1-1-2011

Excess Porteous, Coherent Porteous, and the Hyperelliptic Locus in M3

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ABSTRACT

In the moduli space of curves of genus 3, the locus of hyperelliptic curves forms a divisor, that is a closed subscheme of codimension 1. J. Harris and I. Morrison compute an expression for the class of this divisor, in the Chow ring of the moduli space, using a map of vector bundles and by applying the Thom-Porteous formula to determine an expression for a certain degeneracy locus of this map. One would like to extend their idea in order to compute an expression for the divisor associated to the closure of the hyperelliptic locus, in the Chow ring of the moduli space of stable curves (of genus 3.)

Recent work due to S. Diaz allows one to define the degeneracy class of a map between coherent sheaves, and gives explicit means for computing this class. Diaz uses his technique to partially extend the above mentioned computation, but he points out that in order to complete the computation one must combine his techniques with an Excess Thom-Porteous formula. This thesis completes this computation by combining the work of Diaz with this Excess Thom-Porteous formula.

Excess Porteous, Coherent Porteous, and the Hyperelliptic Locus in $\overline{\mathfrak{M}}_3$

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DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

in the Graduate School of Syracuse University

August 2011

Contents

List of Illustrations

Acknowledgments

I would like to thank my advisor, Steven Diaz, for introducing me to my thesis problem and for the time he spent guiding me through its solution. I would also like to thank my committee members for agreeing to take the time to read my thesis. Finally, I would like to thank my wife, Bianca, for years of support.

Chapter 1

Background

1.1 Introduction

In [HM, p.162] the authors consider a family $\pi : X \to B$ of smooth curves of genus 3, not all of which are hyperelliptic, and a map $\sigma : \mathcal{E} \to \mathcal{F}$ of vector bundles (of ranks 3 and 2, respectively) on X. They show that this map fails to be surjective exactly at the hyperelliptic Weierstrass points of hyperelliptic fibers of π. They then use the Thom-Porteous formula for vector bundles to determine an expression for the class of

$$D_1(\sigma) = \{x \in X \mid \mathrm{rank}(\sigma_x) \leq 1\}$$

in the Chow group of X. The authors then use this result to obtain an expression in $\mathrm{Pic}_{\mathrm{fun}}(\mathfrak{M}_3) \otimes \mathbb{Q}$ (the group of divisor classes on the moduli stack) for the class of the locus of hyperelliptic curves.

One would like to extend this technique to determine an expression in $\mathrm{Pic}_{\mathrm{fun}}(\overline{\mathfrak{M}}_3)\otimes\mathbb{Q}$ for the closure of the locus of hyperelliptic curves. Unfortunately, if one supposes that $\pi : X \to B$ is a family of stable curves of genus 3, then $\mathcal{F}$ will fail to be locally free at singular points of singular fibers of π. (See [HM, Sec. 3.F] for details.) Harris and Morrison are able to compute this class in $\mathrm{Pic}_{\mathrm{fun}}(\overline{\mathfrak{M}}_3) \otimes \mathbb{Q}$ using different techniques, but one would still like to extend the original technique to compute the class.

The author of [D], by constructing a certain blow-up $g : X' \to X$ and a map $\sigma' : \mathcal{E}' \to \mathcal{F}'$ of vector bundles on X', related to the original map σ, is able to define the degeneracy class for a map of coherent sheaves. The author then applies this process in order to determine an expression in $\mathrm{Pic}_{\mathrm{fun}}(\overline{\mathfrak{M}}_3) \otimes \mathbb{Q}$ for the class of the closure of the hyperelliptic locus in $\overline{\mathfrak{M}}_3 \setminus \Delta_1$. The author points out that at singular curves corresponding to general points of Δ_1, not only will $\mathcal{F}$ fail to be locally free at the singular points, but also the map σ will have rank ≤ 1 at all points of the elliptic component of the fiber. The author suggests that one could combine the process for determining the degeneracy class of a map of coherent sheaves with the excess Porteous formula found in [F, Ex. 14.4.7] to compute an expression in $\mathrm{Pic}_{\mathrm{fun}}(\overline{\mathfrak{M}}_3)\otimes\mathbb{Q}$ for the class of the closure of the hyperelliptic locus in $\overline{\mathfrak{M}}_3$. This is the goal of this thesis.

To this end, I consider a family $\pi : X \to B$ of smooth, nonhyperelliptic curves degenerating to a general element of Δ_1, and I consider the map $\sigma' : \mathcal{E}' \to \mathcal{F}'$ mentioned above. After determining the scheme structure of $D_1(\sigma')$, I then use the excess Porteous formula ([F, Ex. 14.4.7]) to determine the number of times the standard Thom-Porteous formula counts a general element of Δ_1. Finally, I combine this computation with that of [D] to determine an expression in $\mathrm{Pic}_{\mathrm{fun}}(\overline{\mathfrak{M}}_3) \otimes \mathbb{Q}$ for the class of the closure of the hyperelliptic locus in $\overline{\mathfrak{M}}_3$.

1.2 Moduli Spaces of Curves

Definition 1.2.1. Let $f : X \to B$ be a morphism of schemes over Spec $\mathbb{C}$ and let $b \in B$ be a point. Let $k(b)$ be the residue field of b and $\operatorname{Spec} k(b) \to B$ be the natural morphism. Then the *fiber of f over b* is the scheme

$$X_b := X \times_B \operatorname{Spec} k(b)$$

Notation 1.2.2. By a *smooth curve* we will mean a smooth curve over $\mathbb{C}$ that is a complete and connected.

Definition 1.2.3. Let $g \in \mathbb{Z}^+$. Let $f : X \to B$ be a flat morphism of schemes over Spec $\mathbb{C}$ such that X_b is a smooth curve of genus g for every closed point $b \in B$. We call f a *family of smooth curves of genus g over B*.

Fix $g \in \mathbb{Z}^+$, and consider the functor

$$F : \text{schemes} \to \text{sets}$$

that sends a scheme B to the set of all families of smooth curves of genus g over B, modulo the following equivalence relation:

$$X \xrightarrow{f} B \ \sim \ X' \xrightarrow{g} B$$

if there exists an isomorphism $\varphi : X \to X'$ such that $f = g \circ \varphi$.

Remark 1.2.4. Note that $F(\text{Spec } \mathbb{C})$ is the set of isomorphism classes of smooth curves of genus g.

Definition 1.2.5. A scheme $\mathfrak{M}$ and a natural transformation ψ from F to the functor $\mathrm{Mor}(-, \mathfrak{M})$ are a *coarse moduli space* for F if:

i) The map $\psi_{\text{Spec } \mathbb{C}} : F(\text{Spec } \mathbb{C}) \to \mathrm{Mor}(\text{Spec } \mathbb{C}, \mathfrak{M})$ is a set bijection.

ii) Given another scheme $\mathfrak{M}'$ and a natural transformation $\psi' : F \to \mathrm{Mor}(-, \mathfrak{M}')$, there exists a unique morphism $\pi : \mathfrak{M} \to \mathfrak{M}'$ such that the associated natural transformation $\Pi : \mathrm{Mor}(-, \mathfrak{M}) \to \mathrm{Mor}(-, \mathfrak{M}')$ satisfies $\psi' = \Pi \circ \psi$.

Proposition 1.2.6. *If a coarse moduli space exists for F, it is unique up to canonical isomorphism.*

Proof. Suppose $\mathfrak{M}$ and $\mathfrak{M}'$ are both coarse moduli spaces for F, with corresponding natural transformations ψ and ψ', respectively, then there exists a unique morphism $\pi : \mathfrak{M} \to \mathfrak{M}'$ such that the associated natural transformation Π (as in (1.2.5)) satisfies $\psi' = \Pi \circ \psi$.

Similarly, there exists a unique morphism $\pi' : \mathfrak{M}' \to \mathfrak{M}$ such that the associated natural transformation Π' satisfies $\psi = \Pi' \circ \psi'$.

But now we have $\pi' \circ \pi : \mathfrak{M} \to \mathfrak{M}$ is an isomorphism with associated natural transformation $\Pi' \circ \Pi$. Moreover, by above we have

$$\begin{aligned} \psi &= \Pi' \circ \psi' \\ &= \Pi' \circ (\Pi \circ \psi) \\ &= (\Pi' \circ \Pi) \circ \psi. \end{aligned}$$

But we know that there is a unique morphism $\varphi : \mathfrak{M} \to \mathfrak{M}$ whose associated natural transformation, Φ, satisfies $\psi = \Phi \circ \psi$. Since $\mathrm{id}_{\mathfrak{M}}$ is such a morphism, we see that $\pi' \circ \pi = \mathrm{id}_{\mathfrak{M}}$. Similarly, $\pi \circ \pi' = \mathrm{id}_{\mathfrak{M}'}$. Thus $\mathfrak{M} \cong \mathfrak{M}'$. Moreover, by the uniqueness of π, we see that the isomorphism is canonical. □

Theorem 1.2.7. *For $g \geq 2$, there exists a coarse moduli space for F, $\mathfrak{M}_g$. Moreover, it is an irreducible quasi-projective variety of dimension $3g-3$ over $\mathbb{C}$.*

Proof. See [M, Thm 5.11]. □

Remark 1.2.8. $\mathfrak{M}_g$ is neither projective nor affine. (See [HM, p.45].)

Definition 1.2.9. A *stable curve* is a complete connected curve that has only nodes as singularities and has only finitely many automorphisms.

Remark 1.2.10. A smooth curve of genus g has at most $84(g-1)$ automorphisms and hence is stable. (See [H, IV.Ex.2.5].)

Definition 1.2.11. The *arithmetic genus* of a connected curve C is $\dim_{\mathbb{C}} H^1(C, \mathcal{O}_C)$.

Notation 1.2.12. By [H, III.7.12.2] the arithmetic genus of a smooth curve is equal to its geometric genus (which up until now we have simply called its genus.) Thus we will refer simply to the *genus* of a stable curve with the understanding that for a smooth curve this can be interpreted as before and for a singular stable curve this will refer to its arithmetic genus.

Proposition 1.2.13. *Let C be a stable curve of genus g with δ nodes and ν irreducible components $C_1, \dots, C_\nu$ of genera $g_1, \dots, g_\nu$, then*

$$g = \left(\sum_{i=1}^{\nu} g_i \right) + \delta - \nu + 1$$

Proof. See [HM, 3.1]. □

Definition 1.2.14. Let $g \in \mathbb{Z}^+$. Let $f : X \to B$ be a flat morphism of schemes over Spec $\mathbb{C}$ such that X_b is a stable curve of (arithmetic) genus g for every closed point $b \in B$. We call f a *family of stable curves of genus g over B.*

Fix $g \in \mathbb{Z}^+$ and consider the functor F_{stab} : schemes $\to$ sets that sends a scheme B to the set of families of stable curves of genus g over B, modulo the same equivalence relation as above. Similar to (1.2.5) we can define what it means to be a coarse moduli space for F_{stab}.

Theorem 1.2.15. *For $g \geq 2$, there exists a coarse moduli space for F_{stab}, $\overline{\mathfrak{M}}_g$. Moreover, it is a projective variety of dimension $3g - 3$ over $\mathbb{C}$.*

Proof. See [HM, Thm. 2.15]. □

Remark 1.2.16. By (1.2.10), we see that $\mathfrak{M}_g \subset \overline{\mathfrak{M}}_g$.

Definition 1.2.17. We call $\overline{\mathfrak{M}}_g$ the *stable compactification* of $\mathfrak{M}_g$.

Notation 1.2.18. Let $\Delta = \overline{\mathfrak{M}}_g \setminus \mathfrak{M}_g$.

Definition 1.2.19. For $i, = 1, \dots, \lfloor g/2 \rfloor$, define Δ_i to be the closure in $\overline{\mathfrak{M}}_g$ of the locus of curves of genera i and $g - i$ meeting transversely at one point.

Define Δ_0 to be the closure in $\overline{\mathfrak{M}}_g$ of the locus of irreducible curves with a single node.

Proposition 1.2.20. Δ *is a divisor in* $\overline{\mathfrak{M}}_g$. *Moreover,* $\Delta_0, \Delta_1, \ldots, \Delta_{\lfloor g/2 \rfloor}$ *are its irreducible components.*

Proof. See [HM, p.50]. $\square$

1.3 Hyperelliptic Curves

Definition 1.3.1. A smooth curve C of genus $g \geq 2$ is called *hyperelliptic* if there exists a finite morphism $f : C \to \mathbb{P}^1$ of degree 2.

Proposition 1.3.2. *The locus of hyperelliptic curves in* $\mathfrak{M}_g$ *is of dimension* $2g - 1$.

We will use the following three lemmas in the proof of the above proposition.

Lemma 1.3.3. *If C is a hyperelliptic curve, then there is a* unique *finite morphism $f : C \to \mathbb{P}^1$ of degree 2, up to automorphism of $\mathbb{P}^1$.*

Proof. See [H, IV.5.3]. □

Lemma 1.3.4. *If C is a hyperelliptic curve of genus g and $f : C \to \mathbb{P}^1$ is a finite morphism of degree 2, then f is branched at $2g + 2$ points of $\mathbb{P}^1$.*

Proof. By [H, IV.2.4], the degree of the ramification divisor, R, of $f : C \to \mathbb{P}^1$ is $2g + 2$. Moreover, since f is a morphism of degree 2, every point of C appearing in R (i.e., appearing with nonzero coefficient) has a coefficient of 1, and all such points must map to distinct points of $\mathbb{P}^1$. □

Definition 1.3.5. The points of C in 1.3.4 that lie over the branch points in $\mathbb{P}^1$ are called *hyperelliptic Weierstrass points of C*.

Lemma 1.3.6. *Any set of $2g + 2$ distinct elements of $\mathbb{P}^1$ determine a hyperelliptic curve of genus g.*

Proof. First we choose homogeneous coordinates on $\mathbb{P}^1$ so that none of of the $2g+2$ distinct elements equal ∞. Let $\alpha_1, \ldots, \alpha_{2g+2} \in \mathbb{C}$ be distinct. Let

$$F(z) = z^2 - (x-\alpha_1)\ldots(x-\alpha_{2g+2}) \in \mathbb{C}(x)[z],$$

and let

$$K = \frac{\mathbb{C}(x)[z]}{F(z)}.$$

Since the α_i are distinct, F is irreducible, and hence K is a finite extension of $\mathbb{C}(x)$ of degree 2. The inclusion $\mathbb{C}(x) \hookrightarrow K$ then defines a morphism of curves, $f : C \to \mathbb{P}^1$, of degree 2. Moreover, this map will be branched at exactly $\alpha_1, \ldots \alpha_{2g+2} \in \mathbb{P}^1$. By [H, IV.2.4] the genus of C must be g. □

(Proof of 1.3.2). Let C be a hyperelliptic curve of genus G and $f : C \to \mathbb{P}^1$ a morphism of degree 2. We can normalize the $2g+2$ branch points in $\mathbb{P}^1$ by first ordering them and then following f by the automorphism of $\mathbb{P}^1$ that sends the first three branch points to $0, 1,$ and ∞, respectively. By abuse of notation we continue to call the composition of f with this automorphism f. Thus we can assume that f is branched at $0, 1, \infty$ and $2g-1$ points of $\mathbb{C} \setminus \{0,1\}$.

Let S_{2g+2} be the symmetric group on $2g+2$ letters. We define an action on sets of $2g-1$ distinct elements $\beta_1, \ldots, \beta_{2g-1} \in \mathbb{C} \setminus \{0,1\}$:

i) Let $\sigma \in S_{2g+2}$. Reorder the set $0, 1, \infty, \beta_1, \ldots, \beta_{2g-1}$ according to σ.

ii) Renormalize using the automorphism of $\mathbb{P}^1$ that sends $\sigma(0), \sigma(1)$, and $\sigma(\infty)$ back to 0, 1, ∞. The images of $\sigma(\beta_1), \ldots, \sigma(\beta_{2g-1})$ become a new set of $2g-1$ distinct elements in $\mathbb{C} \setminus \{0,1\}$.

We consider the open set given by

$$\left(\cap_{i=1}^{2g-1}\{x_i \neq 0\}\right) \bigcap \left(\cap_{i=1}^{2g-1}\{x_i \neq 1\}\right) \bigcap \left(\cap_{i\neq j}\{x_i \neq x_j\}\right)$$

in $\mathbb{A}^{2g-1}_{x_1,\ldots,x_{2g-1}}$, modulo the S_{2g+2} action described above. Since S_{2g+2} is a finite group, this will be an irreducible variety of dimension $2g-1$; call it V.

By (1.3.6) we see that to any point of V there corresponds a hyperelliptic curve of genus g. By (1.3.4) we see that every hyperelliptic curve of genus g must correspond to some point of V. Finally, by (1.3.3) we see that two hyperelliptic curves of genus g correspond to the same point of V if and only if they are isomorphic. Hence V is a parameter space for hyperelliptic curves of genus g. In particular, the dimension of the locus of hyperelliptic curves in $\mathfrak{M}_g$ must equal the dimension of V. $\square$

Proposition 1.3.7. *The locus of hyperelliptic curves in $\mathfrak{M}_g$ is closed.*

Proof. Let B be a regular, integral scheme and $f : X \to B$ be a family of smooth curves of genus $g \geq 2$ such that for $b \in B \setminus \{b_0\}$, X_b is hyperelliptic. Let $\omega = \omega_{X/B}$ be the relative dualizing sheaf. Since ω restricts to the canonical bundle on each fiber of f, it gives a map $g : X \to \mathbb{P}^{g-1} \times B$. By [H, IV.5.3], the image of X_b, for $b \neq b_0$ is a rational curve and $g(X_{b_0})$ is either a rational curve, if X_{b_0} is hyperelliptic, or is isomorphic to X_{b_0}, if X_{b_0} is not hyperelliptic. Thus $p_2 : g(X) \to B$ is a family of rational curves that degenerates to $g(X_{b_0})$. Since such a family cannot possibly degenerate to a curve of genus $g \geq 2$, we see that $g(X_{b_0})$ is rational. Hence X_{b_0} is hyperelliptic. $\square$

Corollary 1.3.8. *The hyperelliptic locus forms a divisor in* $\mathfrak{M}_3$.

Proof. This follows from (1.2.7), (1.3.2), and (1.3.7). □

Notation 1.3.9. Let X be a projective scheme over $\mathbb{C}$ and $\mathcal{F}$ a coherent sheaf on X. For $i \geq 0$, we let $h^i(X, \mathcal{F}) :=\dim_{\mathbb{C}} H^i(X, \mathcal{F})$. (By [H, III.5.2] this is always finite.)

Notation 1.3.10. Let X be a projective scheme over $\mathbb{C}$, $\mathcal{L}$ an invertible sheaf on X, and D a divisor on X. Then we let $\mathcal{L}(D) := \mathcal{L} \otimes \mathcal{O}(D)$.

Proposition 1.3.11. *Let C be a smooth curve, Q a closed point of C, and ω the canonical bundle of C. Then $h^0(C, \omega(-Q)) = h^0(C, \omega(-2Q))$ if and only if Q is a hyperelliptic Weierstrass point of C. In particular, C is not hyperelliptic if and only if $h^0(C, \omega(-Q)) = h^0(C, \omega(-2Q)) + 1, \forall Q \in C$.*

Proof. By the Riemann-Roch Theorem for curves, $h^0(C, \omega(-Q)) = h^0(C, \omega(-2Q))$ if and only if $h^0(C, \mathcal{O}(2Q)) = h^0(C, \mathcal{O}(Q)) + 1 = 2$. But this is true if and only if $|2Q|$ determines a finite morphism of degree 2 from C onto $\mathbb{P}^1$ (which will automatically be ramified at Q); in other words, if and only if Q is a hyperelliptic Weierstrass point of C.

The second part follows from the first and the Riemann-Roch Theorem. □

1.4 The Class of a Degeneracy Locus

Let $\sigma : \mathcal{E} \to \mathcal{F}$ be a homomorphism of vector bundles of ranks e and f on an n-dimensional variety X, and let $k \leq \min(e, f)$.

Definition 1.4.1. The k^{th} *degeneracy locus of* σ, as a set, is

$$D_k(\sigma) := \{x \in X | \mathrm{rank}(\sigma_x) \leq k\}.$$

Remark 1.4.2. Locally, σ can be represented by an $f \times e$ matrix with entries in an affine coordinate ring of X. One can then consider the ideal generated by the $(k+1) \times (k+1)$ minor determinants of this local representation. These local ideals patch together to give an ideal sheaf, which gives $D_k(\sigma)$ the structure of a closed subscheme of X.

Notation 1.4.3. Set $m = n - (e-k)(f-k)$ and $d = e - k$.

In what follows, we use the construction given in [F, 14.4] for defining the k^{th} degeneracy class of σ.

Let $G_d(\mathcal{E})$ be the Grassmannian of $d-$planes in $\mathcal{E}$; let π be the projection from $G_d(E)$ to X, and let S be the universal subbundle of $\pi^*\mathcal{E}$. The composition

$$S \to \pi^*\mathcal{E} \to \pi^*\mathcal{F}$$

determines a section, s_σ, of $S^\vee \otimes \pi^*\mathcal{F}$. The zero set $Z(s_\sigma)$ maps onto $D_k(\sigma)$; let $\eta : Z(s_\sigma) \to D_k(\sigma)$ be the induced morphism. The localized top Chern class $\mathbb{Z}(s_\sigma)$ is in $A_m(Z(s_\sigma))$. (See [F, 14.1] for a discussion of locallized top Chern classes.)

Definition 1.4.4. We define the k^{th} *degeneracy class of* σ to be

$$\mathbb{D}_k(\sigma) = \eta_*(\mathbb{Z}(s_\sigma)) \in A_m(D_k(\sigma))$$

Notation 1.4.5. Let $c = \sum c_i t^i$ be a formal power series. Then $\Delta_q^{(p)}(c)$ will denote the determinant of the $p \times p$ matrix whose i, j-entry is given by c_{j-i+q}.

Notation 1.4.6. $c(\mathcal{E}) := 1 + c_1(\mathcal{E})t + \ldots + c_e(\mathcal{E})t^e$, denotes the *Chern polynomial* of $\mathcal{E}$, and $s(\mathcal{E}) := 1 + s_1(\mathcal{E})t + s_2(\mathcal{E})t^2 + \ldots$, the *Segre polynomial* of $\mathcal{E}$.
$c(\mathcal{F} - \mathcal{E}) := c(\mathcal{F})s(\mathcal{E})$.

Theorem 1.4.7 (The Thom-Porteous Formula)**. (a)** *The image of* $\mathbb{D}_k(\sigma)$ *in* $A_m(X)$ *is*

$$\mathbb{D}_k(\sigma) = \Delta_{f-k}^{(e-k)}\left(c(\mathcal{F} - \mathcal{E})\right) \cap [X].$$

(b) *Each irreducible component of* $D_k(\sigma)$ *has codimension at most* $(e-k)(f-k)$*, in* X*. If codim* $(D_k(\sigma), X) = (e-k)(f-k)$*, then* $\mathbb{D}_k(\sigma)$ *is a positive cycle whose support is* $D_k(\sigma)$*.*

(c) *If* $\operatorname{codim}(D_k(\sigma), X) = (e-k)(f-k)$*, and* X *is Cohen-Macaulay, then* $D_k(\sigma)$ *is also Cohen-Macaulay and*

$$\mathbb{D}_k(\sigma) - [D_k(\sigma)].$$

Proof. See [F, Theorem 14.4]. □

Theorem 1.4.8 (Excess Porteous Formula)**.** *Let* $\sigma : \mathcal{E} \to \mathcal{F}$ *be as above and let* k *be an integer such that* $D_{k-1}(\sigma) = \emptyset$*. Then on* $D = D_k(\sigma)$*, there is an exact sequence*

$$0 \to \mathcal{K} \to \mathcal{E}_D \to \mathcal{F}_D \to \mathcal{C} \to 0,$$

with $\mathcal{K}$,$\mathcal{C}$ vector bundles of ranks $e - k$, $f - k$. Then

$$\mathbb{D}_k(\sigma) = \{c(K^* \otimes C) \cap s(D_k(\sigma), X)\}_m.$$

Proof. See [F, Example 14.4.7]. □

We would now like to define the class of a degeneracy locus for a map of coherent sheaves. We follow the method of [D].

Lemma 1.4.9. *Let $\mathcal{F}$ be a coherent sheaf on the n-dimensional variety X. Let $Y \subset X$ be the locus where $\mathcal{F}$ fails to be locally free. Restricted to $X - Y$, $\mathcal{F}$ is a vector bundle; call its rank f. Let $\mathcal{I}$ be the f^{th} Fitting ideal sheaf of $\mathcal{F}$ on X. The zero scheme of $\mathcal{I}$ has support equal to Y. Let $h : Z \to X$ be the blow-up of X along $\mathcal{I}$. Let $h^*\mathcal{F}$ be the pullback to Z of $\mathcal{F}$. Then the double dual $(h^*\mathcal{F})^{**}$ is locally free, that is a vector bundle.*

Proof. See [D, Lemma 1]. □

Theorem 1.4.10. *Let X be an n-dimensional variety and $\sigma : \mathcal{E} \to \mathcal{F}$ be a homomorphism of coherent sheaves on X. Then there is a blow-up $g : X' \to X$ with the following properties:*

(i) *If Y is the locus where either $\mathcal{E}$ or $\mathcal{F}$ fails to be locally free, then g gives an isomorphism between $X - Y$ and $g^{-1}(X - Y)$.*

(ii) *The pullbacks of the restrictions of $\mathcal{E}$ and $\mathcal{F}$ to $X - Y$ extend to vector bundles on all of X'.*

(iii) *The pullback of the restriction of σ to $X - Y$ extends to a homomorphism of the vector bundles in (ii) over all of X'.*

Proof. Let $h_1 : Z \to X$ be the blow-up of X along the f^{th} Fitting ideal sheaf of $\mathcal{F}$. On Z we have a homomorphism of coherent sheaves $h_1^*\sigma : h_1^*\mathcal{E} \to h_1^*\mathcal{F}$. Taking double duals gives $(h_1^*\sigma)^{**} : (h_1^*\mathcal{E})^{**} \to (h_1^*\mathcal{F})^{**}$. This extends the vector bundle homomorphism $h_1^*\sigma$ on $Z - h_1^{-1}(Y)$.

By (1.4.9) $(h_1^*\mathcal{F})^{**}$ is a vector bundle. Let e be the rank of $(h_1^*\mathcal{E})^{**}$ on its locally free locus, and let $h_2 : X' \to Z$ be the blow-up of Z along the e^{th} Fitting ideal sheaf of $(h_1^*\mathcal{E})^{**}$. By (1.4.9) we then have a homomorphism of vector bundles

$$(h_2^*(h_1^*\sigma)^{**})^{**} : (h_2^*(h_1^*\mathcal{E})^{**})^{**} \to (h_2^*(h_1^*\mathcal{F})^{**})^{**}$$

Notation 1.4.11. Let $\sigma' : \mathcal{E}' \to \mathcal{F}'$ be the map of vector bundles above.

Let $g = h_2 \circ h_1 : X' \to X$.

Using this notation g is the desired blow-up, $\mathcal{E}'$ and $\mathcal{F}'$ are the desired vector bundles, and σ' is the desired homomorphism. □

Definition 1.4.12. Let $k \le \min(e, f)$ and m be as in (1.4.3). We apply the standard Thom-Porteous formula to obtain a class in $A_m(X')$:

$$\Delta_{f-k}^{(e-k)}\,(c(\mathcal{F}' - \mathcal{E}')) \cap [X'].$$

We then define the k^{th} *degeneracy class of the morphism of coherent sheaves* $\sigma : \mathcal{E} \to \mathcal{F}$ to be

$$g_*(\Delta_{f-k}^{(e-k)}\,(c(\mathcal{F}' - \mathcal{E}')) \cap [X']).$$

Chapter 2

The hyperelliptic locus in $\overline{\mathfrak{M}}_3 \setminus \Delta_1$

2.1 The hyperelliptic locus in $\mathfrak{M}_3$

By (1.3.8) the locus of hyperelliptic curves in $\mathfrak{M}_3$ forms a divisor, which we'll call H. In this section we present the method for computing the class, $[H]$, in $\mathrm{Pic}(\mathfrak{M}_3) \otimes \mathbb{Q}$ given in [HM, 3.E]. To this end, let $\pi : X \to B$ be a one-parameter family of smooth curves of genus 3, not all of which are hyperelliptic, with smooth base B. Let $X_2 := X \times_B X$, with π_i, $i = 1, 2$, the projections, and let $\mathcal{I}_\Delta$ be the ideal sheaf of the diagonal. We consider the natural map $\mathcal{O}_{X_2} \to \mathcal{O}_{X_2}/\mathcal{I}_\Delta^2$. Tensoring with $\pi_2^* \omega_{X/B}$ and pushing forward under π_1 gives a map

$$\sigma : (\pi_1)_* \left(\pi_2^* \omega_{X/B} \otimes \mathcal{O}_{X_2} \right) \to (\pi_1)_* \left(\pi_2^* \omega_{X/B} \otimes \mathcal{O}_{X_2}/\mathcal{I}_\Delta^2 \right).$$

Notation 2.1.1. Let $\mathcal{E}$ and $\mathcal{F}$ be the domain and target, respectively, of the above map.

Proposition 2.1.2. *$\mathcal{E}$ and $\mathcal{F}$ are locally free sheaves, that is vector bundles, on X.*

Proof. Let $p \in X$. We consider two copies of the family $\pi : X \to B$. Let x_i, y_i, $i = 1, 2$, be local coordinates on X at p with π given locally by $x_i = t_i$, so that t_i is a local coordinate on B centered at $\pi(p)$. Near $p \times p$, $X \times_B X$ has local coordinates x_1, y_1, x_2, y_2, $X \times_B X$ has local equation $x_1 = x_2$, and the diagonal has local equation $y_1 = y_2$. The ideal $\mathcal{I}_\Delta^2$ is locally generated by $(y_1 - y_2)^2$. Thus in a neighborhood of $p \times p$, elements of $\mathcal{O}_X/\mathcal{I}_\Delta^2$ can be written as $f(x_1, y_1) + g(x_1, y_1)(y_1 - y_2)$. Hence in a neighborhood of p on the first copy of X, $(\pi_1)_*\mathcal{O}_X/\mathcal{I}_\Delta^2$ is a free module over $\mathcal{O}_X$ generated by 1 and $(y_1 - y_2)$. Since $\pi_2^*\omega_{X/B}$ is locally free of rank 1, we see that $\mathcal{F}$ is locally free.

Since $\omega_{X/B}$ is locally free on X, it's clear that $\mathcal{E}$ is locally free. □

Notation 2.1.3. If T is a vector bundle on a scheme X and $p \in X$, we let T_p be the vector bundle fiber over p. If $\sigma : T \to S$ is a map of vector bundles and $p \in X$, we let $\sigma_p : T_p \to S_p$ denote the corresponding homomorphism of vector spaces.

Proposition 2.1.4. *Let $p \in X$ be a closed point and $b = \pi(p)$. Then $\mathcal{E}_p = H^0(X_b, K_{X_b})$ and $\mathcal{F}_p = H^0(X_b, K_{X_b}/K_{X_b}(-2p))$, the space of differentials in a neighborhood of p in X_b, modulo those vanishing to order 2 at p.*

Proof. See [HM, p.162-163]. □

With this description of $\mathcal{E}$ and $\mathcal{F}$ we see that for a point $p \in X$, σ_p sends each global holomorphic differential on $X_{\pi(p)}$ to its truncated Taylor series at p.

Proposition 2.1.5. *The map* $\sigma_p : \mathcal{E}_p \to \mathcal{F}_p$ *fails to be surjective if and only if* p *is a hyperelliptic Weierstrass point of* $X_{\pi(p)}$. *Moreover, if* p *is a hyperelliptic Weierstrass point of* $X_{\pi(p)}$, *then* $\operatorname{rank}\sigma_p = 1$. *Hence,* $D_1(\sigma)$ as a set *consists exactly of the hyperelliptic Weierstrass points of hyperelliptic fibers of* π.

Proof. Since $|K_{X_b}|$ is base point free for every fiber X_b of π, we see that $\operatorname{rank}\sigma_p \geq 1$, $\forall p \in X$. Moreover, by the Riemann-Roch Theorem, for any $p \in X$ we have $h^0(X_{\pi(p)}, K_{X_{\pi(p)}}(-p)) = 2$. Thus we see that $\operatorname{rank}\sigma_p = 1$ if and only if $h^0(X_{\pi(p)}, K_{X_{\pi(p)}}(-2p)) = 2$. That is if and only if p is a hyperelliptic Weierstrass point of $X_{\pi(p)}$. □

Proposition 2.1.6. *The scheme* $D_1(\sigma)$ *is reduced.*

Proof. See [HM, Exercise (3.116)]. □

Corollary 2.1.7.

$$\begin{aligned} In the above situation, we have [D_1(\sigma)] &= \Delta_2^{(1)}(c(\mathcal{E}^* - \mathcal{F}^*)) \cap [X] \\ &= c_2(\mathcal{E}^* - \mathcal{F}^*), \end{aligned}$$

in $A_0(X)$.

Proof. By assumption the fibers of π are not all hyperelliptic and hence the hyperelliptic Weierstrass points of hyperelliptic fibers are isolated points of X. Thus $\operatorname{codim}(D_1(\sigma), X) = 2 = (\operatorname{rank}\mathcal{E} - 1)(\operatorname{rank}\mathcal{F} - 1)$. Since π is relatively smooth and B is smooth, we see by [H, III.10.1(c)] that X is smooth, hence Cohen-Macaulay. The result now follows from (1.4.7(a),(c)). □

Notation 2.1.8. Let $\lambda(\pi) = c_1(\pi_*\omega_{X/B}) \in A_0(B)$ and $\gamma = c_1(\omega_{X/B}) \in A_1(X)$.

Remark 2.1.9. By [HM, p.155], $\lambda(\pi) = \frac{\pi_*(\gamma^2)}{12}$.

Proposition 2.1.10. *In the above situation, we have*

$$c(\mathcal{E}^*) = 1 - \pi^*\lambda(\pi)t$$

Proof. Since $\mathcal{E} \cong \pi^*(\pi_*\omega_{X/B})$, this follows from [F, Theorem 3.2(d)] and [F, Remark 3.2.39(a)]. □

Proposition 2.1.11. *There exists a short exact sequence of line bundles of the form*

$$0 \to \omega_{X/B}^2 \to \mathcal{F} \to \omega_{X/B} \to 0.$$

Thus we have $c(\mathcal{F}) = 1 + 3\gamma t + 2\gamma^2 t^2$.

Proof. See [HM, (3.115)] for the existence of the short exact sequence. The rest follows from [F, Theorem 3.2(e)]. □

Corollary 2.1.12. *In the above situation, we have*

$$s(\mathcal{F}^*) = 1 + 3\gamma t + 7\gamma^2 t^2 + \dots.$$

Moreover,

$$[D_1(\sigma)] = 7\gamma^2 - 3\gamma\pi^*\lambda(\pi).$$

Proof. Let $s(\mathcal{F}^*) = s_0 + s_1 t + s_2 t^2 + \dots$. Since $c_i(\mathcal{F}^*) = (-1)^i c_i(\mathcal{F})$ and $s(\mathcal{F}^*)c(\mathcal{F}^*) = 1$, we have

$$(s_0 + s_1 t + s_2 t^2 + \dots)(1 - 3\gamma t + 2\gamma^2 t^2) = s_0 + (s_1 - 3\gamma)t + (2\gamma^2 s_0 - 3\gamma s_1 + s_2)t^2 + \dots = 1$$

Solving gives $s_0 = 1$, $s_1 = 3\gamma$, and $s_2 = 7\gamma^2$.

This gives

$$\begin{aligned} c(\mathcal{E}^* - \mathcal{F}^*) &= (1 - \pi^*\lambda(\pi)t)(1 + 3\gamma t + 7\gamma^2 t^2 + \ldots) \\ &= 1 + (3\gamma - \pi^*\lambda(\pi))t + (7\gamma^2 - 3\pi^*\lambda(\pi)\gamma)t^2 + \ldots. \end{aligned}$$

The result now follows from (2.1.7). □

Proposition 2.1.13. *In the above situation, we have*

$$\pi_*([D_1(\sigma)]) = 72\lambda(\pi)$$

Proof.

$$\begin{aligned} \pi_*([D_1(\sigma)]) &= \pi_*(7\gamma^2 - 3\gamma\pi^*\lambda(\pi)) && (2.1.1) \\ &= 84\lambda(\pi) - 12\lambda(\pi) && (2.1.2) \\ &= 72\lambda(\pi). && (2.1.3) \end{aligned}$$

Note that 2.1.2 follows from Remark (2.1.9) and the fact that the degree of γ on any fiber is $2(3) - 2 = 4$. □

Notation 2.1.14. Let h denote the class in $\mathrm{Pic}_{\mathrm{fun}}(\mathfrak{M}_3) \otimes \mathbb{Q}$ associated to H by [HM, Proposition (3.91)].

Let λ be the divisor class in $\mathrm{Pic}_{\mathrm{fun}}(\mathfrak{M}_3)$ that associates to each family $\pi : X \to B$ the divisor $\lambda(\pi)$, as above. (See [HM, 3.D] for a discussion of divisor classes on the moduli stack.)

By abuse of notation, we will also let $\lambda = c_1(\pi_* \omega_{\mathcal{C}_3/\mathfrak{M}_3}) \in \mathrm{Pic}(\mathfrak{M}_3)$. The distinction between the two uses of λ should be clear from the context.

Theorem 2.1.15. *We have the following expression,*

$$[H] = 18\lambda \in \mathrm{Pic}(\mathfrak{M}_3) \otimes \mathbb{Q}$$

Proof. By [H, IV.2.4] there are exactly 8 hyperelliptic Weierstrass points on a hyperelliptic curve of genus 3. Thus by [HM, Proposition (3.91)] we have

$$h = \frac{1}{8}(72\lambda) = 9\lambda \in \mathrm{Pic}_{\mathrm{fun}}(\mathfrak{M}_3)$$

Since a generic hyperelliptic curve has one non-trivial automorphism, by [HM, Proposition (3.93)] we have

$$[H] = 18\lambda \in \mathrm{Pic}(\mathfrak{M}_3) \otimes \mathbb{Q}.$$

□

2.2 The hyperelliptic locus in $\overline{\mathfrak{M}}_3 \setminus \Delta_1$

Let $\overline{H}_0$ denote the closure of H in $\overline{\mathfrak{M}}_3 \setminus \Delta_1$. In this section, we follow the method presented in [D] for computing the class $[\overline{H}_0] \in \mathrm{Pic}(\overline{\mathfrak{M}}_3 \setminus \Delta_1) \otimes \mathbb{Q}$. To this end, let $\pi : X \to B$ be a generic 1-parameter family of stable curves of genus 3, with smooth base B. By shrinking B if necessary, we can assume that there are no fibers of π corresponding to elements of Δ_1. Let $\sigma : \mathcal{E} \to \mathcal{F}$ be as above.

Proposition 2.2.1. *$\mathcal{E}$ is a vector bundle on all of X. $\mathcal{F}$ is a vector bundle away from singular points of singular fibers of π, but fails to be locally free at singular points of singular fibers. Moreover, the second Fitting ideal of $\mathcal{F}$ near a node p of a fiber of π is the maximal ideal at p.*

Proof. Since $\omega_{X/B}$ is locally free on X, it's clear that $\mathcal{E}$ is locally free. The statement about $\mathcal{F}$ follows the proof of (2.1.2) and [D, Lemma 2]. □

Let $g : X' \to X$ be the blow-up of X at the nodes of singular fibers of π and let $\sigma' : \mathcal{E}' \to \mathcal{F}'$ be the map of vector bundles obtained by applying the process of (1.4.10).

Proposition 2.2.2. *The map σ' is surjective at all nonsingular points of singular fibers corresponding to general points of Δ_0 and at all points of the exceptional divisor in the above blow-up.*

Proof. See [D, Lemmas 3, 4]. □

This lemma shows that the $[D_1(\sigma)]$ (see Definition (1.4.12)) counts hyperelliptic curves as we wish, away from Δ_1. (This is due to the fact that $\mathrm{codim}(\Delta_0, \overline{\mathfrak{M}}_3) = 1$, and thus a general member of Δ_0 is not contained in $\overline{H}_0$.) Since $\mathcal{E}$ is already a vector bundle, the Chern classes of $\mathcal{E}'$ are simply the pullbacks of the Chern classes of $\mathcal{E}$ along g. Thus we have

$$c(\mathcal{E}') = 1 - g^*(\pi^*\lambda(\pi))t.$$

To compute the Chern classes of $\mathcal{F}'$, we use the following proposition.

Proposition 2.2.3. *On X' there is a two-term filtration,*

$$0 \to \mathcal{F}'_2 \to \mathcal{F}' \to \mathcal{F}'_1 \to 0,$$

where $\mathcal{F}'_1$ is $g^*\omega_{X/B}$ and $\mathcal{F}'_2$ is $g^*\omega^2_{X/B} \otimes \mathcal{O}(-D)$. Moreover, we then have

$$c(\mathcal{F}') = 1 + (3g^*\gamma - D)t + 2g^*\gamma^2 t^2.$$

Proof. The filtration is given by [D, Lemma 5]. The Chern classes of $\mathcal{F}'$ are then given by [F, Theorem 3.2(e)]. □

Proposition 2.2.4.

$$[D_1(\sigma')] = c_2(\mathcal{E}'^* - \mathcal{F}'^*) = 7g^*\gamma^2 - 3(g^*\gamma)(g^*(\pi^*\lambda(\pi))) + D^2.$$

Proof. The first equality follows from (1.4.7). To obtain the second equality, we compute:

$$\begin{aligned} s(\mathcal{F}^*) &= (1 + (D - 3g^*\gamma)t + 2g^*\gamma^2 t^2)^{-1} \\ &= 1 + (3g^*\gamma - D)t + (7g^*\gamma^2 - 6g^*\gamma D + D^2)t^2 + \ldots. \end{aligned}$$

Hence

$$\begin{aligned} c_2(\mathcal{E}'^* - \mathcal{F}'^*) &= (7g^*\gamma^2 - 6g^*\gamma D + D^2) - g^*(\pi^*\lambda(\pi))(3g^*\gamma - D) \\ &= 7g^*\gamma^2 - 3(g^*\gamma)(g^*(\pi^*\lambda(\pi))) + D^2. \end{aligned}$$

□

Notation 2.2.5. Let $\overline{h}_0(\pi)$ denote the divisor on B associated to $\overline{H}_0$ by [HM, Proposition (3.91).]

Proposition 2.2.6. *We have*

$$\overline{h}_0(\pi) = 9\lambda(\pi) - \delta_0(\pi).$$

Proof. By (2.1.6) and (2.2.2) we have that $\pi_* g_*[D_1(\sigma')] = 8\overline{h}_0$. Thus by (2.2.4) we have

$$\begin{aligned} 8\overline{h}_0 &= \pi_* g_* \left(7g^*\gamma^2 - 3(g^*\gamma)(g^*(\pi^*\lambda(\pi))) + D^2\right) && (2.2.1) \\ &= 7\kappa(\pi) - 12\lambda(\pi) - \delta_0(\pi). && (2.2.2) \end{aligned}$$

(2.2.2) follows from (2.1.9) and from the facts that the degree of γ on any fiber is 4 and that there is one component of D for each curve corresponding to an element of Δ_0, all of which are disjoint and have self-intersection -1. By [HM, eq. (3.110)], we have $\kappa(\pi) = 12\lambda(\pi) - \delta(\pi)$. Ignoring δ_1, we obtain $8\overline{h}_0 = 72\lambda(\pi) - 8\delta_0(\pi)$, and hence $\overline{h}_0 = 9\lambda(\pi) - \delta_0(\pi)$. □

Corollary 2.2.7. *We have the following expression,*

$$[\overline{H}_0] = 18\lambda - 2\Delta_0.$$

Proof. This follows from [HM, Proposition (3.93)]. $\square$

Chapter 3

The class of the hyperelliptic locus in $\overline{\mathfrak{M}}_3$

3.1 The rank of σ'

Let $\pi : X \to B$ be a general 1-parameter family of stable curves of genus 3. Let $\sigma : \mathcal{E} \to \mathcal{F}$ be as in the previous chapter.

Proposition 3.1.1. *$\mathcal{E}$ is a vector bundle on all of X. $\mathcal{F}$ is a vector bundle away from singular points of singular fibers of π, but fails to be locally free at singular points of singular fibers. Moreover, the second Fitting ideal of $\mathcal{F}$ near a node p of a fiber of π is the maximal ideal at p.*

Proof. This is simply a restatement of (2.2.1) to include points of fibers corresponding to general elements of Δ_1, and its proof follows from that of (2.2.1) and a trivial generalization of [D, Lemma 2]. □

As in the previous chapter, let $g : X' \to X$ be the blow-up of X at the nodes of singular fibers of π and let $\sigma' : \mathcal{E}' \to \mathcal{F}'$ be the map of vector bundles obtained by applying the process of (1.4.10).

Theorem 3.1.2. **(a)** *The map σ' has rank 1 at the hyperelliptic Weierstrass points of smooth hyperelliptic fibers of π.*

(b) *The map σ' has rank 1 at the hyperelliptic Weierstrass points of (the proper transform of) the genus 2 component of a fiber corresponding to a general member of Δ_1.*

(c) *The map σ' has rank 1 at all points of (the proper transform of) the elliptic component of a fiber corresponding to a general member of Δ_1.*

(d) *The map σ' has rank 1 at all points of the exceptional divisor lying over the node of a fiber corresponding to a general member of Δ_1.*

(e) *The map σ' is surjective at all other points.*

Proof. (a) follows from (2.1.5), and (e) follows from (2.1.5) and [D, Lemmas 3, 4] for points of smooth fibers and points of fibers corresponding to general members of Δ_0.

Let E be the elliptic component and C the genus 2 component of a fiber, X_b, corresponding to a general member of Δ_1, and let P be their point of intersection. Note that away from P we can prove the above statements for σ, rather than for σ'.

Suppose Q is a closed point of $E - P$. Let ω_b, ω_1, and ω_2 the dualizing sheaves on X_b, E, and C, respectively. We see that σ_Q fails to be surjective iff $h^0(\omega_b(-2Q)) = h^0(\omega_b(-Q))$. Moreover, since ω_b is base point free away from P, the rank of σ_Q is always positive. Let P_i be the point on the curve of genus i lying over P in $\widetilde{X_b}$, $i = 1, 2$. Using the description of the dualizing sheaf of X_b given in [HM, p.82], we have

$$H^0(\omega_b(-Q)) = H^0(\omega_1(-Q + P_1)) \oplus H^0(\omega_2(P_2)),$$

and

$$H^0(\omega_b(-2Q)) = H^0(\omega_1(-2Q + P_1)) \oplus H^0(\omega_2(P_2)).$$

Since sections of $\omega_1(P_1)$ are simply constants, if a section vanishes at Q, it vanishes to infinite order. Thus $H^0(\omega_1(-Q + P_1)) = H^0(\omega_1(-2Q + P_1))$.

Suppose Q is a closed point of $C - P$. Using the notation above, we again see that since ω_b is base point free away from P, σ_Q will always have positive rank, and that σ_Q will fail to be surjective iff $h^0(\omega_b(-2Q)) = h^0(\omega_b(-Q))$. We have

$$H^0(\omega_b(-2Q)) = H^0(\omega_1(P_1)) \oplus H^0(\omega_2(-2Q + P_2)),$$

and

$$H^0(\omega_b(-Q)) = H^0(\omega_1(P_1)) \oplus H^0(\omega_2(-Q + P_2)).$$

Thus σ_Q will fail to be surjective if and only if

$$h^0(\omega_2(-2Q + P_2)) = h^0(\omega_2(-Q + P_2)).$$

Since $H^0(\omega_2(P_2)) = H^0(\omega_2)$, this is true if and only if

$$h^0(\omega_2(-2Q)) = h^0(\omega_2(-Q)).$$

That is, if and only if Q is a hyperelliptic Weierstrass point of C.

Let E_0 be the rational curve lying over P in the blow-up $g : X' \to X$, and let Q be a closed point of E_0. Choose local coordinates x and y on X centered at P, so that the map π is given locally by $xy = t$, where t is a local coordinate on B centered at b. At P, $\mathcal{F}$ is simply the linearizations, at P, of differentials in a neighborhood of P. Thus locally, $\mathcal{F}$ is generated by 1, dx, and dy, but since these are relative differentials, we have the nontrivial relation $dt = 0$. That is

$$d(xy) = y\ dx + x\ dy = 0$$

After blowing-up P on X and extending the sheaves this relation becomes $\tilde{y}\ dx + dy = 0$ on one patch and $dx + \tilde{x}\ dy = 0$ on the other.

Choose $\{\alpha_1, \alpha_2, \alpha_3\}$ as a basis for $H^0(\omega_b)$ (ω_b as above), where α_1 is a nonzero constant function on E, α_2 is a regular differential on C that does not vanish at P_2, and α_3 is a regular differential on C vanishing to order 1 at P_2. After multiplying all the α by suitable constants the map becomes

$$\begin{aligned} \sigma'(\alpha_1) &= 0(1) + 1dx + 0dy \\ \sigma'(\alpha_2) &= 0(1) + 0dx + 1dy \\ \sigma'(\alpha_3) &= 0(1) + 0dx + 0dy. \end{aligned}$$

This completes the proof. $\square$

3.2 The scheme structure of $D_1(\sigma')$

Let $\pi : X \to B$, $g : X' \to X$, and $\sigma' : \mathcal{E}' \to \mathcal{F}'$ be as in the previous section. In the previous section, we determined $D_1(\sigma')$ (see (1.4.1)) *as a set*. In this section, we determine the natural scheme structure on $D_1(\sigma')$.

Proposition 3.2.1. *$D_1(\sigma')$ is reduced away from points of X' lying over points of X contained in fibers corresponding to general members of Δ_1.*

Proof. This follows from (3.1.2) and (2.1.6). □

Proposition 3.2.2. *Let C be the genus 2 component of a fiber of π corresponding to a general member of Δ_1. Then $D_1(\sigma')$ is reduced at the hyperelliptic Weierstrass points of (the proper transform of) C.*

Proof. Since this question is clearly local on B, we can assume that $\pi : X \to B$ is a 1-parameter family of smooth, nonhyperelliptic curves of genus 3 degenerating to a general member of Δ_1, with both X and B smooth. Let $X_{b_0} = E \cup C$ denote the special fiber of this family, $\omega_{X/B}$ be the relative dualizing sheaf, and $\omega_{X/B}(E) := \omega_{X/B} \otimes \mathcal{O}_X(E)$. On smooth fibers $\omega_{X/B}(E)$ restricts to the canonical bundle, but on the special fiber we see that $\omega_{X/B}(E)$ restricts to $\omega_2(2P)$ on C and the trivial bundle on E, where ω_2 is the canonical bundle on C and $P = E \cap C$. Since X_{b_0} is a general member of Δ_1, we can assume that P is not a hyperelliptic Weierstrass point of C. Thus $\omega_{X/B}(E)$ determines a map

$$\varphi : X \to \mathbb{P}^2 \times B$$

that embeds each of the smooth fibers as a planar quartic and maps the special fiber to a cuspidal quartic whose normalization is C; E gets collapsed to the cusp.

Again since this is clearly local on B, we can assume $B = \operatorname{Spec} A$, for some 1-dimensional ring A; moreover, we can shrink B to a smaller affine open set if necessary and will do so without comment. Let t be a local parameter at $b_0 \in B$. Then φ maps X to $\mathbb{P}^2_A$ and we can choose homogeneous coordinates so that the image of X has the form

$$Y^2Z^2 + Z\sum_{i+j=3}\alpha_{i,j}X^iY^j + \sum_{i+j=4}\beta_{i,j}X^iY^j + tG(X,Y,Z) = 0,$$

where G is a homogenous polynomial of degree 4 in $A[X,Y,Z]$.

Since we are interested in the behavior of σ' at the hyperelliptic Weierstrass points of C, and since φ is an embedding away from P, it is enough to determine the behavior of σ' on $\varphi(X)$. Moreover, it's clear by the construction of σ' that $D_1(\sigma')$ will be reduced at $Q_1, \ldots, Q_6$ (the hyperelliptic Weierstrass points of C) if and only if $D_1(\sigma)$ is. Thus we will consider $\sigma : \mathcal{E} \to \mathcal{F}$ on $\varphi(X)$.

To do this we will need to determine three sections of $\omega_{X/B}$ whose restrictions to each fiber form a basis for the space of sections of the dualizing sheaf for that fiber. First, we will find three linearly equivalent divisors on X whose restriction to a smooth fiber is the canonical divisor, to E is linearly equivalent to P, and to C is linearly equivalent to $K_C + P$. By the description of the dualizing sheaf given in [HM, p.82], the invertible sheaf associated to such divisors will be $\omega_{X/B}$, and the hope is that the global sections associated to these divisors will restrict to the desired basis on each fiber.

To this end, we consider the divisors $\{X = 0\}$, $\{Y = 0\}$, and $\{Z = 0\}$ on $\varphi(X)$. Since the canonical bundle on a smooth planar quartic is $\mathcal{O}(1)$, these will restrict to the canonical divisor on each smooth fiber; moreover, the associated sections will give a basis for the space of sections of the canonical bundle on a smooth fiber.

We consider the pullbacks of these divisors to X. Since $\{Z = 0\}$ does not pass through the cusp of the central fiber, it pulls back isomorphically to a divisor we'll call D_Z. $\varphi^*\{X = 0\}$ is supported on E and an irreducible curve which we'll call D_X; with this notation we have $\varphi^*\{X = 0\} = D_X + 2E$. Similarly, $\varphi^*\{Y = 0\}$ is supported on E and an irreducible curve we'll call D_Y; we then have $\varphi^* = D_Y + 3E$. Clearly, these pullbacks still restrict to the canonical bundle on smooth fibers of π, but since the map on the special fiber to $\mathbb{P}^2$ is given by $K_C + 2P$, we see that these pullbacks must restrict to $K_C + 2P$ on C and are linearly equivalent to 0 on E.

Since $E.(E + C) \sim 0$, $C.(E + C) \sim 0$, and $E.C \sim P$, we see that $E.E \sim -P$ and $C.C \sim -P$. Thus we have the following:

Lemma 3.2.3. *Using the notation above, the divisors* $D_X + E$*,* $D_Y + 2E$*, and* $D_Z + C$ *restrict to the canonical divisor on smooth fibers of* π*, to* P *on* E*, and to* $K_C + P$ *on* C*.*

We now would like to consider the sections s_X, s_Y, and s_Z of $\omega_{X/B}$ associated to $D_X + E$, $D_Y + 2E$, and $D_Z + C$, respectively.

Lemma 3.2.4. *The sections s_X, s_Y, and s_Z on each fiber of π restrict to a basis for the space of global sections of the dualizing sheaf.*

Proof. Since X, Y, and Z give a basis for $\mathcal{O}(1)$ on $\mathbb{P}^2$, this is clear for smooth fibers of π. Thus we consider the restrictions of s_X, s_Y and s_Z to X_{b_0}.

Suppose $c_1 s_X + c_2 s_Y + c_3 s_Z = 0$ for some $c_i \in \mathbb{C}$. Restricted to E, s_X and s_Y are both 0, but s_Z is not identically 0. As a result, we must have $c_3 = 0$. We can factor out a local equation, say $x = 0$, for E from $c_1 s_X + c_2 s_Y = 0$. This gives $c_1 \frac{s_X}{x} + c_2 \frac{s_Y}{x} = 0$, but restricted to E, $\frac{s_Y}{x} = 0$ and $\frac{s_X}{x}$ is not identically 0. Thus we must have $c_1 = 0$. Since s_Y is not identically 0, we must then have $c_2 = 0$.

Since the space of global sections of the dualizing sheaf of $C \cup E$ has rank 3, this completes the proof.

□

To give the map σ in local coordinates at a smooth point, x, of a fiber of π we simply determine local equations for s_X, s_Y, and s_Z in a neighborhood of x and then consider their linearizations. Presently we are interested in hyperelliptic Weierstrass points of C. Since X_{b_0} is a general point of Δ_1, we can assume that P is not such a point. Since the family remains unchanged away from E under φ, we can make our computations on $\varphi(X)$.

The hyperelliptic Weierstrass points of C can be determined by looking at lines through the cusp of $\varphi(C)$. A point of $\varphi(C)$ whose tangent line passes through the cusp is a hyperelliptic Weierstrass point. Since the line given by $Y = 0$ intersects the cusp with multiplicity 3, we see that no hyperelliptic Weierstrass points lie along this line. Thus it is enough to consider the affine open set of $\varphi(X)$ given by $Y \neq 0$. The total space of our family on this open set is given in affine coordinates by

$$z^2 + z \sum_{i+j=3} \alpha_{i,j} x^i + \sum_{i+j=4} \beta_{i,j} x^i + tG(x, 1, z) = 0.$$

On this open set local equations for s_X, s_Y, and s_Z are x, 1, and zt, respectively. The linearizations of these sections at a point (x_0, z_0, t_0) are:

$$\begin{aligned} x &= x_0 + dx \\ 1 &= 1 \\ zt &= z_0 t_0 + t_0\, dz + z_0\, dt \end{aligned}$$

Since the family is parameterized by t, we must have $dt = 0$. But we then also have

$$\begin{aligned} 0 &= d\left(z^2 + z \sum_{i+j=3} \alpha_{i,j} x^i + \sum_{i+j=4} \beta_{i,j} x^i + tG(x, 1, z) \right) \\ &= \left(2z + \sum_{i+j=3} \alpha_{i,j} x^i + t \frac{\partial}{\partial z} G(x, 1, z) \right) dz \\ &\quad + \left(z \sum_{i+j=3} i\alpha_{i,j} x^{i-1} + \sum_{i+j=4} i\beta_{i,j} x^{i-1} + t \frac{\partial}{\partial x} G(x, 1, z) \right) dx \end{aligned}$$

Suppose $(x_0, z_0, 0)$ is a point on $\varphi(C)$ such that

$$z_0 \sum_{i+j=3} i\alpha_{i,j} x_0^{i-1} + \sum_{i+j=4} i\beta_{i,j} x_0^{i-1} = 0.$$

Then the tangent line to $\varphi(C)$ at this point is given by $z - z_0 = 0$, or in homogeneous coordinates $Z - z_0 Y = 0$. But this line does not pass through the cusp of $\varphi(C)$, so

$(x_0, z_0, 0)$ cannot be a hyperelliptic Weierstrass point. Thus it suffices to consider the open set given by $z\sum_{i+j=3} i\alpha_{i,j}x^{i-1} + \sum_{i+j=4} i\beta_{i,j}x^{i-1} + t\frac{\partial}{\partial z}G(x,1,z) \neq 0$. In which case we have

$$dx = \frac{-\left(2z + \sum_{i+j=3}\alpha_{i,j}x^i + t\frac{\partial}{\partial z}G(x,1,z)\right)}{z\sum_{i+j=3} i\alpha_{i,j}x^{i-1} + \sum_{i+j=4} i\beta_{i,j}x^{i-1} + t\frac{\partial}{\partial x}G(x,1,z)}dz$$

The linearizations of our sections at a point (x_0, z_0, t_0) of this open set are then:

$$\begin{aligned}
x &= x_0 + \frac{-\left(2z_0 + \sum_{i+j=3}\alpha_{i,j}x_0^i + t_0\frac{\partial}{\partial z}G(x_0,1,z_0)\right)}{z_0\sum_{i+j=3} i\alpha_{i,j}x_0^{i-1} + \sum_{i+j=4} i\beta_{i,j}x_0^{i-1} + t_0\frac{\partial}{\partial x}G(x_0,1,z_0)}dz \\
1 &= 1 \\
zt &= z_0t_0 + t_0\, dz
\end{aligned}$$

This shows that in local coordinates at a point of this open set the map $\sigma : \mathcal{E} \to \mathcal{F}$ is given by the matrix

$$\begin{bmatrix} x & 1 & zt \\ \frac{-\left(2z+\sum_{i+j=3}\alpha_{i,j}x^i + t\frac{\partial}{\partial z}G(x,1,z)\right)}{z\sum_{i+j=3} i\alpha_{i,j}x^{i-1} + \sum_{i+j=4} i\beta_{i,j}x^{i-1} + t\frac{\partial}{\partial x}G(x,1,z)} & 0 & t \end{bmatrix}$$

The ideal generated by the 2×2 minors of this matrix is

$$I = \left(t, 2z + \sum_{i+j=3}\alpha_{i,j}x^i\right) \subset \frac{A[x,z]}{(z^2 + z\sum_{i+j=3}\alpha_{i,j}x^i + \sum_{i+j=4}\beta_{i,j}x^i + tG(x,1,z))},$$

or equivalently

$$I = \left(2z + \sum_{i+j=3}\alpha_{i,j}x^i, z^2 + z\sum_{i+j=3}\alpha_{i,j}x^i + \sum_{i+j=4}\beta_{i,j}x^i\right) \subset \mathbb{C}[x,z]$$

Substituting $z = -\frac{1}{2}\sum_{i+j=3}\alpha_{i,j}x^i$ into the third equation gives

$$I = \left(2z + \sum_{i+j=3}\alpha_{i,j}x^i, h(x)\right) \subset \mathbb{C}[x,z],$$

where $h(x)$ is a polynomial of degree 6. Since there are six hyperelliptic Weierstrass points on C, we see that $h(x)$ must have distinct roots, and thus I is the ideal of these six points. This shows that $D_1(\sigma')$ is reduced at these points.

□

To determine the remaining scheme structure of $D_1(\sigma')$, we explicitly construct a family of smooth nonhyperelliptic curves of genus 3 degenerating to a general member of Δ_1. Using the proof of the previous proposition as a guide, we begin with a family of smooth planar quartics, over (an open subset of) $\mathbb{A}^1_t$ degenerating to a cuspidal quartic. We then explicitly compute the stable reduction of such a family.

To begin let

$$F(X,Y,Z) = Y^2Z^2 + Z\sum_{i+j=3} \alpha_{i,j}X^iY^j + \sum_{i+j=4} \beta_{i,j}X^iY^j,$$

where the $\alpha_{i,j}$ and $\beta_{i,j}$ are such that $F(X,Y,Z) = 0$ is nonsingular away from $[0,0,1]$. Note that by proper choice of coordinates, any cuspidal quartic can be given by such an equation. Moreover, we will assume that $\alpha_{3,0} = -1$. Let C_t be the family of curves, parameterized by $t \in \mathbb{C}$, given by

$$F(X,Y,Z) - at^2XZ^3 - bt^3Z^4 = 0,$$

where $a, b \in \mathbb{C}$ are such that $a, b \neq 0$ and $4a^3 + 27b^2 \neq 0$. The special fiber, C_0, is the cuspidal quartic given by $F(X,Y,Z) = 0$.

If we consider the specific planar curve given by

$$Y^2Z^2 - X^3Z + Y^4 - XZ^3 - Z^4 = 0,$$

one easily checks that this curve is nonsingular. Thus for most choices of $\alpha_{i,j}$, $\beta_{i,j}$, a, b, and t such a curve is nonsingular. Thus for most choices of $\alpha_{i,j}$, $\beta_{i,j}$, a, and b all but finitely many fibers of the family C_t will be smooth. So we will assume that we have chosen $\alpha_{i,j}$, $\beta_{i,j}$, a, b in such a way. Moreover, by restricting t to an open neighborhood of $t = 0$, we can assume that all fibers other than C_0 are smooth.

The elliptic curve that will appear in the stable limit will lie over the cusp of $F(X, Y, Z) = 0$; thus for our purposes it will suffice to consider the family $f(x, y) - at^2x - bt^3 = 0$, where $f(x, y) = F(x, y, 1)$.

By [H, III.10.1(c)] the total space of our family is smooth away from the singular point of C_0. However, the total space of this family does have a singularity at the origin. We will resolve the singularity in the total space and the cusp in the central fiber simultaneously with four successive blow-ups.

First, we blow-up along the linear subspace $x = y = 0$ in $\mathbb{A}^3_{(x,y,t)}$ and take the proper transform of our family. This gives us two patches to consider:

On the first patch (which we'll call (P1)), we make the substitution $y = xy$. The exceptional divisor (by which we will always mean the exceptional divisor in the ambient affine space restricted to the total space of our family), which we'll call E_1, is then given by $x = 0$, and the proper transform of C_t is given by $f(x, xy) - at^2x - bt^3 = 0$. (We'll continue to call this C_t,).

On the second patch (P2), we make the substitution $x = xy$. E_1 is given by $y = 0$ and the proper transform of C_t is given by $f(xy, y) - at^2xy - bt^3 = 0$.

On both patches, the special fiber consists of the union of C (the normalization of $f(x, y) = 0$) and E_1, which has multiplicity 2. On the first patch, these two components are tangent at $x = y = 0$. (They're disjoint on the second.) Both patches contain a codimension 1 singularity along E_1. (See (A.1.1) and (A.1.2).)

Next, we blow-up (P1) along $x = y = t = 0$. This gives three patches to consider:

On the first patch (P1-1), we make the substitutions $y = xy$ and $t = xt$. The exceptional divisor, E_2 is given by $x = 0$, and the proper transform of C_t is given by $\frac{1}{x^3}f(x, x^2y) - at^2 - bt^3 = 0$. The special fiber is given by $xt = 0$ and consists of the union of C and E_2, which do not meet on this patch. The total space of the family on this patch is nonsingular. (See (A.1.3).)

On the second patch (P1-2), we make the substitutions $x = xy$ and $t = yt$. E_2 is given by $y = 0$, and the proper transform of C_t is given by $\frac{1}{y^3}f(xy, xy^2) - at^2x - bt^3 = 0$. The special fiber is given by $yt = 0$ and consists of the union of C, E_1 (which appears with multiplicity 2), and E_2. The total space of the family is still singular along E_1. (See (A.1.4).)

On the third patch (P1-3), we make the substitutions $x = xt$ and $y = yt$. E_2 is given by $t = 0$, and the proper transform of C_t is given by $\frac{1}{t^3} f(xt, xyt^2) - ax - b = 0$. Since $x \neq 0$ on this patch, we see that it is contained in (P1-1); thus we can ignore it.

Note that E_2 is the union of three rational curves (which are distinct by the restrictions placed on a, b) meeting at one point (contained in (P1-2)).

Next, in an effort to obtain a special fiber that is supported on a nodal curve, we will blow-up the point in (P1-2) where E_1, E_2, and C all meet. Again we have three patches to consider:

On the first patch (P1-2-1), we make the substitutions $y = xy$ and $t = xt$. The exceptional divisor, E_3, is given by $x = 0$, and the proper transform of C_t is given by $\frac{1}{x^6 y^3} f(x^2 y, x^3 y^2) - at^2 - bt^3 = 0$. The special fiber is given by $x^2 yt = 0$ and consists of the union of C, E_2 (which is now the *disjoint* union of three rational curves which we'll call E_2', E_2'', and E_2'''), and E_3 (which appears with multiplicity 2). The total space of the family is nonsingular on this patch. (See (A.1.5).)

On the second patch (P1-2-2), we make the substitutions $x = xy$ and $t = yt$. E_3 is given by $y = 0$, and the proper transform of C_t is given by $\frac{1}{y^6} f(xy^2, xy^3) - at^2 x - bt^3 = 0$. The special fiber is given by $y^2 t = 0$ and consists of the union of C, E_1 (which appears with multiplicity 2), and E_3 (which also appears with multiplicity 2). The total space of the family is still singular along E_1. (See (A.1.6).)

On the third patch (P1-2-3), we make the substitutions $x = xt$ and $y = yt$. E_3 is given by $t = 0$, and the proper transform of C_t is given by $\frac{1}{y^3t^6}f(xyt^2, xy^2t^3) - ax - b = 0$. Since $x \neq 0$ on this patch, it is contained in (P1-2-1); so we can ignore it.

Finally, we resolve the singularity in the total space by blowing-up along E_1. Points of E_1 are only contained in (P2) and (P1-2-2); so we only need to blow-up these two patches. We blow-up (P2) first. We do this by blowing-up $\mathbb{A}^3$ along $y = t = 0$ and taking the proper transform of C_t. Again there are two patches to consider:

On the first patch (P2-1), we make the substitution $y = yt$. The exceptional divisor, E_4 is given by $t = 0$ and the proper transform of C_t is given by $\frac{1}{t^2}f(xyt, yt) - atxy - bt = 0$. The special fiber is given by $t = 0$ and consists only of E_4 (which appears with multiplicity 2). The total space is nonsingular on this patch. (See (A.1.7).)

On the second patch (P2-2), we make the substitution $t = yt$. E_4 is given by $y = 0$ and the proper transform of C_t is given by $\frac{1}{y^2}f(xy, y) - at^2xy - bt^3y = 0$. The special fiber is given by $yt = 0$ and consists only of C. Again, the total space is nonsingular. (See (A.1.8).)

Next we blow-up (P1-2-2). Again this is done by blowing-up $\mathbb{A}^3$ along $x = t = 0$

and taking the proper transform of C_t. There are two patches to consider:

On the first patch (P1-2-2-1), we make the substitution $t = xt$. E_4 is given by $x = 0$, and the proper transform of C_t is given by $\frac{1}{x^2y^6}f(xy^2, xy^3) - at^2x - bt^3x = 0$. The special fiber is given by $xy^2t = 0$ and consists of the union of C and E_3 (which appears with multiplicity 2). The total space of the family is nonsingular on this patch. (See (A.1.9).)

On the second patch (P1-2-2-2), we make the substitution $x = xt$. E_4 is given by $t = 0$, and the proper transform of C_t is given by $\frac{1}{y^6t^2}f(xy^2t, xy^3t) - atx - bt = 0$. The special fiber is given by $y^2t = 0$ and consists of the union of E_4 (which appears with multiplicity 2) and E_3 (which also appears with multiplicity 2). The total space of the family is nonsingular. (See (A.1.10).)

The total space of our family is now nonsingular and the special fiber is supported on a nodal curve, but contains components of multiplicity 2 that we must deal with. A schematic drawing of the special fiber is given below:

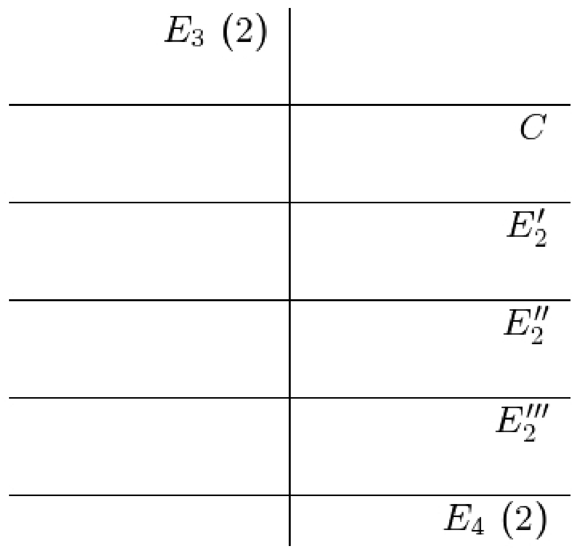

Figure 3.1: A schematic drawing of the special fiber

We deal with the components of multiplicity 2 by making a base change of order 2 branched over $t = 0$. This base change will introduce new singularities into the total space, so we package it with the normalization of the resulting surface. The effect of this will be to take the branched cover of the total space branched along the union of C, E_2', E_2'', and E_2''' (see [HM], pp.124-125). Since this branch divisor is smooth, the resulting surface will be smooth as well. Since E_3 meets the branch locus in four points, its inverse image will be a double cover of $E_3 \cong \mathbb{P}^1$ branched at four points, that is, a single elliptic curve that we will call E. E_4, on the other hand, is disjoint from the branch locus, so its inverse image will be an unramified double cover of $E_4 \cong \mathbb{P}^1$; that is, two disjoint rational curves that we'll call E_4' and E_4''.

The pullback of the special fiber to the new family will then be

$$2\widetilde{C} + 2E_2' + 2E_2'' + 2E_2''' + 2E_4' + 2E_4'' + 2E$$

But the special fiber of the new family is exactly one-half of this divisor. Thus the special fiber is

$$\widetilde{C} + E_2' + E_2'' + E_2''' + E_4' + E_4'' + E$$

Since E_2', E_2'', E_2''', E_4', and E_4'' are all rational curves with self-intersection -1, they can be blown-down. The special fiber then becomes the union of E and C, meeting transversely at one point, as desired.

For our purposes, we are only interested in points of E. Thus we will only explicitly compute the base change described above on those patches containing points of E_3, specifically (P1-2-1), (P1-2-2-1), and (P1-2-2-2). Moreover, all the points of E_3 in (P1-2-2-1) are contained in one of the other two open sets; so we don't need to consider (P1-2-2-1). For ease of notation we will rename the open sets (P1-2-1) and (P1-2-2-2), U and V, respectively.

Recall, the total space on U is given by $\frac{1}{x^6y^3}f(x^2y, x^3y^2) - at^2 - bt^3 = 0$. The special fiber is given by $x^2yt = 0$ and consists of the union of $\widetilde{C}$, E_2', E_2'', E_2''', and E_3 (which appears with multiplicity 2).

The total space on V is given by $\frac{1}{y^6t^2}f(xy^2t, xy^3t) - atx - bt = 0$. The special fiber is given by $y^2t = 0$ and consists of the union of E_4 (which appears with multiplicity

2) and E_3 (which also appears with multiplicity 2).

We now explicitly perform the calculations described above on these two open sets.

After the base change of order 2 on U, the total space of the family is given by

$$\{\frac{1}{x^6y^3}f(x^2y, x^3y^2) - at^2 - bt^3 = 0\} \cap \{u^2 - x^2yt = 0\} \subseteq \mathbb{A}^4_{(x,y,t,u)},$$

with the special fiber given by $u = 0$. Then we normalize by setting $u = xv$. This gives:

$$\{\frac{1}{x^6y^3}f(x^2y, x^3y^2) - at^2 - bt^3 = 0\} \cap \{v^2 - yt = 0\} \subseteq \mathbb{A}^4_{(x,y,t,v)},$$

with the special fiber given by $xv = 0$. We continue to call this open set U.

On the open subset $\{t \neq 0\} \subset U$ we have $y = v^2/t$, so that the total space is given by

$$\{\frac{t^8}{x^6v^6}f(x^2v^2t^{-1}, x^3v^4t^{-2}) - at^7 - bt^8 = 0\} \cap \{t \neq 0\} \subseteq \mathbb{A}^3_{(x,t,v)},$$

with the special fiber still given by $xv = 0$. In this case $x = 0$ is a local equation for E, while $v = 0$ is a local equation for $E_2' \cup E_2'' \cup E_2'''$. Thus these rational curves can be blown-down using the relation $xv \mapsto x$. We thus arrive at

$$\{\frac{t^8}{x^6}f(x^2t^{-1}, x^3vt^{-2}) - at^7 - bt^8 = 0\} \cap \{t \neq 0\} \subseteq \mathbb{A}^3_{(x,t,v)},$$

with the special fiber given by $x = 0$, which is a local equation for E. We will call this open set U_0.

We will let U_1 denote the open subset $\{y \neq 0\} \subset U$. On this open set we have

$t = v^2/y$, so the total space is given by

$$\{\frac{1}{x^6} f(x^2y, x^3y^2) - ayv^4 - bv^6 = 0\} \cap \{y \neq 0\} \subseteq \mathbb{A}^3_{(x,y,v)},$$

with special fiber once again given by $xv = 0$. On this open set we can blow-down using the relation $xy \mapsto x$. We then arrive at

$$\{y^4 + y^3 \sum_{i+j=3} \alpha_{i,j} x^j + y^2 \sum_{i+j=4} \beta_{i,j} x^{j+2} - ayv^4 - bv^6 = 0\} \cap \{y \neq 0\} \subseteq \mathbb{A}^3_{(x,y,v)}$$

After the base change of order 2 on V, the total space of the family is given by

$$\{\frac{1}{y^6t^2} f(xy^2t, xy^3t) - atx - bt = 0\} \cap \{u^2 - y^2t = 0\} \subseteq \mathbb{A}^4_{(x,y,t,u)},$$

with the special fiber given by $u = 0$. We normalize by setting $u = xyv$. This gives:

$$\{\frac{1}{y^6t^2} f(xy^2t, xy^3t) - atx - bt = 0\} \cap \{x^2v^2 - t = 0\} \subseteq \mathbb{A}^4_{(x,y,t,v)},$$

with the special fiber given by $xyv = 0$. Clearly this can be simplified to

$$\{\frac{1}{x^6y^6v^4} f(x^3y^2v^2, x^3y^3v^2) - axv^2 - bv^2 = 0\} \subseteq \mathbb{A}^3_{(x,y,v)}.$$

On this open set $x = 0$ is a local equation for $E_4' \cup E_4''$ and $y = 0$ is a local equation for E. Thus we can blow-down using the relation $xy \mapsto y$. This gives

$$\{\frac{1}{y^6v^4} f(xy^2v^2, y^3v^2) - axv^2 - bv^2 = 0\} \subseteq \mathbb{A}^3_{(x,y,v)},$$

with special fiber given by $yv = 0$.

The question remains as to which elements of Δ_1 can appear in this stable limit. On the open set U_0, the elliptic curve is given by the equation $v^2 - t - at^3 - bt^4 = 0$ in the t, v-plane, and the double cover of $\mathbb{P}^1$ that appears in the stable reduction process is given by $(t, v) \mapsto t$, where we consider t as an affine coordinate on $\mathbb{P}^1$. Moreover, U_0 contains all but three points of the elliptic curve: $C \cap E$, $E_4' \cap E$, and $E_4'' \cap E$, and it's clear where these points map to: $C \cap E$ maps to 0 while the other two points map to infinity. Thus we see that the map is branched at the points of $\mathbb{P}^1$ satisfying $t + at^3 + bt^4 = 0$.

If we compose this map with the automorphism of $\mathbb{P}^1$ that sends t to $\frac{1}{t}$. This new map is then branched at infinity and points satisfying $t^3 + at + b = 0$. Thus the elliptic curve that appears in the stable limit is isomorphic to an elliptic curve in the x, y-plane given by $y^2 = x^3 + ax + b$. The j-invariant of such a curve is easily calculated as

$$j = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Since every elliptic curve is isomorphic to one of the form $y^2 = x^3 + Ax + B$ in the plane, we see that the elliptic curve appearing in our stable limit is a general elliptic curve.

If C is any smooth curve of genus 2 and Q is a point of C that is not a hyperelliptic Weierstrass point, then $|K + 2Q|$ determines a map from C to $\mathbb{P}^2$. This maps C to a cuspidal quartic which is smooth away from the cusp. Moreover, after an automorphism of $\mathbb{P}^2$, we can assume that the cuspidal quartic is given by $F(X, Y, Z) = 0$, for some choice of $\alpha_{i,j}$ and $\beta_{i,j}$. Thus we see that an open dense subset of the genus 2 curves can appear in the limit, and that the point $E \cap C$ can be any point of C other than one of the six hyperelliptic Weierstrass points. This shows that there is an open dense subset of Δ_1, any of whose points can appear as the stable limit of our family.

We now use this family to prove the following theorem:

Theorem 3.2.5. *Using the notation as in the proof of (3.1.2), let $\phi : E \to \mathbb{P}^1$ be the double cover of $\mathbb{P}^1$ determined (up to automorphism of $\mathbb{P}^1$) by $|2P_1|$. Let S_1, S_2, and S_3 be the points of E, other than P_1, that are ramified over $\mathbb{P}^1$. Then, as a scheme, $D_1(\sigma')$ is reduced except at S_1, S_2, S_3, and P_2, where the ideal locally defining $D_1(\sigma')$ is the product of the maximal ideal at S_i and the ideal defining E, $i = 1, 2, 3$, (or the the maximal ideal at P_2 and the ideal defining E_0, respectively).*

Proof. We use the family, $\pi : X \to \mathbb{A}^1_u$, just constructed. It's clear that we can apply (3.2.3) and (3.2.4) to our situation, were D_X, D_Y, and D_Z are simply the proper transforms of $\{X = 0\}$, $\{Y = 0\}$, and $\{Z = 0\}$. Thus we first find local equations for s_X, s_Y, and s_Z on each of the open sets U_0, U_1, and V.

On U_0 we have $s_X = x$, $s_Y = x^2 v$, and $s_Z = t$; on U_1 we have $s_X = x$, $s_Y = x^2 y$, and $s_Z = v$; and on V we have $s_X = yx$, $s_Y = y^2$, and $s_Z = 1$.

Now we can explicitly compute the map at each point of $D_1(\sigma')$. Note that all points of E are contained in U_0, accept those points that are the result of blowing-down E_4' and E_4'' (contained in V) and the point where E meets C (contained in

U_1).

We consider U_0 first. Recall that on this open set the total space of the family is given by

$$\{\frac{t^8}{x^6}f(x^2t^{-1},x^3vt^{-2})-at^7-bt^8=0\}\cap\{t\neq 0\}\subseteq \mathbb{A}^3_{(x,t,v)},$$

with a fiber of the family given by $x=u$. Thus we have

$$dx=0,$$

and

$$d\left(\frac{t^8}{x^6}f(x^2t^{-1},x^3vt^{-2})-at^7-bt^8\right)=0$$

This allows us to write

$$0=p(x,t,v)\;dt+q(x,t,v)\;dv,$$

where

$$\begin{aligned}
p(x,t,v) &= \frac{\partial}{\partial t}\left(\frac{t^8}{x^6}f(x^2t^{-1},x^3vt^{-2})-at^7-bt^8\right)\\
&= 4v^2t^3+\sum_{i+j=3}(5-j)\alpha_{i,j}x^jv^jt^{4-j}+\sum_{i+j=4}(4-j)\beta_{i,j}x^{2+j}v^jt^{3-j}-7at^6-8bt^7\\
q(x,t,v) &= \frac{\partial}{\partial v}\left(\frac{t^8}{x^6}f(x^2t^{-1},x^3vt^{-2})-at^7-bt^8\right)\\
&= 2vt^4+\sum_{i+j=3}j\alpha_{i,j}x^jv^{j-1}t^{5-j}+\sum_{i+j=4}j\beta_{i,j}x^{2+j}v^{j-1}t^{4-j}
\end{aligned}$$

Suppose $(0,\gamma,\zeta)$ is a point on E with $\zeta\neq 0$. Since $t\neq 0$ on all of U_0, we have $2\zeta\gamma^4\neq 0$, and hence $q(0,\gamma,\zeta)\neq 0$. Thus in a neighborhood of $(0,\gamma,\zeta)$ we have

$$dv=\frac{-p(x,t,v)}{q(x,t,v)}dt$$

Let (x_0, t_0, v_0) be a point in such neighborhood. The linearizations of x, x^2v, and t at this point are

$$\begin{aligned} x &= x_0 + dx \\ &= x_0 \\ x^2v &= x_0^2v_0 + 2x_0v_0dx + x_0^2dv \\ &= x_0^2v_0 - x_0^2\left(\frac{-p(x_0,t_0,v_0)}{q(x_0,t_0,v_0)}\right)dt \\ t &= t_0 + dt \end{aligned}$$

Thus locally, the map σ' can be given by the matrix:

$$\begin{bmatrix} x & x^2v & t \\ 0 & -x^2\left(\frac{-p(x,t,v)}{q(x,t,v)}\right) & 1 \end{bmatrix}$$

The ideal of

$$\frac{\mathbb{C}[x,t,v,t^{-1}]}{\left(\frac{t^8}{x^6}f(x^2t^{-1},x^3vt^{-2}) - at^7 - bt^8\right)}$$

generated by the 2×2 minor determinants of this matrix is (x); in particular, $D_1(\sigma')$ is reduced at such points.

Next we consider the points $(0,\gamma,0)$ of E. In this case we have $-7a\gamma^6 - 8b\gamma^7 \neq 0$, and hence $p(0,\gamma,0)\neq 0$. Thus in a neighborhood of $(0,\gamma,0)$ we have

$$dt = \frac{-q(x,t,v)}{p(x,t,v)}dv$$

Again, let (x_0,t_0,v_0) be a point in such a neighborhood. The linearizations of x, x^2v

and t at this point are

$$\begin{aligned}
x &= x_0 + dx \\
&= x_0 \\
x^2 v &= x_0^2 v_0 + 2x_0 v_0 dx + x_0^2 dv \\
&= x_0^2 v_0 + x_0^2 dv \\
t &= t_0 + dt \\
&= t_0 + \frac{-q(x_0, t_0, v_0)}{p(x_0, t_0, v_0)} dv
\end{aligned}$$

So locally, the map σ' can be given by the matrix:

$$\begin{bmatrix} x & x^2 v & t \\ 0 & x^2 & \frac{-q(x,t,v)}{p(x,t,v)} \end{bmatrix}$$

The ideal of

$$\frac{\mathbb{C}[x, t, v, t^{-1}]}{\left(\frac{t^8}{x^6} f(x^2 t^{-1}, x^3 v t^{-2}) - a t^7 - b t^8\right)}$$

generated by the 2×2 minor determinants of this matrix is

$$\left(x^3, \frac{xq(x,t,v)}{p(x,t,v)}, x^2 \left(\frac{vq(x,t,v)}{p(x,t,v)} + t \right) \right)$$

When we pass to the complete local ring at $(0, \gamma, 0)$, we see that since $\frac{vq(x,t,v)}{p(x,t,v)} + t \neq 0$ and $p(x,t,v) \neq 0$, we have that the ideal is given by

$$(xq(x,t,v), x^2) = (xvt^4, x^2) = (xv, x^2)$$

In particular, $D_1(\sigma')$ is non-reduced at such points.

Next we consider the points of V that are not contained in $U_0 \cup U_1$. There are only two such points, those obtained from blowing-down E_4' and E_4''. Recall that on V the total space of the family is given by

$$\{\frac{1}{y^6v^4}f(xy^2v^2, y^3v^2) - axv^2 - bv^2 = 0\} \subseteq \mathbb{A}^3_{(x,y,v)},$$

with a fiber given by $yv = u$. Thus we have

$$d(yv) = y\ dv + v\ dy = 0.$$

Since $v \neq 0$, this gives

$$dy = \frac{-y}{v}dv.$$

We also have

$$d\left(\frac{1}{y^6v^4}f(xy^2v^2, y^3v^2) - axv^2 - bv^2\right) = 0$$

These relations allow us to write

$$0 = p(x,y,v)\ dx + q(x,y,v)\ dv,$$

where

$$\begin{aligned}
p(x,y,v) &= \frac{\partial}{\partial x}\left(\frac{1}{y^6v^4}f(xy^2v^2,y^3v^2)-axv^2-bv^2\right)\\
&= \sum_{i+j=3} i\alpha_{i,j}x^{i-1}y^jv^2+\sum_{i+j=4} i\beta_{i,j}x^{i-1}y^{2+j}v^4-av^2\\
&= v^2\left(\sum_{i+j=3} i\alpha_{i,j}x^{i-1}y^j+\sum_{i+j=4} i\beta_{i,j}x^{i-1}y^{2+j}v^2-a\right)\\
q(x,y,v) &= \frac{\partial}{\partial v}\left(\frac{1}{y^6v^4}f(xy^2v^2,y^3v^2)-axv^2-bv^2\right)\\
&\quad -\frac{y}{v}\frac{\partial}{\partial y}\left(\frac{1}{y^6v^4}f(xy^2v^2,y^3v^2)-axv^2-bv^2\right)\\
&= \sum_{i+j=3} 2\alpha_{i,j}x^iy^jv+\sum_{i+j=4} 4\beta_{i,j}x^iy^{2+j}v^3-2axv-2bv\\
&\quad -\sum_{i+j=3} j\alpha_{i,j}x^iy^jv-\sum_{i+j=4}(2+j)\beta_{i,j}x^iy^{2+j}v^3\\
&= \sum_{i+j=3}(2-j)\alpha_{i,j}x^iy^jv+\sum_{i+j=4}(2-j)\beta_{i,j}x^iy^{2+j}v^3-2axv-2bv
\end{aligned}$$

The points of V we wish to consider are $(0,0,\zeta)$, where $1-b\zeta^2=0$. At such a point we have

$$p(0,0,\zeta)=-a\zeta^2\neq 0$$

Thus in a neighborhood of such points we have

$$dx=\frac{-q(x,y,v)}{p(x,y,v)}dv$$

Let (x_0,y_0,v_0) be a point in such a neighborhood. The linearizations of xy, y^2

and 1 at this point are

$$\begin{aligned}
xy &= x_0y_0 + y_0dx + x_0dy \\
&= x_0y_0 - \left(y_0\frac{q(x_0,y_0,v_0)}{p(x_0,y_0,v_0)} + \frac{x_0y_0}{v_0}\right) dv \\
y^2 &= y_0^2 + 2y_0dy \\
&= y_0^2 - 2\frac{y_0^2}{v_0}dv \\
1 &= 1 + 0\ dv
\end{aligned}$$

So locally, the map σ' can be given by the matrix:

$$\begin{bmatrix} xy & y^2 & 1 \\ -y\frac{q(x,y,v)}{p(x,y,v)} - \frac{xy}{v} & \frac{-2y^2}{v} & 0 \end{bmatrix}$$

The ideal of

$$\frac{\mathbb{C}[x,y,v]}{\left(\frac{1}{y^6v^4}f(xy^2v^2, y^3v^2) - axv^2 - bv^2\right)}$$

generated by the 2×2 minor determinants of this matrix is

$$\left(y\left(\frac{q(x,y,v)}{p(x,y,v)} + \frac{xy}{v}\right), y^2\right).$$

But

$$\frac{q(0,0,\zeta)}{p(0,0,\zeta)} = \frac{2b\zeta}{a} \neq 0$$

Thus in the complete local ring at $(0,0,\zeta)$, the ideal is given simply by (y); in particular, $D_1(\sigma')$ is reduced at the points $(0,0,\zeta)$.

Finally, we consider the point $P = (0, 1, 0)$ of U_1. Recall that the total space of the family on U_1 is given by

$$\{y^4 + y^3 \sum_{i+j=3} \alpha_{i,j} x^j + y^2 \sum_{i+j=4} \beta_{i,j} x^{j+2} - ayv^4 - bv^6 = 0\} \cap \{y \neq 0\} \subseteq \mathbb{A}^3_{(x,y,v)},$$

with a fiber given by $xv = u$. So in a neighborhood of P we have

$$\begin{aligned} 0 &= d\left(y^4 + y^3 \sum_{i+j=3} \alpha_{i,j} x^j + y^2 \sum_{i+j=4} \beta_{i,j} x^{j+2} - ayv^4 - bv^6\right) \\ &= p(x, y, v)\, dx + q(x, y, v)\, dy + r(x, y, v)\, dv \end{aligned}$$

where

$$\begin{aligned} p(x, y, v) &= y^3 \sum_{i+j=3} j\alpha_{i,j} x^{j-1} + y^2 \sum_{i+j=4} (j+2)\beta_{i,j} x^{j+1} \\ q(x, y, v) &= 4y^3 + 3y^2 \sum_{i+j=3} \alpha_{i,j} x^j + 2y \sum_{i+j=4} \beta_{i,j} x^{j+2} - av^4 \\ r(x, y, v) &= -4ayv^3 - 6bv^5 \end{aligned}$$

But we have $q(0, 1, 0) = 4 + 3\alpha_{3,0} = 1$. (Recall that $\alpha_{3,0}$ was assumed to be -1.) Thus in a neighborhood of P we have

$$dy = \frac{-p(x, y, v)}{q(x, y, v)} dx - \frac{r(x, y, v)}{q(x, y, v)} dv$$

We also have the relation

$$0 = d(xv) = v\, dx + x\, dv$$

Thus $\mathcal{F}$ is locally generated by 1, dx, and dv, but with a non-trivial relation at P.

We consider the map from $\mathcal{E}(U_1)$ to the free module generated by 1, dx, and dv. Let (x_0, y_0, v_0) be a point in a neighborhood of P. The linearizations of x, x^2y, and v at this point are

$$\begin{aligned}
x &= x_0 + dx \\
x^2y &= x_0^2y_0 + 2x_0y_0\ dx + x_0^2\ dy \\
&= x_0^2y_0 + 2x_0y_0\ dx + x_0^2\frac{-p(x_0,y_0,v_0)}{q(x_0,y_0,v_0)}dx - x_0^2\frac{r(x_0,y_0,v_0)}{q(x_0,y_0,v_0)}dv \\
&= x_0^2y_0 + \left(2x_0y_0 - x_0^2\frac{p(x_0,y_0,v_0)}{q(x_0,y_0,v_0)}\right)dx - x_0^2\frac{r(x_0,y_0,v_0)}{q(x_0,y_0,v_0)}dv \\
v &= v_0 + dv
\end{aligned}$$

Locally this map is given by

$$\begin{bmatrix} x & x^2y & v \\ 1 & 2xy - x^2\frac{p(x,y,v)}{q(x,y,v)} & 0 \\ 0 & -x^2\frac{r(x,y,v)}{q(x,y,v)} & 1 \end{bmatrix} \tag{3.2.1}$$

Since $\mathcal{F}$ is not free at P, we apply the process of [D] (described above). As mentioned above, the smallest nonzero Fitting ideal of $\mathcal{F}$ is the maximal ideal of P. Thus we blow-up along the maximal ideal of P, pullback $\mathcal{E}$, $\mathcal{F}$, and σ and take their double duals. This gives the map $\sigma' : \mathcal{E}' \to \mathcal{F}'$. We have two patches to consider:

On the first patch we have the relation $v = xv$. The result of pulling back (3.2.1) is

$$\begin{bmatrix} x & x^2y & xv \\ 1 & 2xy - x^2\frac{p(x,y,v)}{q(x,y,v)} & 0 \\ 0 & -x^2\frac{r(x,y,v)}{q(x,y,v)} & 1 \end{bmatrix}$$

But on this patch the relation $0 = v\ dx + x\ dv$, after taking the double dual of the pullback of $\mathcal{F}$, becomes $0 = v\ dx + dv$. Thus the map σ' is given locally by

$$\begin{bmatrix} x & x^2y & xv \\ 1 & 2xy - x^2\frac{p(x,y,v)}{q(x,y,v)} + x^2v\frac{r(x,y,v)}{q(x,y,v)} & -v \end{bmatrix} \tag{3.2.2}$$

The ideal generated by the 2×2 minor determinants in

$$\frac{\mathbb{C}[x,y,v,y^{-1}]}{(y^4 + y^3\sum_{i+j=3}\alpha_{i,j}x^j + y^2\sum_{i+j=4}\beta_{i,j}x^{j+2} - ax^4yv^4 - bx^6v^6)}$$

is

$$\left(xv, x^2\left(y - x\frac{p(x,y,v)}{q(x,y,v)}\right)\right)$$

In the complete local ring at any point of this patch along the exceptional divisor, this ideal becomes

$$(xv, x^2)$$

If $v \neq 0$, then this ideal is simply (x), and we see that $D_1(\sigma')$ is reduced.

On the second patch we have the relation $x = xv$. The result of pulling back

(3.2.1) is

$$\begin{bmatrix} xv & x^2yv^2 & v \\ 1 & 2xyv - x^2v^2\frac{p(xv,y,v)}{q(xv,y,v)} & 0 \\ 0 & -x^2v^2\frac{r(xv,y,v)}{q(xv,y,v)} & 1 \end{bmatrix}$$

But on this patch the relation $0 = v\ dx + x\ dv$, after taking the double dual of the pullback of $\mathcal{F}$, becomes $0 = dx + x\ dv$. Thus the map σ' is given locally by

$$\begin{bmatrix} xv & x^2yv^2 & v \\ -x & -x^2v^2\frac{r(xv,y,v)}{q(xv,y,v)} - 2x^2yv + x^3v^2\frac{p(xv,y,v)}{q(xv,y,v)} & 1 \end{bmatrix} \tag{3.2.3}$$

The ideal generated by the 2×2 minor determinants in

$$\frac{\mathbb{C}[x, y, v, y^{-1}]}{(y^4 + y^3 \sum_{i+j=3} \alpha_{i,j} x^j v^j + y^2 \sum_{i+j=4} \beta_{i,j} x^{j+2} v^{j+2} - ayv^4 - bv^6)}$$

is (xv). This shows that at $(0, 1, 0)$, $D_1(\sigma')$ is simply the union of the exceptional divisor and E.

Combining this with (3.2.2) completes the proof of (3.2.5).

$\square$

3.3 Excess Porteous

Let $\pi : X \to B$ be a generic 1-parameter family of smooth, nonhyperelliptic curves of genus 3 degenerating to a general element of Δ_1; let E and C be the elliptic and genus 2 curves, respectively, meeting transversely at P.

Let $\sigma : \mathcal{E} \to \mathcal{F}$ be the map of coherent sheaves described above; let $g : X' \to X$ be the blow-up of X at the maximal ideal of P, with E_0 the exceptional divisor; and let $\sigma' : \mathcal{E}' \to \mathcal{F}'$ be the map of vector bundles on X' described above.

Let $D = D_1(\sigma')$. The expected codimension of D is $(3-1)(2-1) = 2$, but by (3.1.2) D is the union of E_0, E, and the six hyperelliptic Weierstrass points of C $(Q_1, \ldots, Q_6.)$ We'd like to compute the class $\mathbb{D}_1(\sigma')$ in $A_0(D)$; since the codimension of D is less than 2, we use (1.4.8). Specifically, since $D_0(\sigma') = \emptyset$, there exist vector bundles $\mathcal{K}$ and $\mathcal{C}$, of ranks 2 and 1 respectively, on D, and an exact sequence

$$0 \to \mathcal{K} \to \mathcal{E}'_D \to \mathcal{F}'_D \to \mathcal{C} \to 0,$$

where $\mathcal{E}'_D$ and $\mathcal{F}'_D$ are the restrictions of $\mathcal{E}'$ and $\mathcal{F}'$ to D. Then we have

$$\mathbb{D}_1(\sigma') = \{c(\mathcal{K}^\vee \otimes \mathcal{C}) \cap s(D, X')\}_0$$

Proposition 3.3.1. *Let P_1 and P_2 be as in the proof of (3.1.2)), S_1, S_2, and S_3 as in (3.2.5), and Q_i, $i = 1, \ldots, 6$, the hyperelliptic Weierstrass points of C. Then*

$$s(D, X') = [E_0] + [E] + \sum_{i=1}^{6}[Q_i] + \sum_{i=1}^{3}[S_i] + [P_1] + [P_2].$$

In particular,

$$\int_D s(D, \widetilde{X}) = 11$$

Proof. Let $f : \widetilde{X}' \to X'$ be the blow-up of X' along D. By [H, Ex.II.7.11(b)] we can identify this blow-up with the blow-up along $Q_1 \cup \ldots \cup Q_6 \cup S_1 \cup S_2 \cup S_3 \cup P_2$. Let E' be the exceptional divisor of this blow-up and Q_i', S_i', and P_2' the components lying above Q_i, S_i, and P_2, respectively. Then $D' := f^{-1}(D) = f^*(E_0 + E) + E'$. Let $h : D' \to D$ be the projection. By [F, Cor. 4.2.2] we have

$$
\begin{aligned}
s(D, X') &= h_*[D'] - h_*(D' \cdot [D']) \\
&= [E_0] + [E] - h_*(f^*(E_0 + E) \cdot [f^*(E_0 + E)] + 2f^*(E_0 + E) \cdot [E'] + E' \cdot [E']) \\
&= [E_0] + [E] - (E_0 + E) \cdot [E_0 + E] - h_*(2f^*(E_0 + E) \cdot [E'] + E' \cdot [E']) \\
&= [E_0] + [E] - (E_0 + E) \cdot [E_0 + E] + \sum_{i=1}^{6} [Q_i] + \sum_{i=1}^{3} [S_i] + [P_2] \\
&\quad - h_*(2f^*(E_0 + E) \cdot [E']) \\
&= [E_0] + [E] - (E_0 + E) \cdot [E_0 + E] + \sum_{i=1}^{6} [Q_i] + \sum_{i=1}^{3} [S_i] + [P_2]
\end{aligned}
$$

Lines 2 and 3 are by [F, Prop. 2.3(c)]. Line 4 is due to the fact that each component of E' is a rational curve, with self-intersection -1, disjoint from the other components. Line 5 is shown as follows:

$$
\begin{aligned}
f^*(E_0 + E) \cdot [E'] &= (E_0 + E + S_1' + S_2' + S_3' + P_2') \cdot \\
&\qquad [Q_1' + \ldots + Q_6' + S_1' + S_2' + S_3' + P_2'] \\
&= E_0 \cdot [P_2'] + \sum_{i=1}^{3} E \cdot [S_i'] + \sum_{i=1}^{3} S_i' \cdot [S_i'] + P_2' \cdot [P_2'] \\
&= (E_0 + P_2') \cdot [P_2'] + \sum_{i=1}^{3} (E + S_i') \cdot [S_i'] \\
&= 0,
\end{aligned}
$$

where by abuse of notation we identify E_0 and E with their proper transforms in D'. Furthermore, we have

$$E_0 \cdot [E_0] = -[P_1],\ E_0 \cdot [E] = [P_1],\ \text{and}\ E_1 \cdot [E] = -2[P_1]$$

Thus we have

$$(E_0 + E) \cdot ([E_0 + E]) = E_0 \cdot [E_0] + 2E \cdot [E_0] + E \cdot [E] = -[P_1]$$

□

In order to determine the equivalence of D in $\mathbb{D}_1(\sigma')$, we need only look at $c_1(\mathcal{K}^* \otimes \mathcal{C}) \cap ([E_0] + [E_1])$.

Notation 3.3.2. Since $\mathcal{F}'_D \to \mathcal{C}$ is surjective, its kernel is a vector bundle (of rank 1). But the kernel of this map is the image of $\mathcal{E}'_D \to \mathcal{F}'_D$, which we'll call $\mathcal{A}$.

Proposition 3.3.3. *In the above situation, we have*

$$c_1(\mathcal{K}^* \otimes \mathcal{C}) \cap ([E_0] + [E_1]) = (2c_1(\mathcal{F}'_D) - c_1(\mathcal{A})) \cap ([E_0] + [E_1]).$$

Proof. By [F, Ex. 3.2.2] we have

$$c_1(\mathcal{K}^* \otimes \mathcal{C}) = 2c_1(\mathcal{C}) + c_1(\mathcal{K}^*) = 2c_1(\mathcal{C}) - c_1(\mathcal{K}).$$

Also, from the exact sequence

$$0 \to \mathcal{K} \to \mathcal{E}'_D \to \mathcal{F}'_D \to \mathcal{C} \to 0,$$

we have the relation

$$c_1(\mathcal{K}) + c_1(\mathcal{F}'_D) = c_1(\mathcal{E}'_D) + c_1(\mathcal{C}).$$

Combining we have

$$c_1(\mathcal{K}^* \otimes \mathcal{C}) = c_1(\mathcal{C}) + c_1(\mathcal{F}'_D) - c_1(\mathcal{E}'_D).$$

Moreover, $\mathcal{E}'_D$ is the trivial bundle on both E_0 and E, so $c_1(\mathcal{E}'_D) \cap ([E_0] + [E_1]) = 0$.

But we also have

$$c_1(\mathcal{C}) + c_1(\mathcal{A}) = c_1(\mathcal{F}'_D).$$

Combining this with above gives the desired result. □

Proposition 3.3.4. *In the above situation, we have*

$$\int_D 2c_1(\mathcal{F}'_D) \cap ([E_0] + [E]) = 6.$$

Proof. The proof of [D, Lemma 5] immediately generalizes to our situation to show that

$$c_1(\mathcal{F}'_D) = 3\gamma_D - E_0,$$

where $\gamma_D := g^*(c_1(\omega_{X/B})) \cdot D$. The restriction of $\omega_{X/B}$ to E is $K_1(P_1)$, where K_1 is the canonical bundle on E, so γ_D has degree 1 on this curve. Also the restriction of $\omega_{X/B}$ to C is $K_2(P_2)$ where K_2 is the canonical bundle on C, so γ_D has degree 3 on this curve. Moreover, the degree of γ_D on any member of the family is 4, so that the degree on E_0 is 0. Also, we see that $\sharp(E_0 \cdot [E_0]) = -1$ and $\sharp(E_0 \cdot [E_1]) = 1$. Thus

$$\begin{aligned} \int_D 2c_1(\mathcal{F}'_D) \cap ([E_0] + [E]) &= \int_D 2(3\gamma_D - E_0) \cdot ([E_0] + [E]) \\ &= 2\int_D 3\gamma_D \cdot [E_0] - E_0 \cdot [E_0] + 3\gamma_D \cdot [E] - E_0 \cdot [E] \\ &= 2(0 - (-1) + 3(1) - 1) \\ &= 6. \end{aligned}$$

□

It remains to determine $c_1(\mathcal{A}) \cap ([E_0] + [E])$. We use the following two propositions:

Proposition 3.3.5. *In the above situation, we have*

$$\mathcal{A}_{E_0} \cong \mathcal{O}(1)$$

Proof. We consider the map $\mathcal{E}' \to \mathcal{F}'$ on E_0 given by restricting the matrices (3.2.2) and (3.2.3), from the proof of (3.2.5), to E_0. On one affine patch of E_0 we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -v \end{bmatrix},$$

and on the other

$$\begin{bmatrix} 0 & 0 & 0 \\ -x & 0 & 1 \end{bmatrix},$$

where v and x are affine parameters on their respective patches. On the first patch $v = 0$ is a local equation for C, and on the second $x = 0$ is a local equation for E.

Since C and E both meet E_0 transversely, we see that s_1 and s_3 map to sections with a simple zero on E_0. $\square$

Notation 3.3.6. Let $\mathcal{F}_1' = g^*\omega_{X/B}$ and $\mathcal{F}_2' = g^*\omega_{X/B}^2 \otimes \mathcal{O}(-E_0)$.

Proposition 3.3.7. *There exists a short exact sequence of vector bundles*

$$0 \to \mathcal{F}_2' \to \mathcal{F}' \to \mathcal{F}_1' \to 0,$$

on X'.

Proof. This is a trivial generalization of [D, Lemma 5]. $\square$

Proposition 3.3.8. *In the above situation,* $\mathcal{A}_E \cong \mathcal{O}_E$*. In particular,* $c_1(\mathcal{A}) \cap E = 0$.

Proof. We consider the composition of maps $\mathcal{E}' \to \mathcal{F}' \to \mathcal{F}_1'$. For a point $x \in E$ this map takes sections of the relative dualizing sheaf, expands them about a local coordinate at x, first maps them to the constant and linear term, and then maps them to the constant term. We can choose a basis for the sections of the dualizing sheaf such that two of them, when expanded about $x \in E$ are 0, and the other is a section of $K_1(P_1)$ where K_1 is the canonical bundle of E. Thus $\mathcal{E}_x' \to (\mathcal{F}_1')_x$ is surjective if and only if the section on E fails to vanish at x. This is the case for all $x \neq P_1$. When we consider the map $\mathcal{E}' \to \mathcal{F}_1'(-P_1)$, we see that away from P_1 this is the same as before. But at P_1 this map will take the linear term of the section of the dualizing sheaf instead of the constant term, and thus is surjective at all points of E_1. Thus $\mathcal{A}_E \to (\mathcal{F}_1')_E(-P_1)$ is a surjective map of vector bundles of the same rank, and thus is an isomorphism. But $\mathcal{F}_1'$ is simply the relative dualizing sheaf of the family, so $\mathcal{F}_1'$ is $K_1(P_1)$ on E. Thus $\mathcal{A}_E \cong K_1 = \mathcal{O}_E$. □

Proposition 3.3.9. *Thus we have*

$$\begin{aligned} c_1(\mathcal{A}) \cap ([E_0] + [E]) &= c_1(\mathcal{O}(1)) \cap [E_0] \\ \sharp(c_1(A)) \cap ([E_0] + [E])) &= 1. \end{aligned}$$

Proof. This follows immediately from (3.3.5) and (3.3.8). □

We have now proven the following result:

Theorem 3.3.10. *In the above situation,*

$$\int_D \mathbb{D}_1(\sigma') = 16$$

Proof. By (3.3.1), (3.3.4), and (3.3.9), we have

$$\begin{aligned} \int_D s(D, X') &= 11 \\ \int_D 2c_1(\mathcal{F}_D') \cap ([E_0] + [E]) &= 6 \\ \int_D c_1(\mathcal{A}) \cap ([E_0] + [E])) &= 1. \end{aligned}$$

Thus by (3.3.1) and (3.3.3), we have

$$\int_D \mathbb{D}_1(\sigma') = 11 + 6 - 1 = 16.$$

□

3.4 The Main Result

Let H be the hyperelliptic locus in $\mathfrak{M}_3$ and $\overline{H}$ its closure in $\overline{\mathfrak{M}}_3$. Let $\overline{h} \in \mathrm{Pic}_{\mathrm{fun}}(\overline{\mathfrak{M}}_3)$ be the rational divisor class on the moduli stack associated to $\overline{H}$ by [HM, Prop. (3.88)]. We wish to combine the above result with that of [D] to obtain an expression for $\overline{h}$ in terms of the generators λ, δ_0, and δ_1 of $\mathrm{Pic}_{\mathrm{fun}}(\overline{\mathfrak{M}}_3)$.

Let $\pi : X \to B$ be a generic 1-parameter family of stable curves of genus 3. Let $\sigma' : \mathcal{E}' \to \mathcal{F}'$ be the map described above on X', where $g : X' \to X$ is the blow-up along the nodes of singular fibers of π. The result of applying (1.4.7) gives

$$[D_1(\sigma')] = c_2(\mathcal{E}'^* - \mathcal{F}'^*)$$

From the proof of [D, Lemma 5], we have

$$c(\mathcal{E}') = 1 - \lambda$$

$$c(\mathcal{F}') = 1 + 3\gamma - E_0 + 2\gamma^2,$$

where E_0 is the exceptional divisor of the blow-up $g : X' \to X$. This gives

$$[D_1(\sigma')] = 7\omega^2 - 3\omega\lambda + E_0^2.$$

But there is one component of E_0 for each fiber of π from δ_0 and one component for each fiber from δ_1, and each component has square -1. Hence

$$(\pi g)_*([D_1(\sigma')]) = 7\kappa - 12\lambda - \delta_0 - \delta_1.$$

We use $\lambda = (\kappa + \delta_0 + \delta_1)/12$ from [HM, (3.110)], to obtain

$$(\pi g)_*([D_1(\sigma')]) = 72\lambda - 8\delta_0 - 8\delta_1.$$

We notice that generic members of Δ_1 are not contained in $\overline{H}$, but by (3.3.10) we know that (1.4.7) will count generic members of Δ_1 16 times each. Thus we need to subtract $16\delta_1$ from the result above. This gives

$$72\lambda - 8\delta_0 - 24\delta_1.$$

Since each smooth hyperelliptic curve contains 8 hyperelliptic Weierstrass points, we divide this by 8, giving

$$\overline{h} = 9\lambda - \delta_0 - 3\delta_1.$$

By [HM, Prop. (3.92)], we have

$$[\overline{H}] = 18\lambda - 2\Delta_0 - 3\Delta_1.$$

Appendix A

Stable Reduction Calculations

A.1 Total Space Singularities

Proposition A.1.1. *(P1) is singular along E1 and nonsingular away from E1.*

Proof. On (P1) the total space of the family is given by

$$f(x, xy) - at^2x - bt^3 = 0$$

Writing this out we have

$$x^2y^2 + \sum_{i+j=3} \alpha_{i,j}x^3y^j + \sum_{i+j=4} \beta_{i,j}x^4y^j - at^2x - bt^3 = 0$$

The first order partial derivatives of this family are

$$\frac{\partial}{\partial x} = 2xy^2 + \sum_{i+j=3} 3\alpha_{i,j}x^2y^j + \sum_{i+j=4} 4\beta_{i,j}x^3y^j - at^2 \tag{A.1.1}$$

$$\frac{\partial}{\partial y} = 2x^2y + \sum_{i+j=3} j\alpha_{i,j}x^3y^{j-1} + \sum_{i+j=4} j\beta_{i,j}x^4y^{j-1} \tag{A.1.2}$$

$$\frac{\partial}{\partial t} = -2atx - 3bt^2 \tag{A.1.3}$$

Since (P1) is the result of blowing up the orginal family along an ideal whose support is the only singular point of the original family, we see that away from E_1 (P1) is nonsingular.

On (P1), E_1 is given by $x = 0$. From the equation for the total space of the family on this patch, we see that $t = 0$ whenever $x = 0$. Moreover, all three partial derivatives vanish at all points of the form $(0, y_0, 0)$. So (P1) is singular along E_1. $\square$

Proposition A.1.2. *(P2) is singular along E1 and nonsingular away from E1.*

Proof. On (P2) the total space of the family is given by

$$f(xy, y) - at^2xy - bt^3 = 0$$

Writing this out we have

$$y^2 + \sum_{i+j=3} \alpha_{i,j} x^i y^3 + \sum_{i+j=4} \beta_{i,j} x^i y^4 - at^2xy - bt^3 = 0$$

The first order partial derivatives of this family are

$$\frac{\partial}{\partial x} = \sum_{i+j=3} i\alpha_{i,j} x^{i-1} y^3 + \sum_{i+j=4} i\beta_{i,j} x^{i-1} y^4 - at^2 y \tag{A.1.4}$$

$$\frac{\partial}{\partial y} = 2y + \sum_{i+j=3} 3\alpha_{i,j} x^i y^2 + \sum_{i+j=4} 4\beta_{i,j} x^i y^3 - at^2 x \tag{A.1.5}$$

$$\frac{\partial}{\partial t} = -2atxy - 3bt^2 \tag{A.1.6}$$

As in the previous proof, (P2) is the result of blowing up the orginal family along an ideal whose support is the only singular point of the original family, and thus away from E_1 (P2) is nonsingular.

On (P2), E_1 is given by $y = 0$. From the equation for the total space of the family, we see that $t = 0$ whenever $y = 0$, and all three partial derivatives vanish at points of the form $(x_0, 0, 0)$. So (P2) is singular along E_1. □

Proposition A.1.3. *(P1-1) is nonsingular.*

Proof. On (P1-1) the total space of the family is given by

$$\frac{1}{x^3} f(x, x^2 y) - at^2 - bt^3 = 0$$

Writing this out we have

$$xy^2 + \sum_{i+j=3} \alpha_{i,j} x^j y^j + \sum_{i+j=4} \beta_{i,j} x^{1+j} y^j - at^2 - bt^3 = 0$$

The first order partial derivatives of this family are

$$\frac{\partial}{\partial x} = y^2 + \sum_{i+j=3} j\alpha_{i,j} x^{j-1} y^j + \sum_{i+j=4} (1+j)\beta_{i,j} x^j y^j \tag{A.1.7}$$

$$\frac{\partial}{\partial y} = 2xy + \sum_{i+j=3} j\alpha_{i,j} x^j y^{j-1} + \sum_{i+j=4} j\beta_{i,j} x^{1+j} y^{j-1} \tag{A.1.8}$$

$$\frac{\partial}{\partial t} = -2at - 3bt^2 \tag{A.1.9}$$

(P1-1) is the result of blowing up a point lying on E_1, the singular locus of (P1). Thus away from E_1 and E_2 (P1-1) is nonsingular.

E_2 is given locally by $x = 0$. From the equation of the total space of the family on this patch, we see that when $x = 0$ we have $-1 - at^2 - bt^3 = 0$. (Recall that $\alpha_{3,0} = -1$.) Hence $t \neq 0$ when $x = 0$. We set all partial derivatives equal to 0 and set $x = 0$ and attempt to solve for y and t. The equation for $\partial/\partial t$ gives $2at + 3bt^2 = 0$.

Since $t \neq 0$ this implies that $t = -2a/3b$. Thus we have

$$\begin{aligned} 0 &= 1 + at^2 + bt^3 \\ &= 1 + a\left(\frac{-2a}{3b}\right)^2 + b\left(\frac{-2a}{3b}\right)^3 \\ &= 1 + \frac{4a^3}{9b^2} - \frac{8a^3}{27b^2} \\ -1 &= \frac{4a^3}{27b^2} \end{aligned}$$

This is a contradiction by the restrictions placed on a, b. Hence, (P1-1) is nonsingular along E_2. Since E_1 does not meet this patch, (P1-1) is nonsingular.

□

Proposition A.1.4. *(P1-2) is singular along E1 and nonsingular away from E1.*

Proof. On (P1-2) the total space of the family is given by

$$\frac{1}{y^3} f(xy, xy^2) - at^2x - bt^3 = 0$$

Writing this out we have

$$x^2y + \sum_{i+j=3} \alpha_{i,j} x^3 y^j + \sum_{i+j=4} \beta_{i,j} x^4 y^{1+j} - at^2x - bt^3 = 0$$

The first order partial derivatives of this family are

$$\frac{\partial}{\partial x} = 2xy + \sum_{i+j=3} 3\alpha_{i,j} x^2 y^j + \sum_{i+j=4} 4\beta_{i,j} x^3 y^{1+j} - at^2 \tag{A.1.10}$$

$$\frac{\partial}{\partial y} = x^2 + \sum_{i+j=3} j\alpha_{i,j} x^3 y^{j-1} + \sum_{i+j=4} (1+j)\beta_{i,j} x^4 y^j \tag{A.1.11}$$

$$\frac{\partial}{\partial t} = -2atx - 3bt^2 \tag{A.1.12}$$

As in the previous proposition (P1-2) is nonsingular away from E_1 and E_2. Moreover, since (P1) was singular at every point of E_1 and only one point of E_1 was blown up, we see that (P1-2) must still be singular along E_1. Thus since E_1 is given locally by $x = 0$, we will consider the open subset given by $x \neq 0$.

Setting each of the partial derivatives equal to 0 and restricting to E_2 (given locally by $y = 0$), we have

$$0 = -3x^2 - at^2 \tag{A.1.13}$$

$$0 = x^2 + \alpha_{2,1}x^3 + \beta_{4,0}x^4 \tag{A.1.14}$$

$$0 = -2atx - 3bt^2 \tag{A.1.15}$$

Since we are assuming $x \neq 0$, (A.1.15) implies that $t \neq 0$ and $x = -3bt/2a$. Substituting into (A.1.13) gives

$$0 = -3\left(\frac{-3b}{2a}\right)^2 t^2 - at^2 = \left(\frac{-27b^2 - 4a^3}{4a^2}\right)t^2$$

Since $t \neq 0$ this implies that $-27b^2 = 4a^3$, which is a contradiction by the restrictions placed on a, b. Thus (P1-2) is nonsingular along points of E_2 not meeting E_1. $\square$

Proposition A.1.5. *(P1-2-1) is nonsingular.*

Proof. On (P1-2-1) the total space of the family is given by

$$\frac{1}{x^6y^3}f(x^2y, x^3y^2) - at^2 - bt^3 = 0$$

Writing this out we have

$$y + \sum_{i+j=3} \alpha_{i,j}x^jy^j + \sum_{i+j=4} \beta_{i,j}x^{2+j}y^{1+j} - at^2 - bt^3 = 0$$

The first order partial derivatives of this family are

$$\frac{\partial}{\partial x} = \sum_{i+j=3} j\alpha_{i,j}x^{j-1}y^j + \sum_{i+j=4}(2+j)\beta_{i,j}x^{1+j}y^{1+j} \tag{A.1.16}$$

$$\frac{\partial}{\partial y} = 1 + \sum_{i+j=3} j\alpha_{i,j}x^{j}y^{j-1} + \sum_{i+j=4}(1+j)\beta_{i,j}x^{2+j}y^{j} \tag{A.1.17}$$

$$\frac{\partial}{\partial t} = -2at - 3bt^2 \tag{A.1.18}$$

This patch is the result of blowing up (P1-2) at the point where E_1 and E_2 meet. Since (P1-2) was singular along E_1 and nonsingular elsewhere, the only possible singular points of this patch are those lying on E_1 and E_3, but E_1 does not meet this patch. Thus we need only check for singularities along E_3, which is given by $x = 0$ on this patch. But one easily see that (A.1.17) does not vanish along $x = 0$. Thus (P1-2-1) is nonsingular. □

Proposition A.1.6. *(P1-2-2) is singular along E1 and nonsingular away from E1.*

Proof. On (P1-2-2) the total space of the family is given by

$$\frac{1}{y^6}f(xy^2, xy^3) - at^2x - bt^3 = 0$$

Writing this out we have

$$x^2 + \sum_{i+j=3}\alpha_{i,j}x^3y^j + \sum_{i+j=4}\beta_{i,j}x^4y^{2+j} - at^2x - bt^3 = 0$$

As above, this patch is the result of blowing up (P1-2) at the point where E_1 and E_2 meet. It's clear that it will still be singular along E_1. Moreover, the only point of E_3 contained in this patch that is not contained in (P1-2-1) is where E_1 and E_3 meet, which must be a singular point. By the previous proposition, we see that (P1-2-2) must thus be nonsingular away from E_1. $\square$

Proposition A.1.7. *(P2-1) is nonsingular.*

Proof. On (P2-1) the total space of the family is given by

$$\frac{1}{t^2}f(xyt, yt) - atxy - bt = 0$$

Writing this out we have

$$y^2 + \sum_{i+j=3} \alpha_{i,j}x^i y^3 t + \sum_{i+j=4} \beta_{i,j}x^i y^4 t^2 - atxy - bt = 0$$

The first order partial derivatives of this family are

$$\frac{\partial}{\partial x} = \sum_{i+j=3} i\alpha_{i,j}x^{i-1}y^3 t + \sum_{i+j=4} i\beta_{i,j}x^{i-1}y^4 t^2 - aty \tag{A.1.19}$$

$$\frac{\partial}{\partial y} = 2y + \sum_{i+j=3} 3\alpha_{i,j}x^i y^2 t + \sum_{i+j=4} 4\beta_{i,j}x^i y^3 t^2 - atx \tag{A.1.20}$$

$$\frac{\partial}{\partial t} = \sum_{i+j=3} \alpha_{i,j}x^i y^3 + \sum_{i+j=4} 2\beta_{i,j}x^i y^4 t - axy - b \tag{A.1.21}$$

This patch is the result of blowing up (P2) along E_1, and thus the only possible singular points are those of E_4, which is given locally by $t = 0$. From the equation for the total space of the family, we see that $t = 0$ implies $y = 0$. Setting both y and t equal to 0, as well as (A.1.21), gives $b = 0$, a contradiction since we assume $b \neq 0$. Thus (P2-1) is nonsingular. $\square$

Proposition A.1.8. *(P2-2) is nonsingular.*

Proof. On (P2-2) the total space of the family is given by

$$\frac{1}{y^2}f(xy,y) - at^2xy - bt^3y = 0.$$

Writing this out we have

$$1 + \sum_{i+j=3} \alpha_{i,j}x^iy + \sum_{i+j=4} \beta_{i,j}x^iy^2 - at^2xy - bt^3y = 0$$

As above, this patch is the result of blowing up (P2) along E_1. Thus the only possible singular points are along E_4, which is given locally by $y = 0$. $y = 0$ does not meet this patch, we see that it is nonsingular. □

Proposition A.1.9. *(P1-2-2-1) is nonsingular.*

Proof. On (P1-2-2-1) the total space of the family is given by

$$\frac{1}{x^2y^6}f(xy^2,xy^3) - at^2x - bt^3x = 0$$

Writing this out we have

$$1 + \sum_{i+j=3} \alpha_{i,j}xy^j + \sum_{i+j=4} \beta_{i,j}x^2y^{2+j} - at^2x - bt^3x = 0$$

As in the previous two patches, the only possible singular points lie along E_4, which is given locally by $x = 0$. Thus E_4 does not meet this patch, and we see immediately that it is nonsingular. □

Proposition A.1.10. *(P1-2-2-2) is nonsingular.*

Proof. On (P1-2-2-2) the total space of the family is given by

$$\frac{1}{y^6t^2}f(xy^2t,xy^3t) - atx - bt = 0$$

Writing this out we have

$$x^2 + \sum_{i+j=3} \alpha_{i,j} x^3 y^j t + \sum_{i+j=4} \beta_{i,j} x^4 y^{2+j} t^2 - atx - bt = 0$$

The first order partial derivatives of this family are

$$\frac{\partial}{\partial x} = 2x + \sum_{i+j=3} 3\alpha_{i,j} x^2 y^j t + \sum_{i+j=4} 4\beta_{i,j} x^3 y^{2+j} t^2 - at \tag{A.1.22}$$

$$\frac{\partial}{\partial y} = \sum_{i+j=3} j\alpha_{i,j} x^3 y^{j-1} t + \sum_{i+j=4} (2+j)\beta_{i,j} x^4 y^{1+j} t^2 \tag{A.1.23}$$

$$\frac{\partial}{\partial t} = \sum_{i+j=3} \alpha_{i,j} x^3 y^j + \sum_{i+j=4} 2\beta_{i,j} x^4 y^{2+j} t - ax - b \tag{A.1.24}$$

Once again, this patch can only be singular along E_4, which is given locally by $t = 0$. From the equation for the total space of our family on this patch we see that $x = 0$ whenever $t = 0$. Setting both x and t equal to 0 in (A.1.24) gives $b = 0$, a contradiction. Thus (P1-2-2-2) is nonsingular. □

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