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Wild hypersurfaces

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Abstract

Complete hypersurfaces of dimension at least 2 and multiplicity at least 4 have wild Cohen-Macaulay type.

Keywords: maximal Cohen–Macaulay module, wild representation type, hypersurface ring
2010 MSC: 16G50, 16G60, 13C14, 13H10, 16G10

Introduction

Let R be a (commutative, Noetherian) local ring. A finitely generated R-module M is called *maximal Cohen–Macaulay* (MCM) provided $\operatorname{depth} M = \dim R$. In particular, R is a Cohen–Macaulay (CM) ring if it is MCM as a module over itself.

This paper is about *CM representation types,* specifically tame and wild CM types. See §1 for the definitions of these properties. In this Introduction, we motivate our main result by recalling the classification of complete equicharacteristic hypersurface rings of *finite CM type.*

Theorem ([2, 13])**.** *Let k be an algebraically closed field of characteristic not equal to 2, 3, or 5. Let $d \geqslant 1$, let $f \in k[\![x_0,\dots,x_d]\!]$ be a non-zero non-unit power series, and let $R = k[\![x_0,\dots,x_d]\!]/(f)$ be the corresponding hypersurface ring. Then there are only finitely many isomorphism classes of indecomposable MCM R-modules if, and only if, we have an isomorphism $R \cong$*

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To Roger Wiegand, our advisor, on the occasion of his N^{th} ascent of the Diamond.

$k[\![x_0,\dots,x_d]\!]/(g(x_0,x_1)+x_2^2+\cdots+x_d^2)$, *where* $g(x_0,x_1)$ *is one of the following polynomials, indexed by the ADE Coxeter–Dynkin diagrams:*

$$\begin{aligned}
&(A_n): && x_0^2+x_1^{n+1}, \quad \textit{some } n\geqslant 1;\\
&(D_n): && x_0^2x_1+x_1^{n-1}, \quad \textit{some } n\geqslant 4;\\
&(E_6): && x_0^3+x_1^4;\\
&(E_7): && x_0^3+x_0x_1^3;\\
&(E_8): && x_0^3+x_1^5.
\end{aligned}$$

A *key step* in the proof of this theorem is [2, Prop. 3.1], which says that if $d\geqslant 2$ and the multiplicity $\mathrm{e}(R)$ is at least 3 (equivalently $f\in(x_0,\dots,x_d)^3$) then R has a family of indecomposable MCM modules parametrized by the points of a cubic hypersurface in $\mathbb{P}^d_k$.

One would like a classification theorem like the one above for, say, hypersurfaces of tame CM type. (Again, see §1 for definitions.) Drozd and Greuel have shown [8] that the one-dimensional hypersurfaces defined in $k[\![x_0,x_1]\!]$ by

$$(T_{pq}) \qquad x_0^p+x_1^q+\lambda x_0^2x_1^2,$$

with $p,q\geqslant 2$, $\lambda\in k\setminus\{0,1\}$, and k an algebraically closed field of characteristic not equal to 2, have tame CM type. (With the exception of the cases $(p,q)=(4,4)$ and $(3,6)$, one may assume $\lambda=1$.) In fact, they show that a curve singularity of infinite CM type has tame CM type if and only if it birationally dominates one of these hypersurfaces. More recently, Drozd, Greuel, and Kashuba [9] have shown that the two-dimensional analogues

$$(T_{pqr}) \qquad x_0^p+x_1^q+x_2^r+x_0x_1x_2$$

with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\leqslant 1$ have tame CM type. Since these hypersurface rings have multiplicity 3 in general, the desired *key step* in a classification of hypersurface rings of tame CM type would have to be of the form "If $d\geqslant 2$ and $\mathrm{e}(R)\geqslant 4$, then R has wild CM type." This result is indeed true for $d=2$, as proved by Bondarenko [1].

In working through Bondarenko's proof, we found a way to simplify the argument somewhat; this simplification allows us to prove the desired *key step* for all $d\geqslant 2$. Thus we prove (Theorem 13)

Main Theorem. *Let* $S=k[\![x_0,\dots,x_d]\!]$ *with* $d\geqslant 2$ *and* f *a non-zero power series of order at least* 4. *Then* $R=S/(f)$ *has wild Cohen-Macaulay type.*

By the original *key step* of [2], the case $d \geqslant 3$ is already known to admit at least a $\mathbb{P}^2$ of indecomposable MCM modules, so is already perhaps known by experts to have wild type. Not being aware of an explicit statement to that effect, we think that a unified statement is desirable.

In §1 we give a brief survey of tame and wild representation types for the commutative-algebraist reader, including Drozd's proof of the essential fact that $k[a_1, \ldots, a_n]$ is finite-length wild for $n \geqslant 2$, and in §2 we prove the Main Theorem.

We are grateful to the anonymous referee, whose careful reading improved the paper.

1. Tameness and Wildness

There are several minor variations on the notions of tame and wild representation type, but the intent is always the same: tame representation type allows the possibility of a classification theorem in the style of Jordan canonical form, while for wild type any classification theorem at all is utterly out of reach. The definitions we will use are essentially those of Drozd [6]; they seem to have appeared implicitly first in [4]. They make precise the intent mentioned above by invoking the classical unsolved problem of canonical forms for n-tuples of matrices up to simultaneous similarity [11] (see Example 3 below).

Definition 1. Let k be an infinite field, R a local k-algebra, and let $\mathscr{C}$ be a full subcategory of the finitely generated R-modules.

(i) We say that $\mathscr{C}$ is *tame,* or *of tame representation type,* if there is one discrete parameter r (such as k-dimension or R-rank) parametrizing the modules in $\mathscr{C}$, such that, for each r, the indecomposables in $\mathscr{C}$ form finitely many one-parameter families and finitely many exceptions. Here a *one-parameter family* is a set of R-modules $\{E/(t-\lambda)E\}_{\lambda \in k}$, where E is a fixed $k[t]$-R-bimodule which is finitely generated and free over $k[t]$.

(ii) We say that $\mathscr{C}$ is *wild,* or *of wild representation type,* if for every finite-dimensional k-algebra Λ (not necessarily commutative!), there exists a *representation embedding* $\mathscr{E}\colon \operatorname{mod}\Lambda \longrightarrow \mathscr{C}$, that is, $\mathscr{E}$ is an exact functor preserving non-isomorphism and indecomposability.

We are mostly interested in two particular candidates for $\mathscr{C}$. When $\mathscr{C}$ consists of the full subcategory of R-modules of finite length, then we say R is *finite-length tame* or *finite-length wild*. At the other extreme, when $\mathscr{C}$ is the full subcategory $\mathrm{MCM}(R)$ of maximal Cohen–Macaulay R-modules, we say R has *tame* or *wild CM type.*

The following Dichotomy Theorem justifies the slight unwieldiness of the definitions. (See also [12] for a more general statement.)

Theorem 2 (Drozd [6, 7], Crawley-Boevey [3])**.** *A finite-dimensional algebra over an algebraically closed field is either finite-length tame or finite-length wild, and not both.*

In this paper we will be most concerned with wildness. It follows immediately from the definition that, to establish that a given module category $\mathscr{C}$ is wild, it suffices to find a single particular example of a wild $\mathscr{C}_0$ and a representation embedding $\mathscr{C}_0 \longrightarrow \mathscr{C}$. To illustrate this idea, as well as for our own use in the proof of the Main Theorem, we give here a couple of examples.

Example 3 ([11])**.** The non-commutative polynomial ring $k\langle a,b\rangle$ over an infinite field k is finite-length wild. To see this, let $\Lambda = k\langle x_1,\dots,x_m\rangle/I$ be an arbitrary finite-dimensional k-algebra and let V be a Λ-module of finite k-dimension n. Represent the actions of the variables $x_1,\dots,x_m$ on V by linear operators $X_1,\dots,X_m \in \mathrm{End}_k(V)$. For m distinct scalars $c_1,\dots,c_m \in k$, define a $k\langle a,b\rangle$-module $M = M_V$ as follows: the underlying vector space of M is $V^{(m)}$, and we let a and b act on M via the linear operators

$$A = \begin{bmatrix} c_1\,\mathrm{id}_V & & & \\ & c_2\,\mathrm{id}_V & & \\ & & \ddots & \\ & & & c_m\,\mathrm{id}_V \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} X_1 & & & \\ \mathrm{id}_V & X_2 & & \\ & \ddots & \ddots & \\ & & \mathrm{id}_V & X_m \end{bmatrix},$$

respectively.

A homomorphism of $k\langle a,b\rangle$-modules from M_V to $M_{V'}$ is defined by a vector space homomorphism $S\colon V^{(m)} \longrightarrow V'^{(m)}$ satisfying $SA = A'S$ and $SB = B'S$, where A and B, resp. A' and B', are the matrices defining the $k\langle a,b\rangle$ structures on M_V, resp. $M_{V'}$. Two modules M_V and $M_{V'}$ are isomorphic via S if and only if S is invertible over k. Similarly, a module M_V is decomposable if and only if there is a non-trivial idempotent endomorphism $S\colon M_V \longrightarrow M_V$.

Assume that $\dim_k V = \dim_k V'$ and let $S\colon V^{(m)} \longrightarrow V'^{(m)}$ be a vector space homomorphism such that $SA = A'S$ and $SB = B'S$. Then we can write $S = (\sigma_{ij})_{1\leqslant i,j\leqslant m}$, with each $\sigma_{ij}\colon V \longrightarrow V'$; we will show that $S = \operatorname{diag}(\sigma_{11},\dots,\sigma_{11})$ and that $\sigma_{11}X_i = X_i'\sigma_{11}$ for each $i = 1,\dots,m$. Thus S is an isomorphism if and only if $\sigma_{11}\colon V \longrightarrow V'$ is an isomorphism of Λ-modules, and S is idempotent if and only if σ_{11} is.

The equation $SA = A'S$ implies $\sigma_{ij}c_j = c_i\sigma_{ij}$ for every i,j. Since the scalars c_i are pairwise distinct, this implies that $\sigma_{ij} = 0$ for all $i \neq j$, so that S is a block-diagonal matrix. Now the equation $SB = B'S$ becomes

$$\begin{bmatrix} \sigma_{11}X_1 & & & & \\ & \sigma_{22} & \ddots & & \\ & & \ddots & \sigma_{m-1,m-1}X_{m-1} & \\ & & & \sigma_{mm} & \sigma_{mm}X_m \end{bmatrix} = \begin{bmatrix} X_1'\sigma_{11} & & & & \\ & \sigma_{11} & \ddots & & \\ & & \ddots & X_{m-1}'\sigma_{m-1,m-1} & \\ & & & \sigma_{m-1,m-1} & X_m'\sigma_{mm} \end{bmatrix},$$

which implies that $\sigma_{ii} = \sigma_{11}$ for each $i = 1,\dots,m$. Denote the common value by σ; then the diagonal entries show that $\sigma X_i = X_i'\sigma$ for each $i = 1,\dots,m$.

Example 4 ([5])**.** Let k be an infinite field, and set $R = k[a,b]/(a^2, ab^2, b^3)$. Then R is finite-length wild. Consequently, the commutative polynomial ring $k[a_1,\dots,a_n]$ and the commutative power series ring $k[\![a_1,\dots,a_n]\!]$ are both finite-length wild as soon as $n \geqslant 2$.

The last sentence follows from the one before, since any R-module of finite length is also a module of finite length over $k[a,b]$ and $k[\![a,b]\!]$, whence also over $k[a_1,\dots,a_n]$ and $k[\![a_1,\dots,a_n]\!]$. Thus by Example 3 above, it suffices to construct a representation embedding of the finite-length modules over $k\langle x,y\rangle$ into $\operatorname{mod} R$.

Let V be a $k\langle x,y\rangle$-module of k-dimension n, with linear operators X and Y representing the $k\langle x,y\rangle$-module structure. We define $(32n \times 32n)$ matrices A and B yielding an R-module structure on $M = M_V = V^{(32)}$. To wit, let $c_1,\dots,c_5 \in k$ be distinct scalars and

$$A = \begin{bmatrix} 0 & 0 & \mathrm{id}_{V^{(15)}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} B_1 & 0 & B_2 \\ 0 & 0 & B_3 \\ 0 & 0 & B_1 \end{bmatrix},$$

where

$$B_1 = \begin{bmatrix} 0 & 0 & \mathrm{id}_{V^{(5)}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ \mathrm{id}_{V^{(5)}} & 0 & 0 \\ 0 & C & 0 \end{bmatrix}, \qquad \text{and} \qquad B_3 = \begin{bmatrix} 0 & D & 0 \end{bmatrix},$$

and finally

$$C = \begin{bmatrix} c_1 \mathrm{id}_V & & & \\ & c_2 \mathrm{id}_V & & \\ & & \ddots & \\ & & & c_5 \mathrm{id}_V \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \mathrm{id}_V & 0 & \mathrm{id}_V & \mathrm{id}_V & \mathrm{id}_V \\ 0 & \mathrm{id}_V & \mathrm{id}_V & X & Y \end{bmatrix}.$$

Observe that, while all the blocks in B_1, B_2, and B_3 are $(5n \times 5n)$, the blocks in A and B are not of uniform size; their four corner blocks are $(15n \times 15n)$, while the center block is $(2n \times 2n)$.

One verifies easily that $AB = BA$ and $A^2 = AB^2 = B^3 = 0$, so A and B do indeed define an R-module structure on M_V.

Let V' be a second n-dimensional $k\langle x, y\rangle$-module, with linear operators X' and Y' defining the $k\langle x, y\rangle$-module structure, and define $M' = M_{V'}$ as above, with linear operators A' and B' giving M' the structure of an R-module. Let $S : V^{(32n)} \longrightarrow V'^{(32n)}$ be a vector space homomorphism such that $SA = A'S$ and $SB = B'S$. We will show that in this case S is a block-upper-triangular matrix (with blocks of size n) having constant diagonal block $\sigma : V \longrightarrow V'$ which satisfies $\sigma X = X'\sigma$ and $\sigma Y = Y'\sigma'$. Thus S is an isomorphism of R-modules if and only if σ is an isomorphism of $k\langle x, y\rangle$-modules, and S is a split surjection if and only if σ is so. It follows that the functor $V \rightsquigarrow M_V$ is a representation embedding, and R is finite-length wild.

Note that A is independent of the module V, so $A = A'$ and $SA = AS$. Write S in block format, with blocks of the same sizes as A,

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix},$$

this means

$$\begin{bmatrix} 0 & 0 & S_{11} \\ 0 & 0 & S_{21} \\ 0 & 0 & S_{31} \end{bmatrix} = \begin{bmatrix} S_{31} & S_{32} & S_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ 0 & S_{22} & S_{23} \\ 0 & 0 & S_{11} \end{bmatrix}.$$

Using now the equation $SB = B'S$, we get

$$\begin{bmatrix} S_{11}B_1 & 0 & S_{11}B_2 + S_{12}B_3 + S_{13}B_1 \\ 0 & 0 & S_{22}B_3 + S_{23}B_1 \\ 0 & 0 & S_{11}B_1 \end{bmatrix} = \begin{bmatrix} B_1'S_{11} & B_1'S_{12} & B_1'S_{13} + B_2'S_{11} \\ 0 & 0 & B_3'S_{11} \\ 0 & 0 & B_1'S_{11} \end{bmatrix}.$$

In particular, $S_{11}B_1 = B_1'S_{11}$. Write the $(15n \times 15n)$ matrix S_{11} in $(5n \times 5n)$-block format as

$$S_{11} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}.$$

Then the definition of B_1 and B_1' gives

$$S_{11} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{11} \end{bmatrix}$$

as above. Now S_{12} is $(15n \times 2n)$, so we write it in $(5n \times 2n)$ blocks as $S_{12} = \begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix}^{\mathrm{tr}}$ and use $B_1'S_{12} = 0$ to get $S_{12} = \begin{bmatrix} U_1 & U_2 & 0 \end{bmatrix}^{\mathrm{tr}}$. We also have $S_{22}B_3 + S_{23}B_1 = B_3'S_{11}$; if we write $S_{23} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$, then this equation reads

$$\begin{bmatrix} 0 & S_{22}D & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & V_1 \end{bmatrix} = \begin{bmatrix} 0 & D'T_{22} & D'T_{23} \end{bmatrix}.$$

It follows that $S_{22}D = D'T_{22}$ and $S_{23} = \begin{bmatrix} D'T_{23} & V_2 & V_3 \end{bmatrix}$.

Finally write

$$S_{13} = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix}$$

and consider the equation

$$S_{11}B_2 + S_{12}B_3 + S_{13}B_1 = B_1'S_{13} + B_2'S_{11}.$$

It becomes

$$\begin{bmatrix} T_{12} & T_{13}C + U_1D & W_{11} \\ T_{22} & T_{23}C + U_2D & W_{21} \\ 0 & T_{11}C & W_{31} \end{bmatrix} = \begin{bmatrix} W_{31} & W_{32} & W_{33} \\ T_{11} & T_{12} & T_{13} \\ 0 & C'T_{22} & C'T_{23} \end{bmatrix}.$$

We read off $T_{22} = T_{11}$ and $T_{11}C = C'T_{11}$. Since $C = C'$ is a diagonal matrix with distinct blocks $c_1\mathrm{id}_V, \ldots, c_5\mathrm{id}_V$, this forces T_{11} to be block-diagonal,

$$T_{11} = \begin{bmatrix} Z_1 & & \\ & \ddots & \\ & & Z_5 \end{bmatrix},$$

with each Z_i an $(n \times n)$ matrix.

We also have $S_{22}D = D'T_{11}$. Write $S_{22} = \left[\begin{smallmatrix} E & F \\ G & H \end{smallmatrix}\right]$ so that this reads

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix} \begin{bmatrix} \mathrm{id}_V & 0 & \mathrm{id}_V & \mathrm{id}_V & \mathrm{id}_V \\ 0 & \mathrm{id}_V & \mathrm{id}_V & X & Y \end{bmatrix} = \begin{bmatrix} \mathrm{id}_V & 0 & \mathrm{id}_V & \mathrm{id}_V & \mathrm{id}_V \\ 0 & \mathrm{id}_V & \mathrm{id}_V & X' & Y' \end{bmatrix} \begin{bmatrix} Z_1 & & \\ & \ddots & \\ & & Z_5 \end{bmatrix}.$$

Carrying out the multiplication, we conclude that $F = G = 0$, so that $E = Z_1 = Z_3 = Z_4 = Z_5$ and $H = Z_2 = Z_3$. Set $\sigma = E = H$. Then $HX = X'Z_4$ and $HY = Y'Z_5$ imply $\sigma X = X'\sigma$ and $\sigma Y = Y'\sigma$, so that σ is a homomorphism of $k\langle x, y\rangle$-modules $V \longrightarrow V'$. Since T_{11} and S_{22} are both block-diagonal with diagonal block σ, we conclude that S is block-upper-triangular with constant diagonal block σ, as claimed.

We restate one part of this example separately for later use.

Proposition 5. *Let $Q = k[a_1, \ldots, a_n]$ or $k[\![a_1, \ldots, a_n]\!]$, with $n \geqslant 2$. If there is a representation embedding of the finite-length Q-modules into a module category $\mathscr{C}$, then $\mathscr{C}$ is wild.* □

2. Proof of the Main Theorem

We use without fanfare the theory of matrix factorizations, namely the equivalence between matrix factorizations of a power series f and MCM modules over the hypersurface ring defined by f ([10], see [14] for a complete discussion). The two facts we will use explicitly are contained in the following Remark and Example.

Remark 6. Let S be a regular local ring and $f \in S$ a non-zero non-unit. Set $T = S[\![u, v]\!]$. Then the functor from matrix factorizations of f over S to matrix factorizations of $f + uv$ over T, defined by

$$(\varphi, \psi) \mapsto \left(\begin{bmatrix} \varphi & -vI \\ uI & \psi \end{bmatrix}, \begin{bmatrix} \psi & vI \\ -uI & \varphi \end{bmatrix} \right),$$

induces an equivalence of stable categories [14, Theorem 12.10]. In particular it gives a bijection on isomorphism classes of MCM modules over $S/(f)$ and $T/(f + uv)$.

Example 7. Let k be a field and set $S_n = k[\![x_1,\dots,x_n,y_1,\dots,y_n]\!]$ and $f_n = x_1y_1+\cdots+x_ny_n$ for $n \geqslant 1$. The ring $R_n = S_n/(f_n)$ is an (A_1) hypersurface singularity, so has finite Cohen-Macaulay type; in fact, there is only one non-free indecomposable MCM R_n-module, or equivalently, one nontrivial indecomposable matrix factorization of f_n. By the remark above, the nontrivial indecomposable matrix factorizations of f_n are in bijection with those of f_{n+1}. For $n = 1$, the element $f_1 = x_1y_1$ has only one nontrivial indecomposable matrix factorization up to equivalence, namely that represented by $(\varphi_1,\psi_1) = (x_1,y_1)$. Defining

$$(\varphi_i,\psi_i) = \left(\begin{bmatrix}\varphi_{i-1} & -y_iI\\ x_iI & \psi_{i-1}\end{bmatrix}, \begin{bmatrix}\psi_{i-1} & y_iI\\ -x_iI & \varphi_{i-1}\end{bmatrix}\right),$$

we have that (φ_n,ψ_n) represents the sole nontrivial indecomposable matrix factorization of f_n over S_n.

Next we see that, at the cost of introducing some parameters, every power series of sufficiently high order can be written in the form of an (A_1) singularity, with some control over the coefficients.

Lemma 8. *Let $f \in k[\![x_1,\dots,x_n,z]\!]$ be a power series of order at least 4, and let $a_1,\dots,a_n$ be parameters. Then f can be written in the form*

$$f = z^2h + (x_1-a_1z)g_1 + \cdots + (x_n-a_nz)g_n \tag{8.1}$$

where $g_1,\dots,g_n,h$ are power series in $x_1,\dots,x_n,z$ with coefficients involving the parameters $a_1,\dots,a_n$, each g_i has order at least 3 in $x_1,\dots,x_n,z$, and h has order at least 2 in $x_1,\dots,x_n,z$.

Proof. Work over $k[\![x_1,\dots,x_n,z]\!]$, with the parameters $a_1,\dots,a_n$ considered as variable elements of k, and consider the ideals $\mathfrak{m} = (x_1,\dots,x_n,z)$ and $I = (x_1-a_1z,\dots,x_n-a_nz)$. We claim that $(z^2)+I\mathfrak{m} = \mathfrak{m}^2$. The left-hand side is clearly contained in the right. For the other inclusion, simply check each monomial of degree 2: $z^2 \in (z^2)+I\mathfrak{m}$ by definition, whence

$$x_iz = (x_i-a_iz)z + a_iz^2 \quad \in (z^2)+I\mathfrak{m}$$

for each i, and

$$x_ix_j = (x_i-a_iz)x_j + a_ix_jz \quad \in (z^2)+I\mathfrak{m}$$

for each i,j. Writing $\mathfrak{m}^4 = \mathfrak{m}^2\mathfrak{m}^2 = ((z^2)+I\mathfrak{m})\mathfrak{m}^2 = z^2\mathfrak{m}^2 + I\mathfrak{m}^3$ completes the proof. □

Given an expression for $f \in k[\![x_1,\dots,x_n,z]\!]$ as in Lemma 8, we obtain from Remark 6 a matrix factorization (φ_n,ψ_n) of f, with

$$(\varphi_0,\psi_0) = \left([z^2],[h]\right), \qquad (\varphi_1,\psi_1) = \left(\begin{bmatrix} z^2 & -g_1 \\ x_1 - a_1 z & h \end{bmatrix}, \begin{bmatrix} h & g_1 \\ -x_1 + a_1 z & z^2 \end{bmatrix}\right)$$

and, in general,

$$(\varphi_n,\psi_n) = \left(\begin{bmatrix} \varphi_{n-1} & -g_n \operatorname{id}_{2^{n-1}} \\ (x_n - a_n z)\operatorname{id}_{2^{n-1}} & \psi_{n-1} \end{bmatrix}, \begin{bmatrix} \psi_{n-1} & g_n \operatorname{id}_{2^{n-1}} \\ (-x_n + a_n z)\operatorname{id}_{2^{n-1}} & \varphi_{n-1} \end{bmatrix}\right).$$

We now describe how to "inflate" these matrix factorizations given a $k[a_1,\dots,a_n]$-module of finite length.

Definition 9. Let $A_1,\dots,A_r$ be pairwise commuting $m \times m$ matrices over the field k. Let $f = f(a_1,\dots,a_r)$ be a power series in variables $x_1,\dots,x_n,z$ involving the parameters $a_1,\dots,a_r$, which we think of as variable elements of k. Let $F = F(A_1,\dots,A_r)$ be the $m \times m$ matrix obtained by replacing in f each scalar $\alpha \in k$ by $\alpha \operatorname{id}_m$, each x_i by $x_i \operatorname{id}_m$, z by $z \operatorname{id}_m$, and each parameter a_i by the corresponding matrix A_i. We call this process *inflating* f.

If $(\varphi,\psi) = (\varphi(a_1,\dots,a_r),\psi(a_1,\dots,a_r))$ is a matrix factorization, again involving parameters $a_1,\dots,a_r$, of an element $f \in k[\![x_1,\dots,x_n,z]\!]$, let $(\Phi,\Psi) = (\Phi(A_1,\dots,A_r),\Psi(A_1,\dots,A_r))$ be the result of inflating each entry of φ and ψ.

Note that in the second half of the definition, f does not involve the parameters. It's easy to check that, since the A_i commute, (Φ,Ψ) is again a matrix factorization of f.

It follows from Lemma 8 that a power series $f \in k[\![x_1,\dots,x_n,z]\!]$ of order at least 4 has, for every n-tuple of commuting $m \times m$ matrices $(A_1,\dots,A_n)$ over k, a matrix factorization

$$(\Phi_n,\Psi_n) = (\Phi(A_1,\dots,A_n),\Psi(A_1,\dots,A_n)) \tag{9.1}$$

of size $m2^n$.

Notation 10. Let $E = [e_{ij}]$ be a matrix with entries in $k[\![x_1,\dots,x_n,z]\!]$. We set $\overline{E} = [\overline{e_{ij}}]$, where $\overline{e_{ij}}$ denotes the image of e_{ij} modulo the square of the maximal ideal $(x_1,\dots,x_n,z)$.

Also, given a monomial $w \in k[\![x_1,\dots,x_n,z]\!]$, let $E\{w\}$ denote the matrix $[e_{ij}\{w\}]$, where $e_{ij}\{w\}$ denotes the coefficient of w in the power series expansion of e_{ij}. We call this the "w-strand" of the matrix E.

For the rest of the paper, we let f be a power series of order at least 4 as in Lemma 8, let $A_1,\dots,A_n$ and $A'_1,\dots,A'_n$ be n-tuples of commuting $m\times m$ matrices over k, and let $(\Phi_n,\Psi_n)=(\Phi(A_1,\dots,A_n),\Psi(A_1,\dots,A_n))$ and $(\Phi'_n,\Psi'_n)=(\Phi(A'_1,\dots,A'_n),\Psi(A'_1,\dots,A'_n))$ be inflated matrix factorizations of f as in (9.1).

Lemma 11. *Let $i\in\{0,\dots,n\}$ and let C, D be two $(m2^i\times m2^i)$ matrices with entries in k. If C and D satisfy*

$$C\overline{\Phi_i}=\overline{\Phi'_i}D \qquad \text{and} \qquad D\overline{\Psi_i}=\overline{\Psi'_i}C\,, \tag{$\dagger$}$$

then

(*i*) *C and D are $(m\times m)$-block lower triangular, i.e. of the form*

$$C=\begin{bmatrix} C_{11} & & & \\ & C_{22} & \mathbf{0} & \\ & & \ddots & \\ & * & & C_{2^i,2^i}\end{bmatrix},\qquad D=\begin{bmatrix} D_{11} & & & \\ & D_{22} & \mathbf{0} & \\ & & \ddots & \\ & * & & D_{2^i,2^i}\end{bmatrix}.$$

(*ii*) *For each $j=1,\dots,2^i$, C_{jj} and D_{jj} are in the set $\{C_{11},\,D_{11}\}$.*

(*iii*) *For each $j=1,\dots,i$, $C_{2^i,2^i}A_j=A'_jD_{2^i-2^{j-1},2^i-2^{j-1}}$.*

Proof. For parts (i) and (ii), we proceed by induction on i. The base case $i=0$ is vacuous. For the inductive step, since in (8.1) $g_i\in(x_1,\dots,x_n,z)^3$, we can express $\overline{\Phi_i}$, $\overline{\Psi_i}$ as

$$\overline{\Phi_i}=\begin{bmatrix}\overline{\Phi_{i-1}} & 0\\ (\overline{x_i}\,\mathrm{id}_m-A_n\overline{z})\mathrm{id}_{2^{i-1}} & \overline{\Psi_{i-1}}\end{bmatrix},\qquad \overline{\Psi_i}=\begin{bmatrix}\overline{\Psi_{i-1}} & 0\\ (-\overline{x_i}\,\mathrm{id}_m+A_n\overline{z})\mathrm{id}_{2^{i-1}} & \overline{\Phi_{i-1}}\end{bmatrix}$$

and $\overline{\Phi'_i}$, $\overline{\Psi'_i}$ similarly, matrices over $k[[x_1,\dots,x_n,z]]/(x_1,\dots,x_n,z)^2$. (We write $(\overline{x_i}\,\mathrm{id}_m-A_n\overline{z})\mathrm{id}_{2^{i-1}}$ to represent a $(m2^{i-1}\times m2^{i-1})$-block matrix with diagonal blocks $\overline{x_i}\,\mathrm{id}_m-A_n\overline{z}$.) Also express C and D in terms of their $(m2^{i-1}\times m2^{i-1})$-blocks

$$C=\begin{bmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{bmatrix},\qquad D=\begin{bmatrix}\delta_{11} & \delta_{12}\\ \delta_{21} & \delta_{22}\end{bmatrix}.$$

From $C\overline{\Phi_i} = \overline{\Phi'_i}D$, we get the equations

$$\gamma_{11}\overline{\Phi_{i-1}} + \gamma_{12}(\overline{x_i}\,\mathrm{id}_m - A_i\overline{z})\mathrm{id}_{2^{i-1}} = \overline{\Phi'_{i-1}}\delta_{11} \tag{11.1}$$

$$\gamma_{22}\overline{\Psi_{i-1}} = (\overline{x_i}\,\mathrm{id}_m - A'_i\overline{z})\mathrm{id}_{2^{i-1}}\,\delta_{12} + \overline{\Psi'_{i-1}}\delta_{22} \tag{11.2}$$

$$\gamma_{21}\overline{\Phi_{i-1}} + \gamma_{22}(\overline{x_i}\,\mathrm{id}_m - A_i\overline{z})\mathrm{id}_{2^{i-1}} = (\overline{x_i}\,\mathrm{id}_m - A'_i\overline{z})\mathrm{id}_{2^{i-1}}\,\delta_{11} + \overline{\Psi'_{i-1}}\delta_{21} \tag{11.3}$$

and from $D\overline{\Psi_i} = \overline{\Psi'_i}C$:

$$\delta_{21}\overline{\Psi_{i-1}} + \delta_{22}(-\overline{x_i}\,\mathrm{id}_m + A_i\overline{z})\mathrm{id}_{2^{i-1}} = (-\overline{x_i}\,\mathrm{id}_m + A'_i\overline{z})\mathrm{id}_{2^{i-1}}\,\gamma_{11} + \overline{\Phi'_{i-1}}\gamma_{21}\,. \tag{11.4}$$

Since $\overline{\Phi_{i-1}}$, $\overline{\Psi_{i-1}}$, $\overline{\Phi'_{i-1}}$, $\overline{\Psi'_{i-1}}$ do not contain instances of $\overline{x_i}$, we conclude

from (11.1):	$\gamma_{12} = 0$	and	$\gamma_{11}\overline{\Phi_{i-1}} = \overline{\Phi'_{i-1}}\delta_{11}$;
from (11.2):	$\delta_{12} = 0$	and	$\gamma_{22}\overline{\Psi_{i-1}} = \overline{\Psi'_{i-1}}\delta_{22}$;
from (11.3):	$\gamma_{22} = \delta_{11}$; and		
from (11.4):	$\delta_{22} = \gamma_{11}$.		

Thus the pair γ_{11}, δ_{11} satisfy (†), so by the induction hypothesis, they satisfy (i) and (ii). Since C and γ_{11} share the same $(1,1)$ $m \times m$-block (and ditto for D and δ_{11}), the inductive proof is complete.

For part (iii), we consider the $(m \times m)$ block in position $(2^i, 2^i - 2^{j-1})$ on either side of the equation $C\overline{\Phi_i} = \overline{\Phi'_i}D$. We get that

$$C_{2^i,2^i}(\overline{x_j}\,\mathrm{id}_m - A_j\overline{z}) = (\overline{x_j}\,\mathrm{id}_m - A -_j \overline{z})D_{2^i-2^{j-1},2^i-2^{j-1}}\,.$$

Examining the $\overline{z}$-strand yields the desired equality. □

Proposition 12. *Let $(S,T)\colon (\Phi_n,\Psi_n) \longrightarrow (\Phi'_n,\Psi'_n)$ be a homomorphism of matrix factorizations. Then $S\{1\}$ and $T\{1\}$ are $(m \times m)$-block lower triangular of the form*

$$S\{1\} = \begin{bmatrix} U & & & & \\ & U & & \mathit{0} & \\ & & U & & \\ & & & \ddots & \\ & * & & & U \end{bmatrix}, \qquad T\{1\} = \begin{bmatrix} U & & & & \\ & U & & \mathit{0} & \\ & & U & & \\ & & & \ddots & \\ & * & & & U \end{bmatrix}, \tag{$*$}$$

where $UA_i = A'_iU$ for $i = 1,\ldots,n$.

Proof. We first show that $S_{11}\{1\} = T_{11}\{1\}$, where S_{ij} and T_{ij} denote the $(m \times m)$ blocks of S and T, respectively, in the $(i,j)^{\text{th}}$ position. For this, we consider the $(m \times m)$ block in position $(1,1)$ on either side of the equation $S\Phi_n = \Phi'_n T$. We get that

$$S_{11}z^2 + \sum_{i=1}^{n} S_{1,2^{i-1}+1}(x_i \,\mathrm{id}_m - A_i z) = z^2 T_{11} + \sum_{i=1}^{n} T_{2^{i-1}+1,1} G'_i,$$

where G'_i are the matrices resulting from "inflating" the power series g'_i. Since the g'_i have order at least 3, each entry of G'_i also has order at least 3, and so the quadratic strands give the following equations:

$$\{z^2\}: \qquad S_{11}\{1\} - \sum_{i=1}^{n} S_{1,2^{i-1}+1}\{z\}A_i = T_{11}\{1\} \tag{12.1}$$

$$\{x_i^2\}: \qquad S_{1,2^{i-1}+1}\{x_i\} = 0 \tag{12.2}$$

$$\{x_i z\}: \qquad S_{1,2^{i-1}+1}\{z\} - \sum_{j=1}^{n} S_{1,2^{j-1}+1}\{x_i\}A_j = 0 \tag{12.3}$$

$$\{x_i x_j\}: \qquad S_{1,2^{j-1}+1}\{x_i\} + S_{1,2^{i-1}+1}\{x_j\} = 0 \tag{12.4}$$

Starting from equation (12.1), we have

$$\begin{aligned} S_{11}\{1\} &= T_{11}\{1\} + \sum_{i=1}^{n} S_{1,2^{i-1}+1}\{z\}A_i \\ &= T_{11}\{1\} + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} S_{1,2^{j-1}+1}\{x_i\}A_j A_i\right) \\ &= T_{11}\{1\}, \end{aligned}$$

the last equality following from equations (12.2), (12.4), and the commutativity of the A_i.

The proof is completed by Lemma 11 above. □

Let M be a $k[a_1,\ldots,a_n]$-module of dimension m over k. After choosing a k-basis for M, the action of each a_i on M can be expressed as multiplication by an $(m \times m)$ matrix A_i over k. (Note that the A_i's must be pairwise commutative.) We may thus identify M with the linear representation $L\colon k[a_1,\ldots,a_n] \longrightarrow \mathrm{Mat}_m(k)$, where $a_i \mapsto A_i$ for $i = 1,\ldots,n$.

A homomorphism from a linear representation $L(A_1,\ldots,A_n)$ to another $L(A'_1,\ldots,A'_n)$ is defined by a matrix U such that $UA_i = A'_i U$ for $i = 1,\ldots,n$.

Representations are thus isomorphic if this matrix U is invertible. A representation $L(A_1,\dots,A_n)$ is decomposable if it has a non-trivial idempotent endomorphism, that is, there exists a matrix U such that $UA_i = A_iU$ for $i = 1,\dots,n$, $U^2 = U$ and $U \neq 0$, id.

Theorem 13. *Let k be an infinite field, let $S = k[\![x_1,\dots,x_n,z]\!]$, and let $f \in S$ be a non-zero element of order at least 4. Set $R = S/(f)$. Then the functor F from finite-length $k[a_1,\dots,a_n]$-modules to MCM R-modules, sending a given linear representation $L(A_1,\dots,A_n)$ to the inflated matrix factorization $(\Phi(A_1,\dots,A_n),\Psi(A_1,\dots,A_n))$, is a representation embedding.*

In particular, if $n \geqslant 2$ then R has wild Cohen-Macaulay type.

Proof. The functor F is defined as follows on homomorphisms of linear representations $U\colon L(A_1,\dots,A_n) \longrightarrow L(A'_1,\dots,A'_n)$. If the A_i are $\ell \times \ell$ matrices and the A'_i are $m \times m$, then U is an $m \times \ell$ matrix over k, and is sent to the block-diagonal $(m2^n \times \ell 2^n)$ matrix $\widetilde{U}$ with U down the diagonal. Since U satisfies the relations $UA_i = A'_iU$, and the blocks of $\Phi(A_1,\dots,A_n)$ and $\Psi(A_1,\dots,A_n)$ are power series in the matrices A_i with coefficients in S, we get $\widetilde{U}\Phi = \Phi'\widetilde{U}$ and $\widetilde{U}\Psi = \Psi'\widetilde{U}$. Now it is clear that F is an exact functor.

Suppose there is an isomorphism between matrix factorizations

$$(S,T)\colon (\Phi(A_1,\dots,A_n),\ \Psi(A_1,\dots,A_n)) \to (\Phi(A'_1,\dots,A'_n),\ \Psi(A'_1,\dots,A'_n)).$$

By Proposition 12, $S\{1\}$ and $T\{1\}$ are of the form in $(*)$, in which U defines a homomorphism from $L(A_1,\dots,A_n)$ to $L(A'_1,\dots,A'_n)$. Since S is invertible, so is U. Thus the representations are isomorphic.

Suppose the matrix factorization $(\Phi(A_1,\dots,A_n),\ \Psi(A_1,\dots,A_n))$ is decomposable, that is, it has an endomorphism (S,T) such that $S^2 = S$, $T^2 = T$ and $(S,T) \neq (0,0),(\mathrm{id},\mathrm{id})$. Again, by Proposition 12, $S\{1\}$ and $T\{1\}$ are of the form in $(*)$, in which the matrix U now defines an idempotent endomorphism of the representation $L(A_1,\dots,A_n)$. Since S and T are idempotent matrices, if $U = 0$, then $S = T = 0$. Similarly, if $U = \mathrm{id}$, then $S = T = \mathrm{id}$. Thus the representation $L(A_1,\dots,A_n)$ must be decomposable.

The final sentence follows from Proposition 5. □

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