_{vl}\partial_{im}(\det X)\delta_{uj}\delta_{km}+\partial_{uk}\partial_{im}(\det X)\delta_{vj}\delta_{lm}\\&\qquad+\partial_{uk}\partial_{vl}\partial_{im}(\det X)x_{jm}\,.\end{aligned}$$

Substituting and evaluating the Kronecker symbols yields

$$\begin{aligned}(B_AA'-r\cdot\mathrm{id}_n)_{ij}&=\sum_{k<l,m}a_{kl}\left(\partial_{ik}\partial_{ml}(\det X)a'_{mj}+\sum_{u<v}\partial_{uk}\partial_{vl}\left(\partial_{im}(\det X)x_{jm}\right)a'_{uv}\right)\\&=\sum_{k<l}a_{kl}\left(\sum_m\partial_{ik}\partial_{ml}(\det X)a'_{mj}+\sum_{j<v}\partial_{vl}\partial_{ik}(\det X)a'_{jv}\right.\\&\qquad\left.+\sum_{u<j}\partial_{uk}\partial_{il}(\det X)a'_{uj}+\sum_{u<v,m}\partial_{uk}\partial_{vl}\partial_{im}(\det X)a'_{uv}x_{jm}\right)\end{aligned}$$

The terms involving only second order derivatives of the determinant cancel. To see this, rename summation indices, and use that $\partial_{mk}\partial_{il}(\det X) = -\partial_{ml}\partial_{ik}(\det X)$ and that A' is alternating, whence its entries satisfy $a'_{mm} = 0, a'_{jm} = -a'_{mj}$. In detail,

$$\begin{aligned}
(B_A A' - r\,\mathrm{id}_n)_{ij} = \sum_{k<l} a_{kl} &\left(\sum_m \partial_{ml}\partial_{ik}(\det X)a'_{jm} + \sum_{j<m} \partial_{ml}\partial_{ik}(\det X)a'_{jm} \right.\\
&\left. + \sum_{m<j} \partial_{mk}\partial_{il}(\det X)a'_{mj} + \sum_{u<v}\sum_m \partial_{uk}\partial_{vl}\partial_{im}(\det X)a'_{uv}x_{jm} \right)\\
= \sum_m &\left(\sum_{k<l,u<v} a_{kl}\partial_{uk}\partial_{vl}\partial_{im}(\det X)a'_{uv} \right) x_{jm}\\
= \sum_m &\left(\sum_{k<l,u<v} (-1)^{u+v+i+k+l+m} a_{kl}[uvi \widehat{|} klm]a'_{uv} \right) x_{jm}\\
= (CX^T)_{ij}&
\end{aligned}$$

where we have evaluated the third order derivatives of the determinant according to 2.6(3).

For the other statement, we observe that

$$\begin{aligned}
A'{}_A B X^T &= A' \operatorname{adj}(X) A\\
&= X^T B_{A'} A\\
&= X^T(r \cdot \mathrm{id}_n + CX^T)\,,
\end{aligned}$$

so that

$$A'{}_A B = r \cdot \mathrm{id}_n + X^T C$$

as well. □

Combining the results from Theorem 2.8 and Proposition 2.10 yields the following.

Theorem 2.11. *Let U, A, A' be $(n \times n)$–matrices over a commutative ring K, with A, A'* alternating. *One then has an equality of matrices*

$$A \operatorname{adj}(U) A' = r \cdot U^T + U^T C U^T\,,$$

where r and $C = C_{A,A'}$ are as specified in Proposition 2.10. *In particular, $A\operatorname{adj}(U)A'$ is both left- and right-divisible by U^T.* □

Remark 2.12. The element $r \in I_1(A) \cdot I_{n-2}(U) \cdot I_1(A') \subseteq R$ is a "*half trace*" of both $B_A A'$ and $A_{A'}B$, as

$$\begin{aligned}\operatorname{tr}(B_A A') &= \sum_{k<l}\sum_{i,j} a_{kl}(-1)^{i+j+k+l}[ij \widehat{\restriction} kl]a'_{ji}\\ &= 2\sum_{k<l}\sum_{i<j} a_{kl}(-1)^{i+j+k+l}[ij \widehat{\restriction} kl]a'_{ji}\\ &= 2r\end{aligned}$$

invoking once again that A' is alternating. Equivalently, $\operatorname{tr}(CU^T) = (2-n)r$.

Remark 2.13. If $n = 2$, all expressions of the form $[uvw \widehat{\restriction} klm]$ vanish, so that Theorem 2.11 (for $U = X$) specializes to the easily established identity

$$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = -ab\begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix}.$$

If $n = 2m$ is even, then over any commutative ring there are *invertible* alternating matrices of size n. For example, the alternating "hyperbolic matrix" $\begin{pmatrix} 0 & \operatorname{id}_m \\ -\operatorname{id}_m & 0 \end{pmatrix}$ has determinant equal to 1 over any ring. The following corollary, immediate from Theorem 2.11, thus gives the factorizations promised in the Introduction.

Corollary 2.14. *If n is* even, *then the adjoint of the generic matrix admits non-trivial factorizations*

$$\operatorname{adj}(X) = YZ = Y'Z'$$

into products of $(n \times n)$–matrices over S with $\det Y = \det Z' = \det X$ up to units of S.

More precisely, any pair of invertible alternating $(n \times n)$–matrices A, A' over S gives rise to such factorizations. With r and C the data associated to A, A' as in

Proposition 2.10*, one may take*

$$Y = A^{-1}X^T \quad \textit{and} \quad Z = (r\cdot \mathrm{id}_n + CX^T)A'^{-1}\,,$$
$$Y' = A^{-1}(r\cdot \mathrm{id}_n + X^TC) \quad \textit{and} \quad Z' = X^TA'^{-1}\,.$$

3. Matrix Factorizations

The rest of this paper is devoted to interpreting the factorizations of Section 2 as extensions of maximal Cohen–Macaulay modules over the hypersurface ring $R = S/(\det X)$. Here we collect some preliminary material, including a brief résumé of the theory of matrix factorizations, after D. Eisenbud [10], and some convenient results on stable homomorphism modules and multilinear algebra.

Definition 3.1. Let S be a commutative Noetherian ring. A *matrix factorization* (φ, ψ, F, G) of an element $f \in S$ is a pair of homomorphisms between finitely generated free S-modules, $\varphi : G \longrightarrow F$ and $\psi : F \longrightarrow G$, satisfying $\varphi\psi = f\cdot \mathrm{id}_F$ and $\psi\varphi = f \cdot \mathrm{id}_G$. We sometimes suppress F and G from the notation and refer to the matrix factorization (φ, ψ).

3.2. Let (φ, ψ, F, G) be a matrix factorization of $f \in S$, and assume that f is a nonzerodivisor. Then we have exact sequences

$$(3.2.1)\qquad \begin{array}{c} 0 \longrightarrow G \xrightarrow{\varphi} F \longrightarrow \operatorname{cok}\varphi \longrightarrow 0 \\ 0 \longrightarrow F \xrightarrow{\psi} G \longrightarrow \operatorname{cok}\psi \longrightarrow 0\,. \end{array}$$

As $f\cdot F = \varphi\psi(F)$ is contained in the image of φ, the cokernel of φ is annihilated by f. Similarly, $f\cdot\operatorname{cok}\psi = 0$. Thus $\operatorname{cok}\varphi$ and $\operatorname{cok}\psi$ are naturally finitely generated modules over $R = S/(f)$. If we write $\overline{?} := ? \otimes_S R$ for reduction modulo f, then the sequence

$$(3.2.2)\qquad \cdots \xrightarrow{\overline{\psi}} \overline{G} \xrightarrow{\overline{\varphi}} \overline{F} \xrightarrow{\overline{\psi}} \overline{G} \xrightarrow{\overline{\varphi}} \overline{F}\ (\longrightarrow \operatorname{cok}\overline{\varphi} \longrightarrow 0\ )$$

is a complex of free R-modules that constitutes a free resolution of $\operatorname{cok}\varphi = \operatorname{cok}\overline{\varphi}$. In particular, $\operatorname{cok}\varphi$ has a periodic resolution of period at most 2.

The reversed pair (ψ, φ) is also a matrix factorization of f. Put $M = \operatorname{cok}\varphi$ and $N = \operatorname{cok}\psi$; then the resolution (3.2.2) exhibits N as a first syzygy of M over R and vice versa:

$$\begin{array}{c} 0 \longrightarrow N \longrightarrow \overline{F} \longrightarrow M \longrightarrow 0 \\ 0 \longrightarrow M \longrightarrow \overline{G} \longrightarrow N \longrightarrow 0 \end{array}$$

are exact sequences of R-modules. As a matter of notation, we write $M = \operatorname{cok}(\varphi, \psi)$ and $N = \operatorname{cok}(\psi, \varphi)$ to emphasize their provenance. There are two distinguished *trivial* matrix factorizations, namely $(1, f, S, S)$ and $(f, 1, S, S)$. Note that $\operatorname{cok}(1, f) = 0$, while $\operatorname{cok}(f, 1) \cong R$.

3.3. Suppose again that $f \in S$ is a nonzerodivisor. Then the free modules F and G in any matrix factorization (φ, ψ, F, G) of f have the same rank n, as can be seen from equation (3.2.1). The homomorphisms φ and ψ, then, can be represented by square matrices over S after choosing bases for F and G.

If in addition f is a prime element of S, so that R is an integral domain, then from $\varphi\psi = f \cdot \mathrm{id}_n$ it follows that both $\det \varphi$ and $\det \psi$ are, up to units, powers of f. Specifically, $\det \varphi = uf^k$ and $\det \psi = u^{-1} f^{n-k}$ for some unit $u \in S$ and $k \le n$. In this case the R-module $\operatorname{cok}(\varphi, \psi)$ has rank k, while $\operatorname{cok}(\psi, \varphi)$ has rank $n - k$. (To see this, localize at the prime ideal (f). Then over the discrete valuation ring $S_{(f)}$, φ is equivalent to $f \cdot \mathrm{id}_k \oplus \mathrm{id}_{n-k}$ and so $\operatorname{cok} \varphi$ has rank k over the field $R_{(f)}$.)

Definition 3.4. Given two matrix factorizations $(\varphi_1, \psi_1, F_1, G_1)$ and $(\varphi_2, \psi_2, F_2, G_2)$ of the same element $f \in S$, a homomorphism of matrix factorizations from (φ_1, ψ_1) to (φ_2, ψ_2) is a pair of homomorphisms of free modules $\alpha : F_1 \longrightarrow F_2$ and $\beta : G_1 \longrightarrow G_2$ rendering

$$\begin{tikzcd}
F_1 \arrow[r, "\psi_1"] \arrow[d, "\alpha"'] & G_1 \arrow[r, "\varphi_1"] \arrow[d, "\beta"] & F_1 \arrow[d, "\alpha"] \\
F_2 \arrow[r, "\psi_2"] & G_2 \arrow[r, "\varphi_2"] & F_2
\end{tikzcd}$$

commutative. Such a diagram induces a homomorphism of R-modules $\operatorname{cok}(\varphi_1, \psi_1) \longrightarrow \operatorname{cok}(\varphi_2, \psi_2)$, which we write as $\operatorname{cok}(\alpha, \beta)$.

Consider the pullback square

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & \operatorname{Hom}_S(F_1, F_2) \arrow[d, "?\varphi_1"] \\
\operatorname{Hom}_S(G_1, G_2) \arrow[r, "\varphi_2?"'] \arrow[ur, phantom, "\lrcorner"] & \operatorname{Hom}_S(G_1, F_2)\,.
\end{tikzcd}$$

The module $\operatorname{Hom}_S((\varphi_1, \psi_1), (\varphi_2, \psi_2)) := C$ consists of pairs

$$(\alpha, \beta) \in \operatorname{Hom}_S(F_1, F_2) \times \operatorname{Hom}_S(G_1, G_2)$$

so that $\alpha\varphi_1 = \varphi_2\beta$. There is a natural map $\mathrm{Hom}_S(F_1, G_2) \longrightarrow C$ sending $\gamma \in \mathrm{Hom}_S(F_1, G_2)$ to $(\varphi_2\gamma, \gamma\varphi_1)$, and an exact sequence

$$\mathrm{Hom}_S(F_1, G_2) \longrightarrow C \longrightarrow \mathrm{Hom}_R(M_1, M_2) \longrightarrow 0\,,$$

which is also exact at the left if φ_2 is injective, *e.g.* if f is a nonzerodivisor in S.

The two matrix factorizations are *equivalent* if there is a homomorphism of matrix factorizations (α, β) as above in which both α and β are isomorphisms of free modules. Direct sums of matrix factorizations are defined in the natural way, and we say that a matrix factorization (φ, ψ) is *reduced* provided it is not equivalent to a matrix factorization with a direct summand of the form $(f, 1)$. Similarly, (φ, ψ) is called *minimal* if it is not equivalent to one with a direct summand of the form $(1, f)$.

The following Proposition, while straightforward to verify, is key in our constructions of matrix factorizations.

Proposition 3.5. *Let* $(\alpha, \beta) : (\varphi_1, \psi_1, F_1, G_1) \longrightarrow (\varphi_2, \psi_2, F_2, G_2)$ *be a homomorphism of matrix factorizations of* $f \in S$*, set* $R = S/(f)$*, and put* $M_i = \mathrm{cok}(\varphi_i, \psi_i), N_i = \mathrm{cok}(\psi_i, \varphi_i)$ *for* $i = 1, 2$*. Then the bottom row of the pushout diagram of* R*-modules*

(3.5.1)
$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\mathrm{cok}(\alpha,\beta)"'] & \overline{G_1} \arrow[r] \arrow[d] & N_1 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M_2 \arrow[r] & Q \arrow[r] & N_1 \arrow[r] & 0 \end{tikzcd}$$

defines an element of $\mathrm{Ext}^1_R(N_1, M_2)$*, which is the image of* $\mathrm{cok}(\alpha, \beta)$ *under the natural surjection* $\mathrm{Hom}_R(M_1, M_2) \longrightarrow \mathrm{Ext}^1_R(N_1, M_2)$*. The module* Q *is again given by a matrix factorization, namely*

$$Q \cong \mathrm{cok}\left(\begin{pmatrix} \varphi_2 & \alpha \\ 0 & \psi_1 \end{pmatrix}, \begin{pmatrix} \psi_2 & -\beta \\ 0 & \varphi_1 \end{pmatrix}\right).$$

3.6. If, in the notation of 3.5, $\mathrm{cok}(\alpha, \beta)$ factors through a projective R-module, then the bottom row of (3.5.1) splits, and vice versa. In this case, $\mathrm{cok}(\alpha, \beta)$ factors through $\overline{G}_1$, and we have $Q \cong M_2 \oplus N_1$.

The main application we have in mind for matrix factorizations is their equivalence with maximal Cohen–Macaulay modules over a hypersurface ring.

Definition 3.7. Let R be a Noetherian ring and M a finitely generated R-module. Recall that M is a *Cohen–Macaulay* module provided $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = \dim M_\mathfrak{p}$ for each prime $\mathfrak{p} \in \operatorname{Spec} R$. In particular, M is *maximal Cohen–Macaulay* if M is Cohen–Macaulay and $\dim M = \dim R$.

3.8. To describe the connection between matrix factorizations and MCM modules, we let S be a regular ring over which all projective modules are free. (In general, one must replace F and G by S-projectives, and φ, ψ by appropriate linear maps.) Let $f \in S$ be a nonzero nonunit and set $R = S/(f)$. Given a matrix factorization (φ, ψ, F, G) of f, we have seen that $\operatorname{cok}(\varphi, \psi)$ has projective dimension 1 over S. By the Depth Lemma, we obtain

$$\operatorname{depth}_R \operatorname{cok}(\varphi, \psi)_\mathfrak{p} = \dim R_\mathfrak{p}$$

for each $\mathfrak{p} \in \operatorname{Spec} R$, so that $\operatorname{cok}(\varphi, \psi)$ is a MCM R-module.

Conversely, let M be a nonzero MCM R-module. Then $\operatorname{pd}_S M = 1$, so that M has a projective resolution of the form

$$0 \longrightarrow G \xrightarrow{\varphi} F \longrightarrow M \longrightarrow 0\,, \tag{3.8.1}$$

with G and F free S-modules of the same finite rank. As M is annihilated by f, the map of complexes from (3.8.1) to itself given by multiplication by f is homotopic to zero. Equivalently, there is a homomorphism $\psi : F \longrightarrow G$ so that $\varphi\psi = f \cdot \mathrm{id}_F$. Since φ is necessarily injective, we have $\psi\varphi = f \cdot \mathrm{id}_G$ as well. Thus (φ, ψ, F, G) is a matrix factorization of f with $\operatorname{cok}(\varphi, \psi) \cong M$.

The matrix factorization (φ, ψ, F, G) is reduced if and only if the R-modules M and N are *stable*, that is, have no nonzero free direct summand. Equivalently, no entry of φ or ψ is a unit.

Theorem 3.9 ([10, Theorem 6.3])**.** *Let S be a regular ring such that all projective S-modules are free and set $R = S/(f)$ for a nonzero nonunit f. The association*

$$(\varphi, \psi, F, G) \longleftrightarrow \operatorname{cok}(\varphi, \psi)$$

induces a bijection between equivalence classes of reduced matrix factorizations of f and isomorphism classes of stable MCM R-modules.

Among other things, this theorem implies that MCM modules over the ring R above have periodic resolutions of period at most 2. In particular, the modules $\mathrm{Ext}^i_R(M,N)$, for M a MCM module, are periodic in i. To make this notion more precise, as well as for later use, we recall the definition of "stable homomorphisms".

Definition 3.10. Let M and N be finitely generated modules over a ring R. Denote by $P(M,N)$ the set of R-homomorphisms from M to N that factor through a projective R-module, and put

$$\underline{\mathrm{Hom}}_R(M,N) = \mathrm{Hom}_R(M,N)/P(M,N)\,.$$

We call $\underline{\mathrm{Hom}}_R(M,N)$ the *stable* Hom*–module*. We also write $\underline{\mathrm{End}}_R(M)$ for $\underline{\mathrm{Hom}}_R(M,M)$, and refer to it as the *stable endomorphism ring*.

Note that $P(M,N)$ is the image of the natural homomorphism

$$q : N \otimes_R \mathrm{Hom}_R(M,R) \longrightarrow \mathrm{Hom}_R(M,N)$$

defined by $q(y\otimes f)(x) = y\cdot f(x)$ for $y\in N$, $f\in \mathrm{Hom}_R(M,R)$, and $x\in M$.

In order to have a uniform notation for the periodicity of Ext over hypersurface rings, we introduce the *ad hoc* notion of stable extension groups. See also [1].

Definition 3.11. Let M and N again be finitely generated modules over a ring R. Define the *stable* Ext *groups* of M by N by

$$\underline{\mathrm{Ext}}^i_R(M,N) = \begin{cases} \underline{\mathrm{Hom}}(M,N) & \text{if } i=0 \\ \mathrm{Ext}^i_R(M,N) & \text{if } i>0. \end{cases}$$

For MCM modules over a hypersurface ring, it follows from 3.6 and the structure of projective resolutions that the stable Ext groups are periodic:

Proposition 3.12. *Let M and N be finitely generated modules over a hypersurface ring R, with M MCM. Then $\underline{\mathrm{Ext}}^i_R(M,N)\cong \underline{\mathrm{Ext}}^{i+2}_R(M,N)$ for all $i\geq 0$.*

To facilitate explicit computations, we review some conventions from the dictionary translating matrices to multilinear algebra — and back. In the statements and proofs to follow, we will use these two languages interchangeably; while the latter is perhaps more elegant, the former makes for faster and more transparent calculations.

3.13. Let S be a ring. If F, G are finite free modules with, say, $\operatorname{rank} F = n$ and $\operatorname{rank} G = m$, then an element $a \in F \otimes_S G^*$ may be viewed as an $(n \times m)$–matrix over S. Namely, in terms of given ordered bases $(f_1, \dots, f_n)$ for F and $(g_1, \dots, g_m)$ for G, the element a can be written $a = \sum_{i,j} a_{ij} f_i \otimes g_j^*$, where $(g_1^*, \dots, g_m^*)$ is the canonical dual basis for $G^* = \operatorname{Hom}_S(G, S)$, and $A = (a_{ij})$ represents the desired matrix. As a matrix, A gives a homomorphism $A : G \longrightarrow F$. Equivalently, one may view a as a linear form $\alpha : G \otimes_S F^* \longrightarrow S$, with $a_{ij} = \alpha(g_j \otimes f_i^*)$.

3.14. In the same vein, an element of the second exterior power $\Lambda^2 F$ can be identified with an alternating $(n \times n)$–matrix, that is, an element of $\operatorname{Alt}_n(S)$. Recall that a square matrix $A = (a_{ij})$ is *alternating* provided $A^T = -A$ and the diagonal elements vanish, $a_{ii} = 0$. The canonical projection $F \otimes_S F \longrightarrow \Lambda^2 F$ becomes in terms of matrices the map $A \mapsto A - A^T$. The kernel of this epimorphism is again a free S-module, denoted $\mathbb{D}_2 F$. Its elements can be viewed as the *symmetric* matrices, $A = A^T$. Continuing with this point of view, the canonical inclusion $\Lambda^2 F \longrightarrow F \otimes_S F$ views an alternating matrix A simply as a matrix, and the cokernel of this map, denoted $\mathbb{S}_2 F$, can be identified with the module of all $(n \times n)$–matrices modulo the alternating ones, or equivalently with the free module of all (at choice: upper or lower) triangular matrices over S.

3.15. Note that the map $?+?^T : F \otimes_S F \longrightarrow F \otimes_S F$ that sends a matrix A to $A + A^T$ kills all alternating matrices and returns a symmetric matrix, thus, induces a canonical map of free S-modules $\mathbb{S}_2 F \longrightarrow \mathbb{D}_2 F$ of equal rank $\binom{n+1}{2}$. However, this map is not an isomorphism if 2 is not a unit in S; rather, it assigns to the (upper) triangular matrix U the symmetric matrix $U + U^T$ with diagonal entries in the ideal generated by 2 in S. Thus, the kernel of that map is $(\operatorname{Ann}_S 2)^n$ and its cokernel is $(S/2S)^n$.

4. Factorizations and Extensions

In this section we construct an explicit correspondence between factorizations $\operatorname{adj}(X) = YZ$ and extensions of maximal Cohen–Macaulay modules over the generic determinantal hypersurface ring.

Notation 4.1. Here is the notation for our default situation throughout the rest of the paper. Let K be a field, n a positive integer, $X = (x_{ij})$ the generic $(n \times n)$–matrix over K, and $S = K[x_{ij}]$. (Virtually everything below remains true if we assume only that K is a "Quillen–Suslin regular" ring, that is, a regular ring over which all projective modules are free. To streamline our exposition, we leave this extension to the interested reader.) Put $F = S^n$, the free module of rank n, with canonical ordered basis $(f_1, \dots, f_n)$, and $G = S^n(-1)$ the free S-module of the same rank, but with ordered basis $(g_1, \dots, g_n)$ whose elements are in degree 1 with respect to the natural $\mathbb{N}$-grading on S. (If an S-module is naturally graded, we shall keep track of its grading. However, not all modules we consider will be graded.)

We write $R := S/(\det X)$, and $\overline{N} := N \otimes_S R$ for the reduction of an S-module N modulo the determinant. The hypersurface ring R is a domain of dimension $n^2 - 1$, and the singular locus of R (in any characteristic) is defined by the partial derivatives of $\det X$, which by Lemma 2.6 are precisely the entries of $\operatorname{adj}(X)$. The ideal generated by these entries, $I_{n-1}(X)$, is prime of height 4 in S [7, 2.5], so that the singular locus $V(I_{n-1}(X))$ has codimension 3 in $\operatorname{Spec} R$. In particular, R is regular in codimension one, and so is a normal domain.

For M a Cohen–Macaulay S-module of codepth t, we set $M^\vee := \operatorname{Ext}^t_S(M, S(-n))$. Note that if M is a MCM (so free) S-module, then $M^\vee \cong \operatorname{Hom}_S(M, S)$, while for a MCM R-module M, we have $M^\vee \cong \operatorname{Hom}_R(M, R)$, up to shifts in grading.

We define the R-modules L, M, respectively $L^\vee, M^\vee$, through the exact sequences of S-modules

$$
\begin{array}{rl}
& 0 \longrightarrow G \xrightarrow{\ X\ } F \longrightarrow L \longrightarrow 0 \\
& 0 \longrightarrow F(-n) \xrightarrow{\operatorname{adj}(X)} G \longrightarrow M \longrightarrow 0 \\
(4.1.1) & 0 \longrightarrow F^\vee \xrightarrow{\ X^T\ } G^\vee \longrightarrow L^\vee \longrightarrow 0 \\
& 0 \longrightarrow G^\vee(-n) \xrightarrow{\operatorname{adj}(X)^T} F^\vee \longrightarrow M^\vee \longrightarrow 0\,;
\end{array}
$$

equivalently, one has exact sequences of R-modules

$$
\begin{array}{rl}
& 0 \longrightarrow M \longrightarrow \overline{F} \longrightarrow L \longrightarrow 0 \\
& 0 \longrightarrow L(-n) \longrightarrow \overline{G} \longrightarrow M \longrightarrow 0 \\
(4.1.2) & 0 \longrightarrow M^\vee \longrightarrow \overline{G}^\vee \longrightarrow L^\vee \longrightarrow 0 \\
& 0 \longrightarrow L^\vee(-n) \longrightarrow \overline{F}^\vee \longrightarrow M^\vee \longrightarrow 0\,.
\end{array}
$$

Each of $L, M, L^\vee, M^\vee$ is a MCM R-module, with associated matrix factorizations $(X, \mathrm{adj}(X), F, G)$, $(\mathrm{adj}(X), X, G, F)$, and so on. By 3.3, L and $L^\vee$ have rank one over R, while M and $M^\vee$ have rank $n-1$. Fixing any $n-1$ columns of X, the module L is isomorphic to the ideal generated by the maximal minors of those rows, while $L^\vee$ is obtained similarly by fixing any $n-1$ columns [11, Thm. A2.14]. In particular, L and $L^\vee$ are indecomposable nonfree R-modules. To see that M and $M^\vee$ are indecomposable as well, localize and use the fact that a syzygy of an indecomposable MCM module over a Gorenstein local ring is again indecomposable [14, Lemma 1.3].

As in 2.2, let $\tau : R \longrightarrow R$ be the K-algebra involution induced by $\tau(x_{ij}) = x_{ji}$. Then τ induces an autoequivalence on the category of R-modules, which we denote τ^*, satisfying $\tau^* L \cong L^\vee$ and $\tau^* M \cong M^\vee$.

Here is the basic link between factorizations of the adjoint and MCM modules.

Proposition 4.2. *Let Y and Z be square matrices over S so that $\mathrm{adj}(X) = YZ$. Then $\operatorname{cok} Y$ and $\operatorname{cok} Z$ are MCM R-modules, and there is a short exact sequence*

$$0 \longrightarrow \operatorname{cok} Z \longrightarrow M \longrightarrow \operatorname{cok} Y \longrightarrow 0\,. \tag{4.2.1}$$

Furthermore, Y and Z are noninvertible if and only if the exact sequence is nonsplit. In this case, $\operatorname{cok} Y$ and $\operatorname{cok} Z$ are nonfree R-modules of rank at most $n-2$.

Proof. We have $\det(\mathrm{adj}(X)) = (\det X)^{n-1}$, and $\det X$ is an irreducible element of S. It follows that, up to unit factors, both $\det Y$ and $\det Z$ are powers of $\det X$. In particular, both Y and Z are one-to-one as linear maps. From (1.0.1), we have $YZX = (\det X) \cdot \mathrm{id}_n$, and multiplying on the right by Y gives $YZXY = (\det X) \cdot Y$. Cancelling Y from the left, we have $ZXY = (\det X) \cdot \mathrm{id}_n$. Since also $XYZ = (\det X) \cdot \mathrm{id}_n$, the pair (Z, XY) is a matrix factorization of $\det X$. Similarly, (Y, ZX) is as well a matrix factorization of $\det X$. Thus $\operatorname{cok} Y$ and $\operatorname{cok} Z$ are MCM R-modules, whose ranks sum to $n-1$ by 3.3.

Taking the canonical basis for S^n, we view the matrix Y as an S-linear homomorphism $S^n \longrightarrow G$ and Z as a homomorphism $F \longrightarrow S^n$. Thus we have the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\mathrm{adj}(X)"] \arrow[d, "Z"'] & G \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & S^n \arrow[r, "Y"'] & G \arrow[r] & \operatorname{cok} Y \arrow[r] & 0
\end{tikzcd}
\tag{4.2.2}
$$

with exact rows. This induces the homomorphism $M \longrightarrow \operatorname{cok} Y$, which is surjective with kernel isomorphic to $\operatorname{cok} Z$ by the Snake Lemma, giving the exact sequence (4.2.1).

Since M is indecomposable, the sequence (4.2.1) splits if and only if either $\operatorname{cok} Y$ or $\operatorname{cok} Z$ is zero, equivalently, one of Y and Z is invertible. Finally, if $\operatorname{cok} Y$ is a nonzero free module, then (4.2.1) clearly splits, and so $\operatorname{cok} Y = 0$. Since R is a Gorenstein ring, free modules are also injective objects in the subcategory of MCM modules, whence (4.2.1) splits as well if $\operatorname{cok} Z$ is free, and then $\operatorname{cok} Z = 0$. □

4.3. According to Bergman's theorem [3], we should only hope to find nontrivial factorizations $\mathrm{adj}(X) = YZ$ satisfying either $\det Y = u \det X$ or $\det Z = u \det X$ for some unit $u \in S$, at least in characteristic zero. Further, we shall from now onward omit mention of u, and tacitly assume the phrase "up to unit factors in S" where necessary. With this in mind, from this point on **we consider only factorizations of the adjoint in which $\det Y = \det X$.** The case $\det Z = \det X$ can be recovered by applying the transpose and the automorphism τ of 2.2: If $\mathrm{adj}(X) = YZ$ with $\det Z = \det X$, then $\mathrm{adj}(X)^T = Z^T Y^T$, and so $\mathrm{adj}(X) = \tau(Z^T)\tau(Y^T)$ is a factorization with $\det \tau(Z^T) = \det X$.

4.4. Since we assume $\det Y = \det X$, the MCM R-module $\operatorname{cok} Y$ has rank one by 3.3. It is also reflexive, so isomorphic to a divisorial ideal of R. The divisor class group of R was computed by Bruns: $\mathrm{Cl}(R) \cong \mathbb{Z}$, generated by the class of $[L] = -[L^\vee]$ (see [4] or [5, 7.3.5]). Furthermore, the symbolic powers $L^{(m)}$ representing elements $m[L] \in \mathrm{Cl}(R)$ are equal to the usual powers L^m, and among these, only L and $L^\vee$ are MCM modules [7, 9.27]. More generally, if K is only a normal domain, then R is still normal and $\mathrm{Cl}(R) \cong \mathrm{Cl}(K) \oplus \mathbb{Z}$. Succinctly: when $\mathrm{Cl}(K) = 0$, the only nonfree MCM R-modules of rank one, up to isomorphism, are L and $L^\vee$. Thus $\det Y = \det X$ implies either $\operatorname{cok} Y \cong L$ or $\operatorname{cok} Y \cong L^\vee$.

This already allows us to rule out all nontrivial factorizations of $\operatorname{adj}(X)$ when $n \leq 3$ and K is a unique factorization domain.

Theorem 4.5. *Let K be a UFD and $X = (x_{ij})$ the generic (3×3)–matrix over K. Then there are no nontrivial factorizations $\operatorname{adj}(X) = YZ$.*

Proof. When $n = 3$, the adjoint $\operatorname{adj}(X)$ has determinant $(\det X)^2$, so that a nontrivial factorization $\operatorname{adj}(X) = YZ$ must have $\det Y = \det Z = \det X$. In particular both $\operatorname{cok} Y$ and $\operatorname{cok} Z$ are of rank one and nonfree, so are each isomorphic to one of $\{L, L^\vee\}$. In the divisor class group $\operatorname{Cl}(R)$, we have $[M] = -[L] = [L^\vee]$ by the defining sequences (4.1.2), and furthermore $[M] = [\operatorname{cok} Y] + [\operatorname{cok} Z]$ from (4.2.1). If either of $\operatorname{cok} Y$ or $\operatorname{cok} Z$ were isomorphic to $L^\vee$, this would force the other to be zero in $\operatorname{Cl}(R)$, a contradiction. If on the other hand both $\operatorname{cok} Y$ and $\operatorname{cok} Z$ were isomorphic to L, then $[M] = 2[L]$ in $\operatorname{Cl}(R)$, again a contradiction. □

4.6. Returning to the case of arbitrary $n \geq 3$ and K a field, let $\operatorname{adj}(X) = YZ$ be a factorization with $\det Y = \det X$. We have the exact sequence

$$0 \longrightarrow \operatorname{cok} Z \longrightarrow M \longrightarrow \operatorname{cok} Y \longrightarrow 0$$

of Proposition 4.2, in which $\operatorname{cok} Y$ is isomorphic to either L or $L^\vee$. We have also the exact sequence

$$0 \longrightarrow M \longrightarrow \overline{F} \longrightarrow L \longrightarrow 0$$

displaying M as a first syzygy of L. Form the pushout diagram:

(4.6.1)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{cok} Z \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \operatorname{cok} Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{cok} Z \arrow[r] & \overline{F} \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The rightmost column is an exact sequence

$$(4.6.2) \qquad 0 \longrightarrow \operatorname{cok} Y \longrightarrow Q \longrightarrow L \longrightarrow 0\,,$$

thus naturally gives an element of $\operatorname{Ext}^1_R(L, \operatorname{cok} Y)$.

Proposition 4.7. *Let* $\operatorname{adj}(X) = YZ$ *be a nontrivial factorization of* $\operatorname{adj}(X)$ *with* $\det Y = \det X$*. Then the exact sequence (4.6.2) is nonsplit, and the middle term* Q *is a MCM* R*-module of rank* 2 *requiring at most* n *generators.*

Proof. If (4.6.2) splits, then $Q \cong L \oplus \operatorname{cok} Y$. Localize at the maximal ideal $\mathfrak{m} = (x_{ij})$. Then $\operatorname{syz}_1(L_\mathfrak{m}) \cong M_\mathfrak{m}$ is isomorphic to a direct summand of $(\operatorname{cok} Z)_\mathfrak{m} \oplus H$ for some free $R_\mathfrak{m}$-module H. Since $M_\mathfrak{m}$ is indecomposable and nonfree, this implies (after passing to the completion to use the Krull-Schmidt theorem) that $M_\mathfrak{m}$ is a direct summand of $(\operatorname{cok} Z)_\mathfrak{m}$, and in particular that $\operatorname{cok} Z$ has rank at least $n-1$, contradicting the nontriviality of the factorization $\operatorname{adj}(X) = YZ$. The statements about Q follow from the diagram (4.6.1). □

4.8. In order to classify factorizations of $\operatorname{adj}(X)$, Proposition 4.7 hints that we should classify certain extensions in $\operatorname{Ext}^1_R(L, \operatorname{cok} Y)$. Since we assume $\det Y = \det X$, we have either $\operatorname{cok} Y \cong L$ or $\operatorname{cok} Y \cong L^\vee$, so we must consider $\operatorname{Ext}^1_R(L, L)$ and $\operatorname{Ext}^1_R(L, L^\vee)$. Specifically, we are concerned with extensions whose middle terms need the minimum number of generators, n. We define such extensions in more generality.

Definition 4.9. Let A be a (commutative, Noetherian) ring and N_1, N_2 finitely generated A-modules. Let $\operatorname{Ext}^1_A(N_1, N_2)_{\min}$ be the subset of $\operatorname{Ext}^1_A(N_1, N_2)$ consisting of equivalence classes of extensions

$$0 \longrightarrow N_2 \longrightarrow E \longrightarrow N_1 \longrightarrow 0$$

in which E requires no more generators than N_1.

Lemma 4.10. *For any factorization* $\operatorname{adj}(X) = YZ$ *with* $\det Y = \det X$*, the natural epimorphism* $\operatorname{Hom}_R(M, \operatorname{cok} Y) \longrightarrow \operatorname{Ext}^1_R(L, \operatorname{cok} Y)$ *induces a* $1-1$ *correspondence between the elements of* $\operatorname{Ext}^1_R(L, \operatorname{cok} Y)_{\min}$ *and surjective homomorphisms* $M \longrightarrow \operatorname{cok} Y$*.*

Proof. We have already seen in Proposition 4.7 that an epimorphism $\varphi \in \operatorname{Hom}_R(M, \operatorname{cok} Y)$ gives rise to an element of $\operatorname{Ext}^1_R(L, \operatorname{cok} Y)_{\min}$. Conversely, given an extension

$$0 \longrightarrow \operatorname{cok} Y \longrightarrow Q \longrightarrow L \longrightarrow 0$$

of $\operatorname{cok} Y$ by L, with Q generated by at most n elements, we construct a commutative diagram of MCM R-modules with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \operatorname{cok} Y \arrow[r] & Q \arrow[r] & L \arrow[r] & 0\,. \end{tikzcd}$$

By Theorem 3.9, there exists a pair of $(n \times n)$–matrices (U, V) giving a matrix factorization of $\det X$, and so that $\operatorname{cok} U \cong L$, $\operatorname{cok} V \cong M'$. It follows that U is matrix-equivalent to the generic matrix X, whence V is equivalent to $\operatorname{adj}(X)$. Thus $M' \cong M$. The homomorphism $M' \longrightarrow \operatorname{cok} Y$ is then surjective by the Snake Lemma.

If a surjection $\varphi \in \operatorname{Hom}_R(M, \operatorname{cok} Y)$ maps to zero in $\operatorname{Ext}^1_R(L, \operatorname{cok} Y)$, that is, gives a split-exact sequence, then $Q \cong L \oplus \operatorname{cok} Y$ requires more than n generators. Thus the map from $\operatorname{Hom}_R(M, \operatorname{cok} Y)$ to $\operatorname{Ext}^1_R(L, \operatorname{cok} Y)$ is injective on surjective homomorphisms. □

The results of this section set out a correspondence between factorizations $\operatorname{adj}(X) = YZ$ with $\det Y = \det X$, surjective homomorphisms $M \longrightarrow \operatorname{cok} Y$, and elements of $\operatorname{Ext}^1_R(L, \operatorname{cok} Y)_{\min}$. Since $\det Y = \det X$ implies either $\operatorname{cok} Y \cong L$ or $\operatorname{cok} Y \cong L^\vee$, we treat the two cases separately. In Section 5 we shall show that in fact $\operatorname{Ext}^1_R(L, L) = 0$, so there are *no* factorizations of the adjoint with $\operatorname{cok} Y \cong L$. The sections thereafter treat the case $\operatorname{cok} Y \cong L^\vee$.

5. The Module $\operatorname{Ext}^1_R(L, L)$ vanishes

In this section we show that $L = \operatorname{cok} X$ has no self-extensions, equivalently, is *rigid* over $R = S/(\det X)$. Our goal follows from a recent result of R. Ile [18]. We include a proof of Ile's theorem here, since it is short and elegant, and the matrix equations of Section 2 simplify the argument slightly. In the interest of broader applicability, we will state the main results in terms of general matrix factorizations

(φ, ψ) over Noetherian rings S, as Ile does, indicating where "specialization to the generic case" simplifies the arguments still further.

Ile's result is couched in terms of the *Scandinavian complex* $\mathcal{S}c(\varphi)$ attached to a matrix factorization (φ, ψ) by T. Gulliksen and O. Negård [13], which we shall have reason to use again in Section 7.

Definition 5.1. Let $\varphi : G \longrightarrow F$ be a homomorphism of free modules of the same (finite) rank n over a Noetherian ring S. Assume that $f = \det \varphi$ is an irreducible nonzerodivisor of S and that $\operatorname{Ann}_S \operatorname{cok} \varphi = (f)$, so that $(\varphi, \operatorname{adj}(\varphi), F, G)$ is a matrix factorization of f. The *Scandinavian complex* $\mathcal{S}c(\varphi)$ is

$$0 \longrightarrow S \xrightarrow{?\,\operatorname{adj}(\varphi)} \operatorname{Hom}_S(F, G) \xrightarrow{(\varphi ?, ?\varphi)} \mathbb{H} \xrightarrow{?\varphi - \varphi ?} \operatorname{Hom}_S(G, F) \xrightarrow{\operatorname{tr}(?\,\operatorname{adj}(\varphi))} S \longrightarrow 0\,,$$

where $\mathbb{H}$ is the homology in the middle of the short complex

$$S \xrightarrow{\Delta} \operatorname{End}_S(F) \oplus \operatorname{End}_S(G) \xrightarrow{\operatorname{tr}(?) - \operatorname{tr}(?)} S\,,$$

$\operatorname{tr}(?)$ denotes the trace function, and Δ is the diagonal map.

The complex $\mathcal{S}c(\varphi)$ is functorial with respect to homomorphisms of matrix factorizations. Here is the main theorem of [13].

Proposition 5.2 ([13]; see also [7])**.** *For φ as above, we have*

$$H_0(\mathcal{S}c(\varphi)) \cong S/I_1(\operatorname{adj}(\varphi)) = S/I_{n-1}(\varphi)\,,$$

and

$$\max\{\, q \mid H_q(\mathcal{S}c(\varphi)) \neq 0\} = 4 - \operatorname{grade} I_{n-1}(\varphi)\,.$$

In particular, if the grade of $I_{n-1}(\varphi)$ on S is 4, the maximum possible value, then $\mathcal{S}c(\varphi)$ is a (minimal, in case S is local or graded and no entry of φ is a unit) S-free resolution of $S/I_{n-1}(\varphi)$.

Remark 5.3. It's well-known (see, for example, [17] or [5, 7.3.1]) that for $\varphi = X$ a generic square matrix of indeterminates, the maximum value, $\operatorname{grade} I_{n-1}(X) = 4$, is achieved.

Ile's main result identifies the deformation theory of $\operatorname{cok} \varphi$ as the homology of $\mathcal{S}c(\varphi)$. The rest of this section will be devoted to the proof of this theorem.

Theorem 5.4 ([18])**.** *Let S be a Noetherian ring and $\varphi : G \longrightarrow F$ a homomorphism between free S-modules of the same rank n, such that $f = \det\varphi$ is an irreducible nonzerodivisor of S. Set $R := S/(f)$ and $M := \operatorname{cok}\varphi$, and assume that $\operatorname{Ann}_S M = (f)$. Then*

$$H_1(\mathcal{S}c(\varphi)) \cong \operatorname{Ext}^1_R(M, M)\,.$$

In particular, if $\operatorname{grade} I_{n-1}(\varphi) = 4$, then $\operatorname{Ext}^1_R(M, M) = 0$; thus M is rigid.

Proof. To compute the homology of $\mathcal{S}c(\varphi)$ at $\operatorname{Hom}_S(G, F)$, consider the diagram

$$\begin{tikzcd}
\mathbb{H} \arrow[r, "?\varphi-\varphi?"] & \operatorname{Hom}_S(G,F) \arrow[r, "\operatorname{tr}(?\operatorname{adj}(\varphi))"] \arrow[d, "\pi"'] & S \arrow[d, "\overline{\cdot\, \mathrm{id}_M}"] \\
& \operatorname{Ext}^1_S(M,M) \arrow[r, "\epsilon"'] & \operatorname{End}_R(M)\,.
\end{tikzcd} \tag{5.4.1}$$

Here π is the natural surjection and ϵ is defined by pulling back cocycles along $\operatorname{adj}(\varphi)$. That is, for $\chi \in \operatorname{Ext}^1_S(M, M)$, we choose a preimage $U \in \operatorname{Hom}_S(G, F)$ and observe that $(U \operatorname{adj}(\varphi), \operatorname{adj}(\varphi)U)$ is a homomorphism of matrix factorizations

$$(U \operatorname{adj}(\varphi), \operatorname{adj}(\varphi)\, U) : (\varphi, \operatorname{adj}(\varphi)) \longrightarrow (\varphi, \operatorname{adj}(\varphi))\,;$$

put $\epsilon(\chi) = \operatorname{cok}(U \operatorname{adj}(\varphi), \operatorname{adj}(\varphi)\, U) \in \operatorname{End}_R(M)$.

We claim first that the square commutes. The following lemma is the crux of the argument.

Lemma 5.5. *For each $U \in \operatorname{Hom}_S(G, F)$, there exists $V \in \operatorname{Hom}_S(F, G)$ such that*

$$U \operatorname{adj}(\varphi) - \varphi V = \operatorname{tr}(U \operatorname{adj}(\varphi)) \cdot \mathrm{id}_F\ .$$

Proof. For the purposes of this proof, we may revert to the generic situation, where $\varphi = X$ is a square matrix of indeterminates. As in Section 2, let $\partial_{ij} = \frac{\partial}{\partial x_{ij}}$ be the partial derivative with respect to the variable x_{ij}; then

$$\begin{aligned}
\partial_{ij}[(\det X) \cdot \mathrm{id}_F] &= \partial_{ij}[X \operatorname{adj}(X)] \\
&= \partial_{ij}(X) \operatorname{adj}(X) + X\partial_{ij}(\operatorname{adj}(X))\,,
\end{aligned}$$

where we apply ∂_{ij} to a matrix entry-by-entry. By Lemma 2.6(1), this can be rewritten as

$$E_{ij} \operatorname{adj}(X) + X\partial_{ij}(\operatorname{adj}(X)) = \operatorname{adj}(X)_{ji} \cdot \mathrm{id}_F\ . \tag{5.5.1}$$

Write $U = (u_{ij})$; multiplying (5.5.1) by u_{ij} and taking the sum over all (i,j) gives

$$U \operatorname{adj}(X) - XV = \left(\sum_{i,j} u_{ij} \operatorname{adj}(X)_{ji} \right) \cdot \mathrm{id}_F \,,$$

with $V = -\sum_{i,j} u_{ij} \partial_{ij}(\operatorname{adj}(X))$. The right-hand side of this last equation is equal to $\operatorname{tr}(U \operatorname{adj}(X)) \cdot \mathrm{id}_F$. $\square$

Returning to the proof of Theorem 5.4, we must show that

$$\overline{\operatorname{tr}(U \operatorname{adj}(\varphi)) \cdot \mathrm{id}_M} = \operatorname{cok}(U \operatorname{adj}(\varphi), \operatorname{adj}(\varphi)\, U) \,,$$

as endomorphisms of M, for each $U \in \operatorname{Hom}_S(G, F)$. By the Lemma, there exists $V \in \operatorname{Hom}_S(F, G)$ so that

$$U \operatorname{adj}(\varphi) - \varphi V = \operatorname{tr}(U \operatorname{adj}(\varphi)) \cdot \mathrm{id}_F \;.$$

In particular, the two sides induce the same endomorphism of M. The term φV factors through F, so gives the zero map on $M = \operatorname{cok} \varphi$; thus $U \operatorname{adj}(\varphi)$ induces $\overline{\operatorname{tr}(U \operatorname{adj}(\varphi)) \cdot \mathrm{id}_M}$.

Next we shall show that $\ker \epsilon \cong \operatorname{Ext}^1_R(M, M)$. Indeed, an S-module extension χ, represented by $U \in \operatorname{Hom}_S(G, F)$, is an extension of R-modules if and only if U is part of a homomorphism of matrix factorizations, *i.e.*, there exists $V \in \operatorname{Hom}_S(F, G)$ so that $U \operatorname{adj}(\varphi) = \varphi V$. This is the case precisely when $U \operatorname{adj}(\varphi)$ factors through G, that is, induces the zero endomorphism of M.

Finally, we claim that π induces an isomorphism $H_1(\mathcal{S}c(\varphi)) \longrightarrow \ker \epsilon$. To see this, first let $[U]$ be a homology class. Then the image of U in $\operatorname{End}_R(M)$ is zero, so that $\pi(U) \in \ker \epsilon$. Next, take $U \in \operatorname{Hom}_S(G, F)$ to be a boundary, so that $U = A\varphi - \varphi B$ for some $(A, B) \in \mathbb{H}$. Then the homomorphism of matrix factorizations induced by $\pi(U)$ is equivalent to $(\varphi B \operatorname{adj}(\varphi), \operatorname{adj}(\varphi) A \varphi)$. Since $\varphi B \operatorname{adj}(\varphi)$ factors through G, this is zero in $\operatorname{End}_R(M)$. Lastly, any $\chi \in \operatorname{Ext}^1_R(M, M)$ lifts to $U \in \operatorname{Hom}_S(G, F)$, which must then be a cycle by the commutativity of the square. This finishes the proof. $\square$

Specializing to the case of a generic matrix, we obtain our main result of this section.

Corollary 5.6. *Let K be a commutative Noetherian normal domain, $X = (x_{ij})$ the generic $(n \times n)$–matrix over K, $S = K[x_{ij}]$, and $R = S/(\det X)$. Set $L := \operatorname{cok} X$. Then*

(1) $\operatorname{End}_R(L) \cong R$ *;*
(2) $\operatorname{Ext}^1_R(L, L) = 0$ *;*
(3) $\operatorname{Ext}^1_S(L, L)$ *is isomorphic to the ideal $I_{n-1}(X)/(\det X)$ of R ;*
(4) $\operatorname{Ext}^2_R(L, L) \cong S/I_{n-1}(X)$ *; and*
(5) $\underline{\operatorname{End}}_R(L) \cong S/I_{n-1}(X)$ *.*

Proof. We have already observed that R is a normal domain (see 4.1). Since L has rank one, the ring $\operatorname{End}_R(L)$ is a finite extension of R contained in its quotient field, so equal to R by normality. Claims (2), (3), and (4) follow from Theorem 5.4 and the diagram (5.4.1): Since $\operatorname{grade} I_{n-1}(X) = 4$, we have $\operatorname{Ext}^1_R(L, L) = 0$, and the image of ϵ is equal to the image of $\operatorname{tr}(? \operatorname{adj}(X))$, that is, $I_{n-1}(X)$. Finally, statement (5) follows from (4) and Proposition 3.12. □

6. The module $\operatorname{Hom}_R(M, L^\vee)$ and Classification of Factorizations

In this section we consider the case $\operatorname{cok} Y \cong L^\vee$. Using the matrix equations of Section 2, we compute the free resolution of the S-module $\operatorname{Hom}_R(M, L^\vee)$, and compare the result to a canonical short exact sequence to show that not only does every factorization $\operatorname{adj}(X) = YZ$ with $\det Y = \det X$ yield an extension of L by $L^\vee$, but we can classify precisely when two factorizations give equivalent extensions.

Our next task is to interpret the matrix-theoretic result Theorem 2.8 in terms of MCM modules over R. Keep the notation of 4.1.

Remark 6.1. Let $A \in \operatorname{Alt}_n(S)$ be an alternating $(n \times n)$–matrix over S and let B_A be the companion matrix of Theorem 2.8, so that

$$A \operatorname{adj}(X) = X^T B_A .$$

This equation defines a commutative diagram of free S-modules

$$\begin{tikzcd}
G(-n) \arrow[r, "X"] \arrow[d, "A"'] & F(-n) \arrow[r, "\operatorname{adj}(X)"] \arrow[d, "B_A"'] & G \arrow[d, "A"] \\
F(-n) \arrow[r, "\operatorname{adj}(X)^T"'] & G \arrow[r, "X^T"'] & F ,
\end{tikzcd}$$

that is, a homomorphism of matrix factorizations

$$(A, B_A) : (\mathrm{adj}(X), X) \longrightarrow (X^T, \mathrm{adj}(X)^T)$$

and thus a homomorphism of MCM R-modules

$$\mathrm{cok}(A, B_A) : M \longrightarrow L^\vee .$$

In other words, we have a homomorphism $\mathrm{Alt}_n(S) \xrightarrow{A \mapsto \mathrm{cok}(A,B_A)} \mathrm{Hom}_R(M, L^\vee)$. Our next result is that this homomorphism is surjective, so that $\mathrm{Hom}_R(M, L^\vee)$ is generated by the alternating matrices, and moreover that $\mathrm{Hom}_R(M, L^\vee)$ is itself a MCM R-module.

Theorem 6.2. *The R-module $\mathrm{Hom}_R(M, L^\vee)$ is MCM of rank $n-1$, minimally generated by $\binom{n}{2}$ elements. More precisely, it has the following free presentation as an S-module*

$$0 \longrightarrow \mathrm{Alt}_n(S) \xrightarrow{U \mapsto X^T U X} \mathrm{Alt}_n(S)(2) \xrightarrow{A \mapsto \mathrm{cok}(A,B_A)} \mathrm{Hom}_R(M, L^\vee) \longrightarrow 0\,;$$

alternatively, in terms of exterior powers, this exact sequence can be written as

$$0 \longrightarrow \Lambda^2 F^\vee \xrightarrow{\Lambda^2 X^T} \Lambda^2 G^\vee \longrightarrow \mathrm{Hom}_R(M, L^\vee) \longrightarrow 0\,.$$

Proof. For U an alternating $(n \times n)$–matrix over S, we have

$$\begin{aligned} X^T B_{X^T U X} &= (X^T U X)\,\mathrm{adj}(X) \\ &= X^T U \cdot (\det X)\,, \end{aligned}$$

so that $B_{X^T U X} = U \cdot (\det X)$. Thus the homomorphism $\mathrm{cok}(X^T U X, B_{X^T U X})$ is zero on the R-module M, and the alleged resolution of $\mathrm{Hom}_R(M, L^\vee)$ is at least a complex.

Put $D := \mathrm{cok}(U \mapsto X^T U X)$. Then D maps to $\mathrm{Hom}_R(M, L^\vee)$ and we must show that this map is an isomorphism. Note first that D is a MCM R-module, with matrix factorization

$$(U \mapsto X^T U X, A \mapsto B_A)\,.$$

Indeed, we have seen that $B_{X^T U X} = U \cdot (\det X)$, and also $X^T B_A X = A\,\mathrm{adj}(X) X = A \cdot (\det X)$. Thus in particular D is a reflexive R-module, and $U \mapsto X^T U X$ is an injective endomorphism of the module of alternating matrices.

The free module $\mathrm{Alt}_n(S)$ has rank $\binom{n}{2}$, so the determinant of the endomorphism $U \mapsto X^T U X$ is homogeneous of degree $n(n-1)$ in the variables x_{ij}. Since it must also be a unit times $(\det X)^{\operatorname{rank} D}$, we see that D has rank $n-1$ as an R-module, equal to that of $\mathrm{Hom}_R(M, L^\vee)$.

Outside the singular locus $V(I_{n-1}(X))$ of R, at least one maximal minor of X^T is a unit. Thus after elementary transformations and linear changes of variables, $X^T = \mathrm{diag}(0, 1, \ldots, 1)$ and so $\mathrm{adj}(X)^T = E_{11}$, the elementary matrix with 1 at position $(1,1)$ and zeros elsewhere. Now any homomorphism α from the cokernel of E_{11} to the cokernel of $\mathrm{diag}(0, 1, \ldots, 1)$ is induced by an alternating $(n \times n)$–matrix, namely any alternating matrix with first row α. That is, outside the singular locus of R, $\mathrm{Hom}_R(M, L^\vee)$ is indeed generated by homomorphisms $\mathrm{cok}(A, B_A)$ for alternating A. The map $D \longrightarrow \mathrm{Hom}_R(M, L^\vee)$ is thus surjective, and since D and $\mathrm{Hom}_R(M, L^\vee)$ have the same rank, is even an isomorphism, outside $V(I_{n-1}(X))$.

Recall that R is normal and $I_{n-1}(X)$ has codimension 3 in $\operatorname{Spec} R$. The homomorphism $D \longrightarrow \mathrm{Hom}_R(M, L^\vee)$ is thus a homomorphism between reflexive modules over a normal domain, which is an isomorphism in codimension one. It follows that in fact $D \longrightarrow \mathrm{Hom}_R(M, L^\vee)$ is an isomorphism. $\square$

6.3. Define an S-module C by the pullback diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathrm{Hom}_S(G, G^\vee) \arrow[d, "?\,\mathrm{adj}(X)"] \\ \mathrm{Hom}_S(F(-n), F^\vee) \arrow[r, "X^T ?"'] & \mathrm{Hom}_S(F(-n), G^\vee) \end{tikzcd}$$

Then

$$C = \{(A, B) \mid A\,\mathrm{adj}(X) = X^T B\}\,.$$

By Remark 6.1, there is a natural homomorphism $C \longrightarrow \mathrm{Hom}_R(M, L^\vee)$, sending (A, B) to $\mathrm{cok}(A, B)$. There is also a natural embedding $\mathrm{Hom}_S(G, F^\vee) \longrightarrow C$, given by $U \mapsto (X^T U, U\,\mathrm{adj}(X))$. This gives an exact sequence of S-modules (*cf.* 3.4)

$$0 \longrightarrow \mathrm{Hom}_S(G, F^\vee) \longrightarrow C \longrightarrow \mathrm{Hom}_R(M, L^\vee) \longrightarrow 0\,. \tag{6.3.1}$$

Comparing the resolution given by Theorem 6.2 to (6.3.1) will give our classification of factorizations of $\mathrm{adj}(X)$. To prepare for this, we make a definition.

Definition 6.4. A factorization of the form $\operatorname{adj}(X) = JXZ$ or $\operatorname{adj}(X) = JX^TZ$, with J an invertible $(n \times n)$–matrix, is called a *normalized factorization* of the adjoint.

Note that since, up to equivalence, the only $(n \times n)$–matrices with determinant equal to $\det X$ are X and X^T, we may always normalize a given factorization. Explicitly, if $\operatorname{adj}(X) = YZ$ with $\operatorname{cok} Y \cong L^\vee$, there exist invertible matrices J and J' so that $Y = JX^TJ'$, and replacing Z by $J'Z$ gives a normalized factorization. Similarly, if $\operatorname{cok} Y \cong L$, we may multiply Z on the left by an invertible matrix to achieve a normalized factorization.

Theorem 6.5. *Let* $\operatorname{adj}(X) = YZ = JX^TZ$ *be a normalized factorization of* $\operatorname{adj}(X)$ *with* $\operatorname{cok} Y \cong L^\vee$*. Then there exist a unique invertible alternating* $(n \times n)$*–matrix* A *and a unique* $(n \times n)$*–matrix* U *such that*

$$J^{-1} = A + X^TU \quad \text{and} \quad Z = B_A + U\operatorname{adj}(X)\,.$$

In particular, the homomorphism $\operatorname{cok}(A, B_A) : M \longrightarrow L^\vee$ *is surjective. Two normalized factorizations* JX^TZ *and* $J'X^TZ'$ *give the same epimorphism in* $\operatorname{Hom}_R(M, L^\vee)$ *if and only if* $J^{-1} - J'^{-1} = X^TV$ *for some* $(n \times n)$*–matrix* V*, and then* $Z - Z' = V\operatorname{adj}(X)$*.*

Proof. We have a commutative diagram of S-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^2 F^\vee \arrow[r, "X^T ? X"] \arrow[d, "?X"'] & \Lambda^2 G^\vee \arrow[r] \arrow[d, "A \mapsto (A, B_A)"] & \operatorname{Hom}_R(M, L^\vee) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Hom}_S(G, F^\vee) \arrow[r, "(X^T ?, ? \operatorname{adj}(X))"'] & C \arrow[r] & \operatorname{Hom}_R(M, L^\vee) \arrow[r] & 0
\end{tikzcd}$$

in which the vertical arrows represent monomorphisms. The factorization $\operatorname{adj}(X) = JX^TZ$ yields $J^{-1}\operatorname{adj}(X) = X^TZ$, so gives a homomorphism (J^{-1}, Z) of matrix factorizations. From the diagram, we obtain $A \in \Lambda^2 G^\vee$, unique by the Snake Lemma, so that $J^{-1} = A + X^TU$ for some $U \in \operatorname{Hom}_S(G, F^\vee)$ and $Z = B_A + U\operatorname{adj}(X)$. Since (J^{-1}, Z) and (A, B_A) induce the same homomorphism $M \longrightarrow L^\vee$, A is also invertible. The final statement follows from the uniqueness of A and 3.4 above. $\square$

Corollary 6.6. *There is a one-to-one correspondence between equivalence classes of extensions*

$$0 \longrightarrow L^\vee \longrightarrow Q \longrightarrow L \longrightarrow 0$$

in which Q is a homomorphic image of R^n and equivalence classes of normalized factorizations $\operatorname{adj}(X) = JX^TZ$*, where the equivalence relation is given by*

$$JX^TZ \sim J'X^TZ' \quad \textit{iff} \quad J^{-1} - J'^{-1} = X^TV \textit{ and } Z - Z' = V\operatorname{adj}(X)$$

for some $(n \times n)$–matrix V.

7. The Module $\operatorname{Ext}^1_R(L, L^\vee)$

We now turn to computing $E := \operatorname{Ext}^1_R(L, L^\vee)$. Keep the notation of 4.1.

The epimorphism $\Lambda^2 G^\vee \longrightarrow \operatorname{Hom}_R(M, L^\vee)$ of Theorem 6.2 composes with the natural epimorphism $\operatorname{Hom}_R(M, L^\vee) \longrightarrow E$ to begin a free resolution of E. Let us first identify this map more explicitly. Recall that a finitely generated module N over a normal domain is *orientable* provided $[N] = 0$ in the divisor class group.

Lemma 7.1. *For each alternating $(n \times n)$–matrix A over S, there exists an extension*

$$0 \longrightarrow L^\vee \longrightarrow Q \longrightarrow L \longrightarrow 0\,, \tag{7.1.1}$$

which is the image of $\operatorname{cok}(A, B_A)$ *under the natural epimorphism* $\operatorname{Hom}_R(M, L^\vee) \longrightarrow$ $\operatorname{Ext}^1_R(L, L^\vee)$*. In particular, the module Q is a MCM R-module of rank 2 given by the matrix factorization*

$$Q = \operatorname{cok}\left(\begin{pmatrix} X^T & A \\ 0 & X \end{pmatrix}, \begin{pmatrix} \operatorname{adj}(X)^T & -B_A \\ 0 & \operatorname{adj}(X) \end{pmatrix} \right) .$$

Furthermore, Q is orientable.

Proof. This is a restatement of Remark 6.1 and Proposition 3.5. To see that Q is orientable, observe that $[Q] = [L] + [L^\vee] = [L] - [L] = 0$ in $\operatorname{Cl}(R)$. ☐

Remark 7.2. The matrix factorization given for Q in Lemma 7.1 may not be of minimal size. Indeed, if A is invertible then we have seen that Q requires only n

generators. In this case, we have

$$\begin{pmatrix} \mathrm{id}_n & 0 \\ -XA^{-1} & \mathrm{id}_n \end{pmatrix} \begin{pmatrix} X^T & A \\ 0 & X \end{pmatrix} \begin{pmatrix} \mathrm{id}_n & 0 \\ -A^{-1}X^T & \mathrm{id}_n \end{pmatrix} = \begin{pmatrix} 0 & A \\ -XA^{-1}X^T & 0 \end{pmatrix},$$

so that the given matrix factorization for Q can be reduced to

$$Q \cong \mathrm{cok}(XA^{-1}X^T, B_A)\,.$$

More generally, if a $(k \times k)$–minor of A is invertible, then the given matrix factorization of Q can be reduced to one of size $2n-k$.

For any n, the graded, orientable rank two MCM R-modules are minimally evenly generated [15, 3.1]. In fact, they are presented by yet another alternating matrix over S, as in [2].

Remark 7.3. The orientable MCM module Q of Lemma 7.1 is decomposable if and only if $Q \cong L \oplus L^\vee$, equivalently, the sequence (7.1.1) is split exact. To see this, recall that L and $L^\vee$ are up to isomorphism the only MCM R-modules of rank one. As Q is orientable, the only possible direct-sum decomposition for Q is $L \oplus L^\vee$, and by Miyata's theorem [20], if (7.1.1) is apparently split then it is split.

We first determine the structure of E as an S-module.

Theorem 7.4. *The S-module $E = \mathrm{Ext}^1_R(L, L^\vee)$ has the following graded minimal resolution.*

$$\begin{tikzcd}[column sep=small]
 & & \mathbb{S}_2 G^\vee(-n) \arrow[dr, "-\mathrm{adj}(X)"] & & & \\
0 \arrow[r] & \Lambda^2 F^\vee(-n) \arrow[r, "?X"] & F^\vee \otimes G^\vee(-n) \arrow[u, "X^T?+?^T X"] \arrow[d, "?\,\mathrm{adj}(X)+\mathrm{adj}(X)?^T"'] & & G^\vee \otimes F^\vee \arrow[r, "?X-X^T?^T"] & \Lambda^2 G^\vee \arrow[r] & E \arrow[r] & 0 \\
 & & \mathbb{D}_2 F^\vee \arrow[ur, "X^T?"'] & & &
\end{tikzcd}$$

Proof. The diagram of S-modules

$$\begin{tikzcd}
 & & & & 0 & \\
 & & & & \operatorname{Ext}^1_R(L, L^\vee) \arrow[u] & \\
 & 0 \arrow[r] & \Lambda^2 F^\vee \arrow[r, "X^T ? X"] & \Lambda^2 G^\vee \arrow[r] & \operatorname{Hom}_R(M, L^\vee) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[r] & \operatorname{Hom}_S(F, F^\vee) \arrow[r, "X^T ?"] \arrow[u, "? - ?^T"] & \operatorname{Hom}_S(F, G^\vee) \arrow[r] \arrow[u, "? X - X^T ?^T"'] & \operatorname{Hom}_R(\overline{F}, L^\vee) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Lambda^2 F^\vee(-n) \arrow[r, "? X"'] & F^\vee \otimes_S G^\vee(-n) \arrow[r, "X^T ? + ?^T X"'] \arrow[u, "? \operatorname{adj}(X) + \operatorname{adj}(X)^T ?^T"] & \mathbb{S}_2 G^\vee(-n) \arrow[r] \arrow[u, "? \operatorname{adj}(X)"'] & \operatorname{Hom}_R(L, L^\vee) \arrow[r] \arrow[u] & 0 \\
 & & & & 0 \arrow[u] &
\end{tikzcd}$$

commutes, has exact rows, and has complexes for columns. The ingredients of the diagram are as follows. For rows, it has the resolution of $\operatorname{Hom}_R(M, L^\vee)$ computed in Theorem 6.2, the result of applying $\operatorname{Hom}_S(F, -)$ to the S-module resolution of $L^\vee$, and the resolution of $\operatorname{Hom}_R(L, L^\vee) \cong \mathbb{S}_2 L^\vee$ computed via the Eagon–Northcott complex in [7, Thm. 2.16]. The rightmost column is the result of applying $\operatorname{Hom}_R(-, L^\vee)$ to the R-module resolution of L. The other columns are liftings of the maps in the rightmost column.

Truncate the diagram, retaining only the free S-modules; since the rows are acyclic, the total complex is acyclic as well, with zeroth homology isomorphic to E. This gives a free resolution

$$0 \leftarrow E \longleftarrow \Lambda^2 G^\vee \longleftarrow \begin{matrix} G^\vee \otimes F^\vee \\ \oplus \\ \Lambda^2 F^\vee \end{matrix} \longleftarrow \begin{matrix} \mathbb{S}_2 G^\vee(-n) \\ \oplus \\ F^\vee \otimes F^\vee \end{matrix} \longleftarrow F^\vee \otimes G^\vee(-n) \longleftarrow \Lambda^2 F^\vee(-n) \leftarrow 0$$

of E. By 3.14, the map $F^\vee \otimes F^\vee \xrightarrow{? - ?^T} \Lambda^2 F^\vee$ splits out a direct summand isomorphic to the kernel, $\mathbb{D}_2 F^\vee$, which gives the resolution claimed. $\square$

Corollary 7.5. *As a graded module, the Hilbert series of E is*

$$H_E(t) = \frac{\binom{n}{2} t^2 - n^2 t^3 + \binom{n+1}{2}(t^4 + t^{n+2}) - n^2 t^{n+3} + \binom{n}{2} t^{n+4}}{(1-t)^{n^2}}\,.$$

Proposition 7.6. *As an S-module, E is perfect of grade 4, with support the singular locus $V(I_{n-1}(X))$ of R. More precisely, the annihilator of E is equal to $I_{n-1}(X)$.*

Proof. To see that $I_{n-1}(X)$ annihilates E, fix an index i and denote by $\overline{X}_i$ the $n\times(n-1)$–matrix obtained by deleting the i^{th} column of X. Then $L^\vee$ is isomorphic to $I_{n-1}(\overline{X}_i)$ for each i. The natural epimorphism $\operatorname{Hom}_R(L, R/I_{n-1}(\overline{X}_i)) \longrightarrow E$ shows that $I_{n-1}(\overline{X}_i)E = 0$. On the other hand, L is isomorphic to $I_{n-1}(\overline{X}'_i)$, where $\overline{X}'_i$ is obtained by deleting the i^{th} *row* of X. Thus $E \cong \operatorname{Ext}^2_R(R/I_{n-1}(\overline{X}'_i), L^\vee)$ is also killed by $I_{n-1}(\overline{X}'_i)$. Letting i vary, we see that E is annihilated by every $(n-1)\times(n-1)$–minor of X, that is, by $I_{n-1}(X)$.

By the resolution of Theorem 7.4, $\operatorname{Ann}_S E$ has codimension at most 4. But $I_{n-1}(X) \subseteq \operatorname{Ann}_S E$ and $I_{n-1}(X) = 4$ is a prime ideal of height 4, so that $I_{n-1}(X) = \operatorname{Ann}_S E$. □

It follows from Proposition 7.6 that E is naturally a module over $S/I_{n-1}(X)$. The next result details the structure of E as an $S/I_{n-1}(X)$-module.

Theorem 7.7. *As an $S/I_{n-1}(X)$-module, $E = \operatorname{Ext}^1_R(L, L^\vee)$ is a MCM module of rank one, isomorphic to the ideal generated by the maximal minors of $n-2$ fixed rows of X.*

Proof. Fix r, s with $1 \le r < s \le n$ and define a homomorphism $\xi_{rs} : E \longrightarrow S/I_{n-1}(X)$ as follows. For an alternating matrix A, let $B_A = (b_{ij})$ be the companion matrix of Theorem 2.8, so that

$$A \operatorname{adj}(X) = X^T B_A \,.$$

Set $\xi_{rs}(A) = b_{rs}$. To verify that ξ_{rs} defines a well-defined homomorphism on E, it suffices (in view of Theorem 7.4) to show that $\xi_{rs}(UX - X^TU^T) = 0$ for any $(n\times n)$–matrix U. Since the companion $B_{UX-X^TU^T}$ satisfies

$$(UX - X^TU^T)\operatorname{adj}(X) = X^T B_{UX-X^TU^T}$$

and is the unique matrix with this property, we see that

$$B_{UX-X^TU^T} = \operatorname{adj}(X)^T U - U^T \operatorname{adj}(X)\,,$$

so that $b_{rs} \in I_1(\operatorname{adj}(X)) = I_{n-1}(X)$ and $\xi_{rs}(UX - X^TU^T) = 0$ in $S/I_{n-1}(X)$.

Recall that

$$b_{rs} = \sum_{k<l} a_{kl}(-1)^{r+s+k+l}[rs \widehat{|} kl]\,.$$

The image of ξ_{rs}, as A ranges over all alternating matrices, is thus equal to the ideal generated by all minors $[\widehat{rs\,|}\,kl]$ for $1 \leq k < l \leq n$.

To show that ξ_{rs} is a monomorphism, it suffices to show that E has rank one over the integral domain $S/I_{n-1}(X)$, or equivalently that we have an equality of multiplicities $e(E) = e(S/I_{n-1}(X))$. The (graded) Scandinavian complex

$$0 \longrightarrow S(-2n) \xrightarrow{?\,\mathrm{adj}(X)} G^\vee \otimes_S F(-n) \xrightarrow{(X?,?X)} \mathbb{H} \xrightarrow{?X-X?} F^\vee \otimes_S G(-n) \xrightarrow{\mathrm{tr}(?\,\mathrm{adj}(X))} S \longrightarrow 0$$

of Definition 5.1 is a graded minimal S-free resolution of $S/I_{n-1}(X)$, so from it we compute the Hilbert series

$$H_{S/I_{n-1}(X)}(t) = \frac{1 - n^2t^{n-1} + (2n^2-2)t^n - n^2t^{n+1} + t^{2n}}{(1-t)^{n^2}}$$

and see from Corollary 7.5 that $e(E) = \frac{1}{12}(n^4 - n^2) = e(S/I_{n-1}(X))$ (*cf.* also [16]). $\square$

Consider the homomorphism of free S-modules

$$B_? : \Lambda^2 G^\vee \longrightarrow \Lambda^2 F^\vee$$

that sends an alternating matrix A to its companion matrix B_A, satisfying $A\,\mathrm{adj}(X) = X^TB_A$. Recall from Theorem 6.2 that $\mathrm{Hom}_R(M, L^\vee)$ is isomorphic to the image of $B_? \otimes_S R$. Here is another characterization of $E = \mathrm{Ext}^1_R(L, L^\vee)$ in these terms, which follows from the description of the epimorphism $\Lambda^2G^\vee \longrightarrow E$ in Theorem 7.4.

Proposition 7.8. *The module* $\mathrm{Ext}^1_R(L, L^\vee)$ *is isomorphic to the image of the homomorphism of free* $S/I_{n-1}(X)$*-modules* $B_? \otimes_S S/I_{n-1}(X)$.

Remark 7.9. Of course one has symmetric results for $E' = \mathrm{Ext}^1_R(L^\vee, L)$, exploiting the fact that $\tau^*L \cong L^\vee$ and so $\tau^*E \cong E'$, where τ is again the involution of S defined in 2.2. In particular, E' is also a MCM module of rank one over $R_1 := S/I_{n-1}(X)$, isomorphic to the ideal generated by the maximal minors of any $n-2$ fixed *columns* of X. It follows that in fact E and E' are, up to isomorphism, the only rank-one MCM $S/I_{n-1}(X)$-modules, opposites of each other in $\mathrm{Cl}(R_1)$ [7, 9.27].

8. Extensions of rank-one MCM modules

The results of Section 7 classify the extensions of $L^\vee$ by L up to equivalence. Of course inequivalent extensions may have isomorphic middle terms. In this section we describe the MCM modules over the generic determinantal hypersurface ring which appear as middle terms of the extensions in Theorem 7.4. A complete classification seems out of reach; as soon as $n \geq 3$, the class of such modules cannot be parametrized by the points of any finite-dimensional algebraic variety (Corollary 8.9).

First consider the case $n = 2$. We recover from Theorem 7.7 the following consequence of [8].

Corollary 8.1. *Let K be a field and $X = (x_{ij})$ the generic (2×2)–matrix over K. Then there are no indecomposable MCM modules of rank 2 over $R = K[x_{ij}]/(\det X)$ which are extensions of rank-one MCM R-modules.*

Proof. As $\operatorname{Ext}^1_R(L, L^\vee) \cong K$ is a one-dimensional vector space, there are up to equivalence only two nonsplit extensions of rank-one MCM R-modules: the short exact sequences displaying $L^\vee$ as a first syzygy of L, and vice versa. The middle terms of each of these are free of rank two, not indecomposable. $\square$

Of course, far more is true in this case: When $n = 2$ and K is an algebraically closed field, the *only* indecomposable MCM R-modules are R, L, and $L^\vee$ by the classification in [8], so that R has *finite Cohen–Macaulay type*. This property fails dramatically for $n \geq 3$.

To describe this failure precisely, let us suspend all our notational assumptions for a moment and consider a more general problem.

8.2. Let A and B be finitely generated modules over a (commutative, Noetherian) ring R. Fix free resolutions

$$\cdots \longrightarrow P_2 \xrightarrow{X_2} P_1 \xrightarrow{X_1} P_0 \longrightarrow A \longrightarrow 0$$
$$\cdots \longrightarrow Q_2 \xrightarrow{Y_2} Q_1 \xrightarrow{Y_1} Q_0 \longrightarrow B \longrightarrow 0$$

of A and B. An element $\chi \in \operatorname{Ext}^1_R(A, B)$ is an equivalence class of extensions

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0\,,$$

and the isomorphism class of E is determined by χ. The Horseshoe Lemma provides a free resolution of E

$$\cdots \longrightarrow Q_2 \oplus P_2 \xrightarrow{\left[\begin{smallmatrix} Y_2 & Z_2 \\ 0 & X_2 \end{smallmatrix}\right]} Q_1 \oplus P_1 \xrightarrow{\left[\begin{smallmatrix} Y_1 & Z_1 \\ 0 & X_1 \end{smallmatrix}\right]} Q_0 \oplus P_0 \longrightarrow E \longrightarrow 0\,.$$

Here the Z_i are homomorphisms in $\operatorname{Hom}_R(P_i, Q_{i-1})$ satisfying $Y_i Z_{i+1} + Z_i X_{i+1} = 0$ for all $i \geq 1$.

Definition 8.3. In the situation of 8.2, define a sequence of rings

$$R_i := R/(I_1(X_i) + I_1(Y_i))$$

for $i = 1, 2, \ldots$, where as usual $I_1(U)$ is the ideal of R generated by the entries of U. For each i set

$$\mathcal{J}_i(\chi) = \frac{I_1(Z_i) + I_1(X_i) + I_1(Y_i)}{I_1(X_i) + I_1(Y_i)}\,,$$

an ideal of R_i.

It is straightforward to check that the ideals $\mathcal{J}_i(\chi) \subseteq R_i$ are well-defined. In fact, the $\mathcal{J}_i(\chi)$ are invariants of the isomorphism class of the middle term of χ:

Proposition 8.4. *Let $\chi, \chi' \in \operatorname{Ext}^1_R(A, B)$ have middle terms E, E'. If $E \cong E'$, then $\mathcal{J}_i(\chi) = \mathcal{J}_i(\chi')$ for all i.*

The function $\mathcal{J}_i(?)$ thus defines a map from isomorphism classes of modules E appearing as extensions of B by A to ideals of R_i. We can identify which ideals are in the image of $\mathcal{J}_1$.

Proposition 8.5. *Let $Z_1 : P_1 \longrightarrow Q_0$ and $Z_2 : P_2 \longrightarrow Q_1$ be homomorphisms of free modules such that $Y_1 Z_2 + Z_1 X_2 = 0$. Then there exists $\chi \in \operatorname{Ext}^1_R(A, B)$ such that $\mathcal{J}_1(\chi) = I_1(Z_1) R_1$.*

Proof. Set $E = \operatorname{cok} \left[\begin{smallmatrix} Y_1 & Z_1 \\ 0 & X_1 \end{smallmatrix}\right]$, so that we have a commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & B \arrow[u] \arrow[r] & E \arrow[u] \arrow[r] & A \arrow[u] & \\
0 \arrow[r] & Q_0 \arrow[u] \arrow[r] & Q_0 \oplus P_0 \arrow[u] \arrow[r] & P_0 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & Q_1 \arrow[u, "Y_1"] \arrow[r] & Q_1 \oplus P_1 \arrow[u, "\left[\begin{smallmatrix} Y_1 & Z_1 \\ 0 & X_1 \end{smallmatrix}\right]"'] \arrow[r] & P_1 \arrow[u, "X_1"'] \arrow[r] & 0 \\
0 \arrow[r] & Q_2 \arrow[u, "Y_2"] \arrow[r] & Q_2 \oplus P_2 \arrow[u, "\left[\begin{smallmatrix} Y_2 & Z_2 \\ 0 & X_2 \end{smallmatrix}\right]"'] \arrow[r] & P_2 \arrow[u, "X_2"'] \arrow[r] & 0
\end{tikzcd}$$

with exact rows and columns. The map $E \longrightarrow A$ is surjective by commutativity. To see that $B \longrightarrow E$ is injective, it is equivalent by the Snake Lemma to see that the kernel of $\left[\begin{smallmatrix} Y_1 & Z_1 \\ 0 & X_1 \end{smallmatrix}\right]$ maps onto the kernel of X_1. This is a straightforward calculation using $Y_1Z_2 + Z_1X_2 = 0$. $\square$

8.6. Assume now that $R = S/(f)$ is a hypersurface ring and A, B are MCM modules over R. The free resolutions of A and B are periodic of period 2, given by matrix factorizations of f. Write $A = \operatorname{cok}(\varphi, \psi)$ and $B = \operatorname{cok}(\varphi', \psi')$. Then the sequence of rings R_i is periodic: we have

$$R_i = \begin{cases} S/(I_1(\varphi) + I_1(\varphi')) & \text{for } i \text{ odd, and} \\ S/(I_1(\psi) + I_1(\psi')) & \text{for } i \text{ even.} \end{cases}$$

For $\chi \in \operatorname{Ext}^1_R(A, B)$, the ideals $\mathcal{J}_1(\chi) \subseteq R_1$ and $\mathcal{J}_2(\chi) \subseteq R_2$ are again invariants of the middle term of χ.

8.7. Return now to the generic determinant, with notation as in 4.1. Consider $\operatorname{Ext}^1_R(L, L^\vee)$. Since $L = \operatorname{cok}(X, \operatorname{adj}(X))$ and $L^\vee = \operatorname{cok}(X^T, \operatorname{adj}(X)^T)$, we have

$$R_i = \begin{cases} S/I_1(X) \cong K & \text{for } i \text{ odd, and} \\ S/I_1(\operatorname{adj}(X) = S/I_{n-1}(X) & \text{for } i \text{ even.} \end{cases}$$

By Theorem 7.4 and Lemma 7.1, every element $\chi \in \operatorname{Ext}^1_R(L, L^\vee)$ is of the form

$$\chi : \quad 0 \longrightarrow L^\vee \longrightarrow Q \longrightarrow L \longrightarrow 0$$

with

$$Q \cong \operatorname{cok}\left(\begin{pmatrix} X & A \\ 0 & X^T \end{pmatrix}, \begin{pmatrix} \operatorname{adj}(X) & -B_A \\ 0 & \operatorname{adj}(X)^T \end{pmatrix} \right)$$

for some alternating matrix A over S and its companion matrix B_A. We therefore have $\mathcal{J}_1(\chi) = I_1(A)K$ and $\mathcal{J}_2(\chi) = I_1(B_A)S/I_{n-1}(X)$. In particular, for each ideal of $S/I_{n-1}(X)$ of the form $I_1(B_A)$, where B_A is the companion matrix for some alternating matrix A, there exists an orientable MCM R-module Q of rank 2, and distinct ideals yield nonisomorphic modules Q. More precisely, we have the following result.

Proposition 8.8. *There is a surjective function from the isomorphism classes of rank-two MCM R-modules appearing as the middle terms of extensions of L by $L^\vee$ to the set of principal ideals of the polynomial ring in $(n-2)^2$ variables.*

Proof. Let X' be the generic square matrix of size $n-2$, with entries x'_{ij}, $1 \leq i, j \leq n-2$. Let $S' = K[x'_{ij}]$ be the polynomial ring over K in those indeterminates x'_{ij}, and define $\pi : S \longrightarrow S'$ by $\pi(x_{ij}) = x'_{ij}$ if $i, j \leq n-2$ and $\pi(x_{ij}) = 0$ otherwise. The $(n-1)$-minors of X vanish under π, so we obtain an induced epimorphism $\pi : S/I_{n-1}(X) \longrightarrow S'$. Note that all $(n-2)$-minors of X vanish under π as well, save $[n-1, n \mathbin{\widehat{|}} n-1, n]$, which maps to $\det X'$.

Let $\chi \in \operatorname{Ext}^1_R(L, L^\vee)$. Then χ is the image of an alternating matrix A, and the ideal $\mathcal{J}_2(\chi) \subseteq S/I_{n-1}(X)$ is generated by the entries of the companion matrix B_A. Again, $\mathcal{J}_2(\chi)$ depends only on the isomorphism class of the middle term of χ. Recall (Theorem 2.8) that

$$b_{rs} = \sum_{k<l} a_{kl}(-1)^{r+s+k+l}[rs \mathbin{\widehat{|}} kl]\,.$$

The image of $\mathcal{J}_2(\chi)$ in S', then, is generated by the single element $\pi(a_{n-1,n})\cdot\det X'$.

Define $p : \operatorname{Ext}^1_R(L, L^\vee) \longrightarrow \{\text{ideals of } S'\}$ by $p(\chi) = (\pi(a_{n-1,n}))$. Since $\det X'$ is a nonzerodivisor in S', $p(\chi)$ is a well-defined ideal of S'. Letting A vary over all alternating matrices, we see that p is surjective, and by construction $p(\chi)$ depends only on the isomorphism class of the middle term of χ. □

Corollary 8.9. *Let $X = (x_{ij})$ be the generic $(n\times n)$–matrix over the field K, $n \geq 3$. Let $R = K[x_{ij}]/(\det X)$ be the generic determinantal hypersurface ring. Then the rank-two orientable MCM R-modules cannot be parametrized by the points of any finite-dimensional algebraic variety over K.*

Problem 8.10. Our methods afford us no information in general about orientable rank-two MCM R-modules generated by fewer than n elements. These modules correspond to matrix factorizations (φ, ψ) of the generic determinant of size $m < n$, with $\det\varphi = (\det X)^2$ and $\det\psi = (\det X)^{m-2}$ up to unit multiples. When $n \leq 4$, no such module can exist: either $\det\varphi$ or $\det\psi$ must be a unit multiple of $\det X$, so must have cokernel among $\{L, L^\vee\}$, both of which are n-generated. However, we do not know whether there exists a 4-generated rank-two orientable

MCM R-module when $n = 5$. By the correspondence laid out in [15], such a module would correspond to a codimension-3 complete intersection ideal in $K[x_{1,1}, \ldots, x_{5,5}]$ containing $\det X$ as a non-minimal generator.

9. Higher-order extensions

In this final section we consider the higher-order extension theory of the rank-one MCM modules over the generic determinant. We shall see that it is controlled by the "half-trace" of Proposition 2.10 and Remark 2.12.

Maintain the notation of 4.1. Recall from Corollary 5.6 and Proposition 3.12 that we have natural isomorphisms

$$\operatorname{Ext}^2_R(L, L) \cong \underline{\operatorname{End}}_R(L) \cong S/I_{n-1}(X)\,.$$

Dually, we also have

$$\operatorname{Ext}^2_R(L^\vee, L^\vee) \cong \underline{\operatorname{End}}_R(L^\vee) \cong S/I_{n-1}(X)\,.$$

Theorem 9.1. *Let $\chi \in \operatorname{Ext}^1_R(L, L^\vee)$ and $\chi' \in \operatorname{Ext}^1_R(L^\vee, L)$. Let $A = (a_{kl})$ and $A' = (a'_{kl})$ be alternating matrices representing χ and χ', respectively, and let $r \in R$ and C be defined as in Theorem 2.11, so that*

$$r = -\sum_{k<l,u<v} (-1)^{u+v+k+l} a_{kl}[uv \widehat{\mid} kl]a'_{uv} \ \in K$$

and we have

$$A \operatorname{adj}(X) A' = r \cdot X^T + X^T C X^T\,.$$

Then the image of r in $S/I_{n-1}(X)$ represents both the Yoneda products $\chi'\chi \in \operatorname{Ext}^2_R(L, L)$ and $-\chi\chi' \in \operatorname{Ext}^2_R(L^\vee, L^\vee)$.

9.2. Before beginning the proof, we recall the definition and computation of the Yoneda product. For extensions $\alpha \in \operatorname{Ext}^m_R(A, B)$ and $\beta \in \operatorname{Ext}^n_R(B, C)$ over some ring R, represented by an m-fold extension and an n-fold extension of R-modules, respectively, the Yoneda product $\beta\alpha \in \operatorname{Ext}^{m+n}_R(A, C)$ is represented by the $(m+n)$-fold extension obtained by splicing the representatives for α and β together at their common endpoint B. This product is computed as follows ([19, p.91]):

(1) Choose preimages $\sigma \in \operatorname{Hom}_R(\operatorname{syz}^R_m(A), B)$ and $\rho \in \operatorname{Hom}_R(\operatorname{syz}^R_n(B), C)$ for α and β;

(2) Lift σ to $\widetilde{\sigma} \in \operatorname{Hom}_R(\operatorname{syz}^R_{m+n}(A), \operatorname{syz}^R_n(B))$

(3) $\beta\alpha$ is the image of the composition $\rho\widetilde{\sigma}$ in $\operatorname{Ext}^{m+n}_R(A, C)$.

Proof of Theorem 9.1. To compute $\chi'\chi$, we first choose the natural preimages in $\operatorname{Hom}_R(M, L^\vee)$ and $\operatorname{Hom}_R(M^\vee, L)$. Specifically, $\operatorname{cok}(A, B_A) \in \operatorname{Hom}_R(M, L^\vee)$ is a preimage for χ, and $\operatorname{cok}(A', B_{A'}) \in \operatorname{Hom}_R(M^\vee, L)$ is a preimage for χ'. We lift $\operatorname{cok}(A, B_A)$ naturally to $\operatorname{cok}(B_A, A) \in \operatorname{Hom}_R(L, M^\vee)$. Then $\chi'\chi$ is computed by the image in $\operatorname{Ext}^2_R(L, L)$ of $A'B_A \in \operatorname{Hom}_R(\overline{F}, \overline{F})$. By Proposition 2.10, we have

$$B_A A' = r \cdot \operatorname{id}_n + C X^T\,.$$

As A', B_A are alternating, transposing both sides yields

$$A' B_A = r \cdot \operatorname{id}_n + X C^T\,.$$

The image of the term XC^T in $\operatorname{Ext}^2_R(L, L)$ is zero, as XC^T factors through X. Thus $\chi'\chi$ is the image in $\operatorname{Ext}^2_R(L, L)$ of $r \cdot \operatorname{id}_n$, which agrees in $S/I_{n-1}(X)$ with r.

A symmetric calculation reveals that r also represents $-\chi\chi' \in \operatorname{Ext}^2_R(L^\vee, L^\vee)$. □

9.3. The map

$$\begin{aligned}
E = \operatorname{Ext}^1_R(L, L^\vee) \quad &\longrightarrow \quad R_1 := S/I_{n-1}(X)\\
\chi = [A] \quad &\longmapsto \quad r(\tau(\chi), \chi) = r(\tau(A), A)\\
&\qquad = -\sum_{k<l,u<v} (-1)^{u+v+k+l} \tau(a_{kl})[uv \,\widehat{|}\, kl] a_{uv}
\end{aligned}$$

is R_1-quadratic. We use this quadratic form to define the stable Ext-algebra.

Definition 9.4. The *stable* Ext-*algebra of the rank-one MCM R-modules* is the positively graded algebra $\mathcal{E}$ with homogeneous components

$$\mathcal{E}^i = \underline{\operatorname{Ext}}^i_R(L \oplus L^\vee, L \oplus L^\vee)\,,$$

and multiplication induced by the Yoneda product.

The graded components of $\mathcal{E}$ depend only on parity, so we may consider instead $\underline{\mathcal{E}} := \mathcal{E}^0 \oplus \mathcal{E}^1$ as a graded algebra over $\mathbb{Z}/2\mathbb{Z}$. The structure of $\underline{\mathcal{E}}$ can then be arranged as (2×2)–matrices:

$$\mathcal{E}^0 = \underline{\mathrm{End}}_R(L\oplus L^\vee) \cong \begin{pmatrix} R_1 & 0 \\ 0 & R_1 \end{pmatrix} ;$$

$$\mathcal{E}^1 = \underline{\mathrm{Ext}}^1_R(L\oplus L^\vee, L\oplus L^\vee) \cong \begin{pmatrix} 0 & E' \\ E & 0 \end{pmatrix}$$

with multiplication in $\mathcal{E}^1$ defined by the quadratic form r. Here we have observed that $\underline{\mathrm{Hom}}_R(L, L^\vee) = 0$ from the resolution in [7, Theorem 2.16]; see also the proof of Theorem 7.4.

Here is a summary of these observations.

Theorem 9.5. *With structure as defined above, the stable* Ext*-algebra* $\underline{\mathcal{E}}$ *is a graded-commutative,* $\mathbb{Z}/2\mathbb{Z}$*-graded algebra, with each homogeneous component an orientable MCM module of rank two over* $S/I_{n-1}(X)$*. Moreover, the multiplication yields*

$$\mathcal{E}^0/(\mathcal{E}^1)^2 \cong \begin{pmatrix} S/I_{n-2}(X) & 0 \\ 0 & S/I_{n-2}(X) \end{pmatrix} ,$$

so that the quadratic form degenerates precisely on the singular locus of R_1*.*

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Dept. of Comp. and Math. Sciences, University of Toronto at Scarborough (UTSC), Toronto, Ont. M1A 1C4, Canada

E-mail address: `ragnar@math.utoronto.ca`

Dept. of Math., Syracuse University, Syracuse NY 13244, USA

E-mail address: `gjleusch@math.syr.edu`

URL: `http://www.leuschke.org/`