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PROJECTIONS AND IDEMPOTENTS WITH FIXED DIAGONAL AND THE HOMOTOPY PROBLEM FOR UNIT TIGHT FRAMES

JULIEN GIOL, LEONID V. KOVALEV, DAVID LARSON, NGA NGUYEN AND JAMES E. TENER

Abstract. We investigate the topological and metric structure of the set of idempotent operators and projections which have prescribed diagonal entries with respect to a fixed orthonormal basis of a Hilbert space. As an application, we settle some cases of conjectures of Larson, Dykema, and Strawn on the connectedness of the set of unit-norm tight frames.

1. Introduction

A finite unit-norm tight frame (FUNTF) is a finite sequence of unit vectors $(x_1, \ldots, x_k)$ in an n-dimensional Hilbert space $\mathcal{H}$ which has the following reproducing property:

$$y = \frac{n}{k} \sum_{j=1}^{k} \langle y, x_j \rangle \, x_j \quad \text{for all } y \in \mathcal{H}. \tag{1.1}$$

When $k = n$, the above defines an orthonormal basis in $\mathcal{H}$. The redundancy inherent in the frames with $k > n$ makes them useful in signal processing, as the original signal may be recovered after a partial loss in transmission. We refer to [1, 3, 5, 6, 7] for background on FUNTF and to [9, 10, 16] for the general theory of frames.

We denote the set of all k-vector unit-norm tight frames in an n-dimensional Hilbert space by $\mathcal{F}^{\mathbb{C}}_{k,n}$ or $\mathcal{F}^{\mathbb{R}}_{k,n}$ depending on the base field. When $k = n$, the topology of these sets is well understood. Indeed, $\mathcal{F}^{\mathbb{C}}_{n,n}$ can be identified with the unitary group $U(n)$ and $\mathcal{F}^{\mathbb{R}}_{n,n}$ with the orthogonal group $O(n)$. In particular, $\mathcal{F}^{\mathbb{C}}_{n,n}$ is pathwise connected while $\mathcal{F}^{\mathbb{R}}_{n,n}$ has two connected components. Much less is known about the topology of frames with redundancy,

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i.e., with $k > n$. The third author conjectured in [15] that $\mathcal{F}^{\mathbb{C}}_{k,n}$ is pathwise connected whenever $k > n \geq 1$, or, equivalently, all k-vector unit-norm tight frames are homotopic. Dykema and Strawn proved in [6] that $\mathcal{F}^{\mathbb{C}}_{k,1}$ is pathwise connected for $k \geq 1$ and $\mathcal{F}^{\mathbb{R}}_{k,2}$ is pathwise connected for $k \geq 4$. They conjectured that $\mathcal{F}^{\mathbb{R}}_{k,n}$ is pathwise connected whenever $k \geq n+2 \geq 4$. They also showed that over either field, the number of connected components remains the same when n is replaced with $k-n$. The latter implies that $\mathcal{F}^{\mathbb{C}}_{k,k-1}$ and $\mathcal{F}^{\mathbb{R}}_{k,k-2}$ are also pathwise connected. The other cases of the conjecture remained open.

The Grammian operator [10] of a FUNTF is a scalar multiple of a projection with constant diagonal, see for instance Corollary 2.6 in [6] or Theorem 3.5 in [3]. Furthermore, $\mathcal{F}_{k,n}$ fibers over the set of projections in $B(\mathbb{C}^k)$ or $B(\mathbb{R}^k)$ with all diagonal entries equal to n/k. The fibers are identified with the orthogonal group, which is connected in the complex case and has two connected components in the real case. Thus, the topological structure of $\mathcal{F}_{k,n}$ is largely determined by the structure of the set of projections with a fixed constant diagonal. The latter set is the subject of our first result. We denote by $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{R})$) the set of all $n \times n$ matrices with complex (resp. real) entries. When the choice of $\mathbb{C}$ or $\mathbb{R}$ is unimportant, we write simply M_n.

Theorem 1.1. *The set of projections in $M_{2n}(\mathbb{C})$ with all diagonal entries equal to $1/2$ is pathwise connected for all $n \geq 1$.*

Theorem 1.1 implies that $\mathcal{F}^{\mathbb{C}}_{2n,n}$ is connected for $n \geq 1$. In the case of real scalars Theorem 1.1 remains true if $n \geq 2$, see Remark 3.2. Therefore, $\mathcal{F}^{\mathbb{R}}_{2n,n}$ has at most two connected components when $n \geq 2$, and its quotient under the natural action of the orthogonal group in $\mathbb{R}^n$ is connected.

We denote by $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{R})$) the set of all $n \times n$ matrices with complex (resp. real) entries. When the choice of $\mathbb{C}$ or $\mathbb{R}$ is unimportant, we write simply M_n. Let $D_n \subset M_n$ be the subalgebra of diagonal matrices. There is a natural linear operator (conditional expectation) $E\colon M_n \to D_n$ which acts by erasing off-diagonal entries. Theorem 1.1 concerns the preimage of $(1/2)\mathrm{id}$ under the restriction of E to projections. It is natural to ask if preimages of other matrices are connected as well. We do not have a complete answer, see however Theorem 4.1. The image of the set of projections under E was the subject of recent papers by Kadison [11, 12].

Theorem 8.1 provides a partial extension of Theorem 1.1 to infinite-dimensional spaces, where the notion of connectedness is understood in the sense of norm topology on the space of bounded operators.

Our second main result is a non-self-adjoint version of Theorem 1.1, which applies to idempotent matrices with an arbitrary fixed diagonal.

Theorem 1.2. *For every d in $D_n(\mathbb{C})$, the set of idempotents q in $M_n(\mathbb{C})$ such that $E(q) = d$ is pathwise connected.*

Naturally, the set of idempotents q such that $E(q) = d$ is empty for some matrices d. Diagonal matrices of the form $E(q)$ are characterized in Theorem 5.1. Our proof of Theorem 1.2 involves several results of independent interest. First, we characterize the diagonals of idempotents with given range in terms of the commutator of the range projection $[q]$ (Theorem 6.3). Specifically, the characterization involves the relative commutator $\{[q]\}' \cap D_n$. Along the way we obtain the following rigidity result (Theorem 6.5): for every $d \in D_n$ there exists $\epsilon > 0$ such that the existence of an idempotent q with $\|E(q) - d\| < \epsilon$ implies the existence of another idempotent q_1 with $E(q_1) = d$. A perturbation argument is used to connect an arbitrary idempotent to an idempotent q such that $\{[q]\}' \cap D_n = \mathbb{C}\,\mathrm{id}$ while preserving the diagonal. Finally, we show that idempotents whose range projection has trivial relative commutant form a path-connected set.

The paper concludes with Section 8, where some of our results are extended to operators in separable infinite-dimensional Hilbert spaces. Whether full analogues of Theorem 1.1 and 1.2 hold in infinite dimensions remains open.

2. Preliminaries

2.1. **Projections as 2×2 matrices.** The content of this section is well-known folklore. It is essentially contained in [8, Theorem 2]. We include this discussion for the convenience of the reader, since it is the basis for our proof of Theorem 1.1.

When a Hilbert space comes as an orthogonal direct sum of two Hilbert spaces, say $\mathcal{H} = K \oplus L$, projections p of $B(\mathcal{H})$ can be identified with those 2×2 matrices

$$p = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$$

where a, b, d are operators in $B(K), B(L, K), B(L)$ respectively, such that

(i) $0 \leq a \leq \mathrm{id}$ and $0 \leq d \leq \mathrm{id}$;
(ii) $|b^*| = \sqrt{a(\mathrm{id} - a)}$ and $|b| = \sqrt{d(\mathrm{id} - d)}$;
(iii) $ab = b(\mathrm{id} - d)$.

Then it is readily seen that $\mathrm{Ker}\ \ b^* = \mathrm{Ker}\ \ bb^* = \mathrm{Ker}\ a(\mathrm{id} - a) = \mathrm{Ker}\ \ (\mathrm{id} - a) \oplus \mathrm{Ker}\ \ a$, hence

$$K = \mathrm{Ker}\ \ (\mathrm{id} - a) \oplus \mathrm{Ker}\ \ a \oplus (\mathrm{Ker}\ \ b^*)^{\perp}.$$

Likewise,

$$L = \mathrm{Ker}\ \ d \oplus \mathrm{Ker}\ \ (\mathrm{id} - d) \oplus (\mathrm{Ker}\ \ b)^{\perp}.$$

According to these two decompositions, we can write

$$a = \mathrm{id} \oplus 0 \oplus a' \ , \quad d = 0 \oplus \mathrm{id} \oplus d' \quad \text{and} \quad b = 0 \oplus 0 \oplus b',$$

where b', for instance, denotes the restriction of b to $(\mathrm{Ker}\ \ b)^{\perp}$ which is injective and whose range is dense in $(\mathrm{Ker}\ \ b^*)^{\perp}$.

There is a unique polar decomposition $b' = u'|b'|$, where u' is an isometry from $(\mathrm{Ker}\ \ b)^\perp$ onto $(\mathrm{Ker}\ \ b^*)^\perp$. Note that (ii) and (iii) above entail $|b'| = \sqrt{d'(\mathrm{id} - d')}$ and $a'b' = b'(\mathrm{id} - d')$. Hence $|b'|$ commutes with d' and $a'u'|b'| = u'(\mathrm{id} - d')|b'|$. The range of $|b'|$ being dense in $(\mathrm{Ker}\ \ b)^\perp$, it follows that

$$a'u' = u'(\mathrm{id} - d').$$

Thus the positive injective contractions a' and $\mathrm{id} - d'$ are unitarily equivalent and the same statement holds for $\mathrm{id} - a'$ and d'.

2.2. **Diagonal conditional expectation and minimal block decomposition.** Let $\mathcal{H}$ be a separable Hilbert space and let us fix an orthonormal basis. Let $\{e_i\}_{i\in I}$ denote the corresponding set of rank one projections. An element x of $B(\mathcal{H})$, i.e. a bounded linear operator on $\mathcal{H}$, can be identified with its matrix with respect to this basis. It is then called diagonal if all of its off-diagonal entries are equal to zero, i.e $e_i x e_j = 0$ whenever $i \neq j$. The set D made of these diagonal elements is a maximal abelian self-adjoint algebra in $B(\mathcal{H})$ (it is equal to its commutant). It comes with the so-called *diagonal conditional expectation*

$$E : B(\mathcal{H}) \to D$$

defined as the idempotent map which erases the off-diagonal entries.

Let x in $B(\mathcal{H})$ be fixed and denote $\overset{x}{\sim}$ the smallest equivalence relation on I such that $i \overset{x}{\sim} j$ whenever $e_i x e_j \neq 0$. Summing the projections e_i over each equivalence class, we obtain an orthogonal decomposition of the unit $\{f_j\}_{j\in J}$ within D. We call

$$x = \sum_{j\in J} x f_j$$

the *minimal block decomposition* of x. By construction, the projections f_j commute with x. More precisely, these are the minimal projections of the commutative von Neumann algebra $\{x\}' \cap D$. Note

$$\big(\{x\}' \cap D\big)' \simeq \prod_{j\in J} f_j B(\mathcal{H}) f_j,$$

which justifies the terminology.

Our strategy for the proof of Theorem 1.2 consists in restricting ourselves to idempotents q which share the same diagonal $E(q) = d$ and the property that $\{q\}' \cap D = \mathbb{C}\mathrm{id}$ or, equivalently, $(\{x\}' \cap D)' = B(\mathcal{H})$.

3. Projections with diagonal 1/2

3.1. **Proof of Theorem 1.1.**

Proof. Let p in $M_{2n}(\mathbb{C})$ be a projection such that $E(p) = \mathrm{id}/2$ and write p as a 2×2 matrix

$$p = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \tag{3.1}$$

with coefficients a, b, d in M_n. We will now use implicitly the preliminary remarks of 2.1.

By assumption on the diagonal, we have $\mathrm{Tr}\ a = \mathrm{Tr}\ (\mathrm{id} - d) = n/2$. Since the restriction of a to $(\mathrm{Ker}\ \ b^*)^\perp$ and that of $\mathrm{id} - d$ to $(\mathrm{Ker}\ \ b)^\perp$ are unitarily equivalent, they have equal trace and it follows that $\dim \mathrm{Ker}\ (\mathrm{id} - a) = \dim \mathrm{Ker}\ d$. Considering $\mathrm{id} - p$ instead, the same argument shows that $\dim \mathrm{Ker}\ a = \dim \mathrm{Ker}\ (\mathrm{id} - d)$. In particular, we see that the subspaces $\mathrm{Ker}\ b$ and $\mathrm{Ker}\ b^*$ have the same dimension, hence we can extend the unitary u' to a unitary u in M_n such that

$$p = \begin{pmatrix} a & \sqrt{a(\mathrm{id} - a)}u \\ u^*\sqrt{a(\mathrm{id} - a)} & u^*(\mathrm{id} - a)u \end{pmatrix}.$$

Now if we put $a_t := (\mathrm{id} - t)a + (t/2)\mathrm{id}$ in $B(K)$, it is easily seen that the formula

$$p_t := \begin{pmatrix} a_t & \sqrt{a_t(\mathrm{id} - a_t)}u \\ u^*\sqrt{a_t(\mathrm{id} - a_t)} & u^*(\mathrm{id} - a_t)u \end{pmatrix}.$$

defines a projection-valued path connecting $p_0 = p$ and

$$p_1 = \begin{pmatrix} \mathrm{id}/2 & u/2 \\ u^*/2 & \mathrm{id}/2 \end{pmatrix},$$

and such that $E(p_t) = 1/2$ for all t. The main point is the latter assertion, which readily follows from the linearity of E and the identities

$$a_t = (\mathrm{id} - t)a + t(1/2)\mathrm{id}\ ,$$

$$u^*(\mathrm{id} - a_t)u = (\mathrm{id} - t)u^*(\mathrm{id} - a)u + (t/2)\mathrm{id}.$$

Finally, by connectedness of the unitary group in $M_n(\mathbb{C})$, every projection p in $M_{2n}(\mathbb{C})$ with diagonal $\mathrm{id}/2$ can be connected to

$$q = \begin{pmatrix} \mathrm{id}/2 & \mathrm{id}/2 \\ \mathrm{id}/2 & \mathrm{id}/2 \end{pmatrix}. \qquad \square$$

Remark 3.1. Most of the proof of Theorem 1.1 carries over to a separable Hilbert space $\mathcal{H}$ over real or complex scalars. Indeed, we used the assumption that the space is finite-dimensional only to prove that the subspaces $\mathrm{Ker}\ b$ and $\mathrm{Ker}\ b^*$ are of the same dimension. (The unitary group in $B(\mathcal{H})$ is known to be path-connected and even contractible [13].) Thus we have the following result: if $\mathcal{H}$ is decomposed into a direct sum $K \oplus L$, then the set of all projections of form (3.1) with $E(p) = \mathrm{id}/2$ and $\dim \mathrm{Ker}\ b = \dim \mathrm{Ker}\ b^*$ is path-connected.

Remark 3.2. The set of projections with diagonal $\mathrm{id}/2$ in $M_2(\mathbb{R})$ is not connected, since it consists of just two elements

$$\begin{pmatrix} 1/2 & \pm 1/2 \\ \pm 1/2 & 1/2 \end{pmatrix}$$

However, this set is path-connected in $M_{2n}(\mathbb{R})$ for all $n > 1$. Indeed, the unitary group splits into two components: special unitary group and its complement. If the block b in (3.1) is not invertible, then in the proof of Theorem 1.1 we can choose the unitary u to have determinant 1 or -1. Thus, the existence of a projection p with $E(p) = \mathrm{id}/2$ and noninvertible b implies the connectedness of the set. Such a projection can be easily constructed by including the 4×4 block

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

3.2. **Explicit parametrization in 4 dimensions.** We observe that there are three sets of projections in $M_4(\mathbb{C})$ with diagonal $\mathrm{id}/2$. First, those whose all entries are non-zero can be parametrized by

$$p = \begin{pmatrix} 1/2 & t_1\bar{\xi_1} & t_2\bar{\xi_2} & t_3\bar{\xi_3} \\ t_1\xi_1 & 1/2 & \mp it_3\xi_1\bar{\xi_2} & \pm it_2\xi_1\bar{\xi_3} \\ t_2\xi_2 & \pm it_3\bar{\xi_1}\xi_2 & 1/2 & \mp it_1\xi_2\bar{\xi_3} \\ t_3\xi_3 & \mp it_2\bar{\xi_1}\xi_3 & \pm it_1\bar{\xi_2}\xi_3 & 1/2 \end{pmatrix}$$

with $\sqrt{t_1^2 + t_2^2 + t_3^2} = 1/2$, $t_j > 0$, and ξ_j in $\mathbb{T}$. Then come those with exactly four null entries:

$$p = \begin{pmatrix} 1/2 & t_1\bar{\xi_1} & t_2\bar{\xi_2} & 0 \\ t_1\xi_1 & 1/2 & 0 & t_2\bar{\xi_3} \\ t_2\xi_2 & 0 & 1/2 & -t_1\bar{\xi_1}\xi_2\bar{\xi_3} \\ 0 & t_2\xi_3 & -t_1\xi_1\bar{\xi_2}\xi_3 & 1/2 \end{pmatrix}$$

with $\sqrt{t_1^2 + t_2^2} = 1/2$, $t_j > 0$, ξ_j in $\mathbb{T}$, and the two other families obtained by permutation of the basis. Finally, here are those which have eight null entries:

$$p = \begin{pmatrix} 1/2 & \bar{\xi_1}/2 & 0 & 0 \\ \xi_1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & \bar{\xi_2}/2 \\ 0 & 0 & \xi_2/2 & 1/2 \end{pmatrix}$$

with ξ_j in $\mathbb{T}$, and the two other families obtained by permutation of the basis.

It follows readily that the set of diagonal $1/2$ projections is pathwise connected in $M_4(\mathbb{C})$, giving us an explicit, parametric proof of Theorem 1.1 in that case.

In the real case, the latter set restricts to three sets of four projections. Also, there are no 4×4 diagonal $1/2$ projections whose entries are all non-zero real numbers. And those with four null entries split into twenty-four

paths which connect the twelve extreme projections. For instance, for every $\epsilon_1, \epsilon_2, \epsilon_5, \epsilon_6$ in $\{\pm 1\}$ such that $\epsilon_1\epsilon_2\epsilon_5\epsilon_6 = -1$, the extreme projections

$$p = \begin{pmatrix} 1/2 & \epsilon_1/2 & 0 & 0 \\ \epsilon_1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & \epsilon_6/2 \\ 0 & 0 & \epsilon_6/2 & 1/2 \end{pmatrix}$$

and

$$q = \begin{pmatrix} 1/2 & 0 & \epsilon_2/2 & 0 \\ 0 & 1/2 & 0 & \epsilon_5/2 \\ \epsilon_2/2 & 0 & 1/2 & 0 \\ 0 & \epsilon_5/2 & 0 & 1/2 \end{pmatrix}$$

can be connected by the path

$$\begin{pmatrix} 1/2 & \cos\theta\epsilon_1/2 & \sin\theta\epsilon_2/2 & 0 \\ \cos\theta\epsilon_1/2 & 1/2 & 0 & \sin\theta\epsilon_5/2 \\ \sin\theta\epsilon_2/2 & 0 & 1/2 & \cos\theta\epsilon_6/2 \\ 0 & \sin\theta\epsilon_5/2 & \cos\theta\epsilon_6/2 & 1/2 \end{pmatrix}$$

with θ running from 0 to $\pi/2$.

We let the reader check that any two extreme projections can be connected by at most three paths of this type. In particular, diagonal 1/2 projections in $M_4(\mathbb{R})$ form a pathwise connected set.

4. Further connectedness results for projections with fixed diagonal

4.1. **Amplification of the 2×2 case.** Here is a generalization of Theorem 1.1. The proof is basically the same, so we only insist on the points that differ.

Theorem 4.1. *For every d in D_{2n} of the type $d = \cos^2\theta e + \sin^2\theta e^\perp$ with a rank n projection e in D_{2n} and θ in $[0, \pi/2]$, the set of projections p in $M_{2n}(\mathbb{C})$ such that $E(p) = d$ is pathwise connected.*

Proof. Up to a permutation, we can assume that e is the projection onto the span of the first n vectors of the canonical basis. Now let p be a projection in M_{2n}, written as a 2×2 matrix over M_n like in the previous section. Since $\mathrm{Tr}\ a = \mathrm{Tr}\ (\mathrm{id} - d) = n\cos^2\theta$ and $\mathrm{Tr}\ (\mathrm{id} - a) = \mathrm{Tr}\ d = n\sin^2\theta$, there exists a unitary u in M_n such that

$$p = \begin{pmatrix} a & \sqrt{a(\mathrm{id}-a)}u \\ u^*\sqrt{a(\mathrm{id}-a)} & u^*(\mathrm{id}-a)u \end{pmatrix}.$$

Then we put $a_t := (\mathrm{id} - t)a + t\cos^2\theta 1$ and it simply remains to mimick the rest of the proof of Theorem 1.1. □

5. Diagonals of idempotents

An idempotent is an operator which is equal to its square. For d in D_n to be the diagonal of an idempotent q in M_n, it is necessary that $\mathrm{Tr}\, d = \mathrm{rank}\, q$ belongs to the set of integers $\{0, 1, \ldots, n\}$. Now is this sufficient? The cases $\mathrm{Tr}\ d = 0$ and $\mathrm{Tr}\ d = n$ have to be treated separately. Since 0 and id are the only idempotents with rank 0 and n, respectively, it turns out that 0 and 1 are the only possible diagonals of idempotents with trace 0 and n, respectively. The remainder of this section is devoted to proving that for every d in D_n with $\mathrm{Tr}\ d$ in $\{1, \ldots, n-1\}$ there exists an idempotent q in M_n such that $E(q) = d$.

The case $\mathrm{Tr}\ d = 1$ is very easy. Let $\{d_1, \ldots, d_n\}$ denote the set of values on the diagonal of d so that $\sum_{j=1}^{n} d_j = 1$. Then consider for instance the matrix q in M_n which is defined by its entries $q_{i,j} := d_i$. It is readily seen that q is idempotent and that $E(q) = d$.

We proceed by induction on k.

Assume it has been proven that for all $n \geq k$ and for all d in D_n with $\mathrm{Tr}\ d = k-1$ there exists an idempotent q in M_n such that $E(q) = d$. We now take $n \geq k+1$ and d in D_n with $\mathrm{Tr}\ d = k$. Let $\{d_1, \ldots, d_n\}$ denote the set of values on the diagonal of d.

If $d_{j_0} = 1$ for some j_0, then $\sum_{j \neq j_0} d_j = k-1$ and the induction hypothesis, together with an obvious splicing argument, help us find q.

Since $d \neq \mathrm{id}$, there exist at least two indices i, j such that $d_i + d_j \neq 2$ (otherwise, we find that $d_j = 1$ for all j). Without loss of generality, we can assume that $d_1 + d_2 \neq 2$ and we put $\lambda := (d_2 - 1)/(d_1 + d_2 - 2)$. Since $(d_1 + d_2 - 1) + d_3 + \cdots + d_n = k-1$, we can find by assumption an idempotent r in M_{n-1} such that $E(r)$ has diagonal values $\{d_1 + d_2 - 1, d_3, \ldots, d_n\}$. Now consider the idempotent

$$\widetilde{q} = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_1 + d_2 - 1 & 0 \\ 0 & 0 & * \end{pmatrix}$$

and the invertible element

$$\sigma = \begin{pmatrix} \lambda & \lambda - 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}$$

in M_n. Then a straightforward computation shows that the idempotent $q := \sigma \widetilde{q} \sigma^{-1}$ has diagonal d.

Thus we have proved:

Theorem 5.1. *Let d be in D_n. Then d is the diagonal of an idempotent in M_n if and only if one of the following holds:*

(i) $d = 0$;
(ii) $d = \mathrm{id}$;
(iii) *Tr d belongs to* $\{1, \ldots, n-1\}$.

6. Idempotents with prescribed range and diagonal

Throughout this section, we will work with the conditional expectation E from M_n onto D_n, the set of diagonal $n \times n$ matrices. Given an idempotent p, recall that the set of idempotents which have the same range as p is equal to the affine subset $p + pM_np^\perp$, where $p^\perp = \mathrm{id} - p$. We will now investigate the intersections of the latter with the preimages $E^{-1}(d)$.

First we determine the range of the linear operator $x \longmapsto E(pxp^\perp)$. The following lemma should be compared to Lemma 4.2 in [6], which concerns the rank of the differential of the conditional expectation on the Grassmannian manifold.

Lemma 6.1. *Let p be a projection in M_n with minimal block decomposition $p = \sum_{j=1}^s pf_j$. Then*

$$E(pM_np^\perp) = \{d \in D_n \ : \ Tr\, df_j = 0,\ 1 \leq j \leq s\}.$$

Remark 6.2. If we give $M_n(\mathbb{C})$ its Hilbert-Schmidt (or Euclidean) structure via the inner product Tr a^*b, then the previous lemma can be restated by saying that $E(pM_np^\perp)$ is equal to the orthogonal complement of $\{p\}' \cap D_n$ in D_n.

Proof. Let d belong to $E(pM_np^\perp)$, say $d = E(pxp^\perp)$. Then for all $j = 1 \ldots, s$ we have Tr $df_j =$ Tr $pxp^\perp =$ Tr $p^\perp f_j px$ by commutativity of the trace, hence Tr $df_j = 0$ since $p^\perp f_j p = p^\perp pf_j = 0$. Thus $E(pM_np^\perp)$ is contained in D_n and is orthogonal to the span of the f_j's, namely $\{p\}' \cap D_n$.
Now let d in D_n be orthogonal to $E(pM_np^\perp)$. This means that for all x in M_n, we have Tr $(pxp^\perp)^*d =$ Tr $x^*pdp^\perp = 0$. Hence $pdp^\perp = 0$, i.e. the range of p is invariant under d. By Lagrange interpolation, we can find a polynomial g such that $d^* = g(d)$. Thus $pd^*p^\perp = pg(d)p^\perp = 0$. It follows that d commutes with p, hence d belongs to $\{p\}' \cap D_n$ and the proof is complete. □

Given a projection p, we characterize the diagonals which can be realized as the diagonal of an idempotent with the same range as p.

Theorem 6.3. *Let p be a projection in M_n with minimal block decomposition $p = \sum_{j=1}^s pf_j$. For every diagonal matrix d in D_n, the following assertions are equivalent:*

(i) d belongs to $E(p + pM_np^\perp)$;
(ii) $Tr\, df_j = rank\, pf_j$ for $j = 1, \ldots, s$.

Proof. The first assertion says that $d - E(p)$ belongs to $E(pM_np^\perp)$. By Lemma 6.1, this is equivalent to the fact that Tr $(d - E(p))f_j = 0$ for $j = 1 \ldots, s$. And since Tr $E(p)f_j =$ Tr $pf_j =$ rank pf_j, we get the equivalence with the second assertion. □

The case of diagonal id/2 being our original motivation, let us restate the previous result in this particular situation.

Corollary 6.4. *Let p be a projection in M_{2n} with minimal block decomposition $p = \sum_{j=1}^{s} pf_j$. Then there is a diagonal* $\mathrm{id}/2$ *idempotent in M_{2n} with range equal to that of p if and only if rank $f_j = 2$rank pf_j for $j = 1, \ldots, s$.*

Now that we have characterized the diagonals that belong to $E(p + pM_np^{\perp})$, we will give a uniform lower estimate of the distance between a diagonal and the closed affine subspaces $E(p + pM_np^{\perp})$ of M_n which do not contain it.

Theorem 6.5. *Let d be a diagonal in D_n. Let S be the set of all possible sums of diagonal elements of d, i.e. the set of all Tr (de) when e runs over all diagonal projections. Put $\gamma := 1$ if S is contained in $\mathbb{Z}$ and $\gamma :=$ dist$(S \setminus \mathbb{Z}, \mathbb{Z})$ otherwise. Then for all projections p in M_{2n}, we have the following alternative: either d belongs to $E(p + pM_np^{\perp})$ or*

$$\mathit{dist}(d, E(p + pM_np^{\perp})) \geq \frac{\gamma}{\lfloor n/2 \rfloor}.$$

Proof. Suppose that d does not belong to $E(p + pM_np^{\perp})$ and let $p = \sum_{j=1}^{s} pf_j$ be the minimal block decomposition of p. By Theorem 6.3, there is one j such that Tr $df_j \neq$ rank pf_j. □

Again, we find it worth restating the result above in the special case of diagonal $\mathrm{id}/2$, in a slightly different form.

Corollary 6.6. *Let q be an idempotent in M_{2n}. If $\|E(q) - 1/2\| < \frac{1}{n}$, then there exists an idempotent $\widetilde{q}$ with diagonal* $\mathrm{id}/2$ *and with range equal to that of q.*

Proof. In this case, the constant γ is equal to $1/2$. Let p be the range projection of q. Since $\mathrm{dist}(d, E(p + pM_np^{\perp})) < \frac{1}{n}$, Theorem 6.5 implies that $1/2$ is actually the diagonal of an idempotent $\widetilde{q}$ in $E(p + pM_np^{\perp})$. □

7. Connectedness of idempotents with fixed diagonal

This section is devoted to the proof of Theorem 1.2. Like in the previous section, we work with $n \times n$ matrices and the diagonal conditional expectation $E\colon M_n \to D_n$. But this time, we need to assume that matrices are taken over the complex field (this assumption is used in Lemma 7.3 only).

Given an idempotent q, it will prove convenient to denote $[q]$ its range projection which is given, for instance, by the formula $[q] = q(q + q^* - \mathrm{id})^{-1}$.

The key idea in our strategy is to reduce to the case of idempotents for which the algebra $\{[q]\}' \cap D_n$ is trivial. We begin with a simple observation.

Remark 7.1. The commutative finite-dimensional algebra $\{[q]\}' \cap D_n$ is the span of all diagonal projections which leave the range of q invariant, i.e. those diagonal projections e such that $e[q] = [q]e$ or, equivalently, $q^{\perp}eq = 0$.

We now proceed to the construction that will allow us to implement the reduction claimed above.

Proposition 7.2. *Let q be a nontrivial idempotent in M_n. If dim $\{[q]\}' \cap D_n > 1$, then there exists an idempotent r in M_n such that*

(i) $E(r) = E(q)$;
(ii) $\{[r]\}' \cap D_n \subsetneq \{[q]\}' \cap D_n$;
(iii) there is a piecewise affine path consisting of at most two steps from q to r within the set of idempotents with diagonal constant equal to $E(q)$.

Proof. According to Remark 7.1, the algebra $\{[q]\}' \cap D_n$ is spanned by the diagonal projections e such that $eq = qeq$. By assumption, we can find one such e that is non trivial. We will construct an idempotent r such that $\{[r]\}' \cap D_n \subsetneq \{[q]\}' \cap D_n$, the inclusion being proper because we will arrange for e not to be in $\{[r]\}' \cap D_n$.
The first step is to connect q to an idempotent $\widetilde{q}$ which commutes with e and has same range and diagonal as q. Note that this leaves the algebra $\{[q]\}' \cap D_n = \{[\widetilde{q}]\}' \cap D_n$ unchanged and that one passes from q to $\widetilde{q}$ by a straight line segment. To do this, we set $\widetilde{q} := q - x$ with $x = eqe^{\perp} + e^{\perp}qe$. Since e is a diagonal projection, it is clear that $E(x) = 0$ so that $E(\widetilde{q}) = E(q)$. Using the identity $eq = qeq$, we first check that $\widetilde{q}e = e\widetilde{q} = eqe$. Then we verify that $qx = x$ and $xq = 0$, so that $\widetilde{q}q = q$ and $q\widetilde{q} = \widetilde{q}$, which is the algebraic condition for the idempotents q and $\widetilde{q}$ to have the same range.
Since q is assumed to be non trivial, so is $\widetilde{q}$, i.e. $\widetilde{q} \neq 0$ and $\widetilde{q}^{\perp} \neq 0$. Also, we took e non trivial, i.e. $e \neq 0$ and $e^{\perp} \neq 0$. Now if $\widetilde{q}e = 0$, we have $\widetilde{q}e^{\perp} = \widetilde{q}$ and $\widetilde{q}^{\perp}e = e$. Likewise, if $\widetilde{q}^{\perp}e^{\perp} = 0$, we find that $\widetilde{q}e^{\perp} = e^{\perp}$ and $\widetilde{q}^{\perp}e = \widetilde{q}^{\perp}$. As a consequence, up to replacing e by $e^{\perp}$, we can further assume that $\widetilde{q}e \neq 0$ and $\widetilde{q}^{\perp}e^{\perp} \neq 0$, so that $\widetilde{q}^{\perp}e^{\perp}M_n\widetilde{q}e \neq \{0\}$. We pick now an element $y \neq 0$ in the latter. Note that $y = e^{\perp}ye = \widetilde{q}^{\perp}y\widetilde{q}$.
For the second step, we will exhibit an idempotent r with same nullspace and diagonal as $\widetilde{q}$, and such that $\{[r]\}' \cap D_n \subsetneq \{[\widetilde{q}]\}' \cap D_n$. To this aim, we consider the parametrized family of idempotents given by $r_t := \widetilde{q} + ty$. Since $y = e^{\perp}ye$, we have $E(y) = 0$ hence $E(r_t) = E(\widetilde{q}) = d$ for all t. Since $y = \widetilde{q}^{\perp}y\widetilde{q}$, we see that $\widetilde{q}y = y$ and $y\widetilde{q} = 0$, hence r_t is an idempotent with the same nullspace as $\widetilde{q}$ for all t. Also, for all $t \neq 0$, we observe that e does not belong to $\{[r]_t\}' \cap D_n$, since $r_t^{\perp}er_t = -ty \neq 0$. So it only remains to find a value of $t \neq 0$ for which $\{[r_t]\}' \cap D_n \subset \{[q]\}' \cap D_n$ and we will suffice to take the corresponding r_t for the desired r. Actually, we will show that all but finitely many values of t will do.
Let f be a diagonal projection and consider the map $g : t \longmapsto r_t^{\perp}fr_t$. Since each matrix coefficient is a polynomial of degree not greater than 2, g is either constant equal to zero or vanishes for at most two distinct values of t. So if f does not belong to $\{[q]\}' \cap D_n$ or, in other terms, if $g(0) \neq 0$, we see that f belongs to $\{[r_t]\}' \cap D_n$ for two values of t at most. Because there are only finitely many diagonal projections, we deduce that for all but finitely many values of t, the projections that lie in $\{[r_t]\}' \cap D_n$ also belong to $\{[q]\}' \cap D_n$, hence, in view of Remark 7.1, $\{[r_t]\}' \cap D_n \subset \{[q]\}' \cap D_n$. $\square$

Lemma 7.3. *Let us fix $0 < k < n$ and let $G(k,n)$ denote the set of projections with rank k in $M_n(\mathbb{C})$. The set $\Omega := \{p \in G(k,n) \ : \ \{p\}' \cap D_n = \mathbb{C}\mathrm{id}\}$ is an open, dense, pathwise connected subset of $G(k,n)$.*

Proof. As is well-known, the Grassmannian $G(k,n)$ is a connected complex manifold of dimension $k(n-k)$. Now let us take p in $G(k,n)$ and observe that p belongs to Ω if and only if there is no nontrivial diagonal projection which commutes with p. Hence $G(k,n) \setminus \Omega$ is equal to the union, over all nontrivial projections e in D_n, of the subsets $\{p \in G(k,n) \ : \ ep = pe\}$. The latter can in turn be decomposed into the disjoint union of the subsets $F_l = \{p \in G(k,n) \ : \ ep = pe, \mathrm{rank} ep = l\}$, l running from 0 to k. Each set F_l can be identified with $G(l, \mathrm{rank}\ e) \times G(k-l, n - \mathrm{rank}\ e)$, which is a complex manifold of dimension not greater than $k(n-k)-1$.
This shows in particular that $G(k,n) \setminus \Omega$ is closed and has empty interior. Being a proper analytic subset of the connected complex manifold $G(k,n)$, this set has pathwise connected complement (cf. Proposition 3 of Section 2.2 in [4]). □

Thanks to this result and to the Lemma 6.1 of the previous section, we will now prove that the sets $\Omega \cap E^{-1}(d)$ are either connected or empty.

Theorem 7.4. *For every diagonal d in D_n, the set of idempotents q such that $E(q) = d$ and $\{[q]\}' \cap D_n = \mathbb{C}\,\mathrm{id}$ is pathwise connected whenever it is not empty.*

Proof. Let q and r be two idempotents in the set under consideration, if not empty. By Lemma 7.3 with $k = \mathrm{rank}\ q$, we can find a projection-valued path p_t connecting $[q]$ and $[r]$ within Ω, i.e. such that $\{p_t\}' \cap D_n = \mathbb{C}\,\mathrm{id}$ for all t. Then it follows from Lemma 6.1 that for all t, $E(p_t M_n p_t^\perp)$ is equal to $D_n \cap \mathrm{Ker\ Tr}$, the orthogonal complement of $\mathbb{C}\,\mathrm{id}$ in D_n with respect to the Hilbert-Schmidt inner product. Hence each operator

$$D_t \colon M_n \to D_n = D_n \cap \mathrm{Ker\ Tr} \ \oplus \mathbb{C}\,\mathrm{id}\ , \quad x \longmapsto E(p_t x p_t^\perp)$$

is such that $D_t D_t^*$ realizes an isomorphism from $D_n \cap \mathrm{Ker\ Tr}$ onto itself. Thus

$$C_t := D_t^*(D_t D_t^*)^{-1} : D_n \cap \mathrm{Ker\ Tr} \to M_n$$

defines a continuous path of right inverses for D_t, seen as operators from M_n to $D_n \cap \mathrm{Ker\ Tr}$.
Now consider the path $x_t := C_t(d - E(p_t))$ in M_n, which is continuous and satisfies $E(p_t x_t p_t^\perp) = d - E(p_t)$ for all t. Setting $q_t := p_t + p_t x_t p_t^\perp$, we obtain an idempotent-valued path within the desired set, from $q_0 = [q]$ to $q_1 = [r]$. Since E is linear, it only remains to connect the latter to q and r respectively by straight line segments and we are done. □

Proof of Theorem 1.2. Let q and r be idempotents with diagonal d. By Proposition 7.2, we can connect them, within a finite number of affine steps in the set of idempotents with diagonal d, to two idempotents, $\widetilde{q}$ and $\widetilde{r}$

respectively, such that, moreover, $\{[q]\}' \cap D_n = \{[r]\}' \cap D_n = \mathbb{C}\,\mathrm{id}$. The latter pair can now be connected by Theorem 7.4 and the proof is complete. $\square$

8. Extensions to infinite dimensions

In this section we extend some of the preceding results to operators on a separable Hilbert space $\mathcal{H}$ over complex or real scalars. Recall that by Theorem 1.1 the set of projections with $1/2$ on the diagonal is path-connected in $M_{2n}(\mathbb{C})$ or in $M_{2n}(\mathbb{R})$ for $n > 1$. Theorem 8.1 is a partial extension of this result to $B(\mathcal{H})$ equipped with the operator norm topology. To state it we need the following definition: an operator $x \in B(\mathcal{H})$ with $\|x\| = 1$ is 2-*pavable* if there exists a diagonal projection e such that $\|exe\| < 1$ and $\|e^\perp x e^\perp\| < 1$. Note that for any projection p the operator $2p - \mathrm{id}$ has norm 1; in fact, it is a symmetry (i.e., self-adjoint unitary). See [2] for recent results on paving of projections.

Theorem 8.1. *The projections $p \in B(\mathcal{H})$ such that $E(p) = \mathrm{id}/2$ and $2p - \mathrm{id}$ is pavable are pathwise connected within the set of all projections with diagonal $\mathrm{id}/2$.*

Proof. Let f be a diagonal projection with infinite rank and nullity. Given a projection p as in the statement, we must find a path (within the set of projections with diagonal $\mathrm{id}/2$) from p to the block matrix

$$p_0 = \begin{pmatrix} \mathrm{id}/2 & \mathrm{id}/2 \\ \mathrm{id}/2 & \mathrm{id}/2 \end{pmatrix}$$

in which the blocks correspond to $\operatorname{ran} f$ and $\operatorname{ran} f^\perp$. Let e be a projection that paves $2p - \mathrm{id}$. Note that e has infinite rank and nullity. Replacing e with $e^\perp$ if necessary, we can ensure that both ef and $e^\perp f^\perp$ have infinite rank. Let $\{e_i\}_{i\in\mathbb{N}}$ be the standard basis of $\mathcal{H}$. Let $\sigma \in B(\mathcal{H})$ be a zero-diagonal involution that acts by permuting the basis elements so that (i) $\{e_i, \sigma e_i\} \subset \operatorname{ran}(ef)$ for infinitely many values of $i \in \mathbb{N}$ and (ii) $\{e_i, \sigma e_i\} \subset \operatorname{ran}(e^\perp f^\perp)$ for infinitely many values of $i \in \mathbb{N}$. Let us write the projection $p_1 := (\mathrm{id} + \sigma)/2$ in the block form

$$p_1 = \begin{pmatrix} a_1 & b_1 \\ b_1^* & d_1 \end{pmatrix} \tag{8.1}$$

with respect to the decomposition $\mathcal{H} = \operatorname{ran} f \oplus \operatorname{ran} f^\perp$. The block b_1 has infinite-dimensional kernel which contains all vectors $e_i + \sigma e_i$ such that $\{e_i, \sigma e_i\} \subset \operatorname{ran} f^\perp$. Similarly, the kernel of b_1^* contains all vectors $e_i + \sigma e_i$ such that $\{e_i, \sigma e_i\} \subset \operatorname{ran} f$. By Remark 3.1 the projection p_1 can be connected by an appropriate path to p_0.

Now let p_1 be represented as in (8.1) but with respect to decomposition $\mathcal{H} = \operatorname{ran} e \oplus \operatorname{ran} e^\perp$. Replacing f with e in the preceding paragraph, we again find that b_1 and b_1^* have infinite dimensional kernels. Writing p in

block form with the same decomposition $\mathcal{H} = \operatorname{ran} e \oplus \operatorname{ran} e^{\perp}$, we obtain

$$p = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$$

with $\|2a - \mathrm{id}\| < 1$ and $\|2d - \mathrm{id}\| < 1$. It follows that both $a(\mathrm{id} - a)$ and $d(\mathrm{id} - d)$ are invertible. As was noted in section 2.1, this implies the invertibility of b (and b^*). By Remark 3.1 the projection p can be connected by an appropriate path to p_1, and we are done. □

Recall that $B(\mathcal{H})$ is the dual of S_1, the set of trace-class operators on $\mathcal{H}$. This duality induces w^*-topology on $B(\mathcal{H})$. Recall the definition of the minimal block decomposition of an operator from section 2.2.

Theorem 8.2. *Given a projection $p \in B(\mathcal{H})$, let $\sum_{j\in J} pf_j$ be its minimal block decomposition. The w^*-closure of the set*

$$\{E(q)\colon q^2 = q,\ \operatorname{ran} q = \operatorname{ran} p\}$$

consists of all operators $d \in D(\mathcal{H})$ such that $\operatorname{tr}(df_j) = \operatorname{rk}(pf_j)$ whenever f_j has finite rank.

Since the idempotents q in Theorem 8.2 are all of the form $p + pxp^{\perp}$, the conclusion of Theorem 8.2 can be deduced from the following lemma.

Lemma 8.3. *Let $p \in B(\mathcal{H})$ be a projection. Define $\mathcal{D}_p\colon B(\mathcal{H}) \to B(\mathcal{H})$ by $\mathcal{D}_p(x) = E(pxp^{\perp})$. The w^*-closure of $\operatorname{ran}\mathcal{D}_p$ is the space*

$$\{d \in D(\mathcal{H})\colon\ \operatorname{tr}(dc) = 0\ \forall c \in \{p\}'_D \cap S_1\}. \tag{8.2}$$

Proof. Let N be the space in (8.2). First we prove that $\operatorname{ran}\mathcal{D}_p \subseteq N$. If $d = \mathcal{D}_p(x)$ for some $x \in B(\mathcal{H})$, then for each $c \in \{p\}'_D \cap S_1$ we have

$$\operatorname{tr}(dc) = \operatorname{tr}(pxp^{\perp}c) = \operatorname{tr}(p^{\perp}cpx) = \operatorname{tr}(p^{\perp}pcx) = 0$$

which means that $d \in N$.

Next, suppose that $c \in D(\mathcal{H}) \cap S_1$ annihilates $\operatorname{ran}\mathcal{D}_p$. This means that for any $x \in B(\mathcal{H})$ we have $\operatorname{tr}(pxp^{\perp}c) = 0$. Since

$$\operatorname{tr}(pxp^{\perp}c) = \operatorname{tr}(p^{\perp}cpx), \quad \forall x \in B(\mathcal{H}),$$

it follows that

$$p^{\perp}cp = 0, \tag{8.3}$$

i.e., $\operatorname{ran} p$ is invariant under c. Using continuous functional calculus, we can write $c^* = F(c)$, where $F(z) = \bar{z}$ on the spectrum $\sigma(c)$. Note that $\sigma(c)$ has empty interior and connected complement. By Mergelyan's theorem there exists a sequence of polynomials P_n such that $P_n \to F$ uniformly on $\sigma(c)$. It follows from (8.3) that $p^{\perp}P_n(c)p = 0$ for all n. Letting $n \to \infty$, we obtain $p^{\perp}c^*p = 0$. Taking adjoints, we find that $pc = pcp$. Since $pcp = cp$ by (8.3), c and p commute. Thus c annihilates N. □

Remark 8.4. In general, $\operatorname{ran}\mathcal{D}_p$ is not w^*-closed. Indeed, if p or $p^\perp$ has finite rank, then $\operatorname{ran}\mathcal{D}_p$ is contained in S_1, although it may be w^*-dense in $D(\mathcal{H})$. Therefore, Theorem 8.2 does not completely describe the possible diagonals of idempotents with a given range. The difficulty of obtaining such a description can be illustrated by the following fact: there exists a nonzero idempotent with zero diagonal [14, Theorem 3.7].

Concerning the connectedness of idempotents sharing the same diagonals, we have the following result which generalizes Proposition 7.2. Recall that $\{[r]\}'_D$ means the range of the diagonal expectation E restricted to the commutant of the range projection of r, i.e $\{[r]\}'_D = \{[r]\}' \cap D(\mathcal{H})$.

Proposition 8.5. *For any idempotent $q \in B(\mathcal{H}) \setminus \{0,1\}$ there exists an idempotent $r \in B(\mathcal{H})$ such that $\{[r]\}'_D = \mathbb{C}\,\mathrm{id}$ and there is a piecewise linear path from q to r within the set of idempotents with diagonal $E(q)$.*

The proof is preceded by two lemmas.

Lemma 8.6. *For any idempotent $q \in B(\mathcal{H})$ there exists an idempotent $\tilde{q} \in (\{[q]\}'_D)'$ such that $\operatorname{ran}\tilde{q} = \operatorname{ran} q$ and $E(\tilde{q}) = E(q)$.*

Proof. Let $\{f_j\}_{j\in J}$ be the set of minimal projections in $\{[q]\}'_D$. Let $\tilde{q} = \sum_{j\in J} qf_j$ be the expectation of q with respect to the block-diagonal algebra $(\{[q]\}'_D)'$. It is easy to see that $\tilde{q}$ is an idempotent and $E(\tilde{q}) = E(q)$. Since $q\tilde{q} = \tilde{q}$ and $\tilde{q}q = q$, we have $\operatorname{ran}\tilde{q} = \operatorname{ran} q$. Finally, $\tilde{q}f_j = f_jqf_j = f_j\tilde{q}$ for all $j \in J$, which means $\tilde{q} \in (\{[q]\}'_D)'$. □

Lemma 8.7. *Suppose that $t \mapsto x(t)$ is a real analytic map from $\mathbb{R}$ to $B(\mathcal{H})$. Then there exists a countable set $C \subset \mathbb{R}$ such that all operators $x(t)$, $t \in \mathbb{R} \setminus C$, have the same minimal block decomposition.*

Proof. Each entry of the matrix representing $x(t)$ in the canonical basis of $\mathcal{H}$ is a real-analytic function of t. Recall that a scalar-valued real-analytic function has at most countably many zeroes unless it vanishes identically. Therefore, the set of nonzero entries in the matrix of $x(t)$ is the same for all but countably many values of t. Since the set of nonzero entries determines the minimal block decomposition, the claim follows. □

Proof of Proposition 8.5. Let $\{f_j\}_{j\in J}$ be the set of minimal projections in $\{[q]\}'_D$. By virtue of Lemma 8.6 we may assume that

$$q \in (\{[q]\}'_D)', \quad \text{i.e., } q = \sum_{j\in J} f_jqf_j. \tag{8.4}$$

For $k, l \in J$ we set $y_{kl} = f_kq^\perp x_{kl}qf_l$, where $x_{kl} \in B(\mathcal{H})$ is chosen as follows. If either $k = l$, $q^\perp f_k = 0$, or $qf_l = 0$, then set $x_{kl} = 0$. Otherwise, choose x_{kl} so that $0 < \|y_{kl}\| < 2^{-k-l}$. Let $n = \sum_{k,l\in J} y_{kl}$.

One can easily check that $qn = 0$, $nq = n$, and $E(n) = 0$. Therefore, $q_t := q + tn$ is an idempotent for all $t \in \mathbb{R}$, and $E(q_t) = E(q)$. The range

projection of q_t,

$$[q_t] = (q + tn)\,(q + q^* + t(n + n^*) - \mathrm{id})^{-1}$$

is real analytic in t. By Lemma 8.7 $[q_t]$ has the same minimal block decomposition for all $t \in \mathbb{R} \setminus C$ where C is countable. If this decomposition consists of just one block, then we can set $r = q + tn$ for some $t \in \mathbb{R} \setminus C$.

Suppose that the minimal block decomposition of $[q_t]$, $t \in \mathbb{R} \setminus C$, is nontrivial. Then there exists a diagonal projection $f \notin \{0, \mathrm{id}\}$ that commutes with $[q_t]$ for all $t \in \mathbb{R} \setminus C$, hence for all $t \in \mathbb{R}$. This can be expressed as

$$(q^\perp - tn) f (q + tn) = 0, \quad t \in \mathbb{R}. \tag{8.5}$$

The coefficient of t in (8.5) must be zero, hence

$$q^\perp f n - n f q = 0.$$

Since $f \in [q]'_D$, we have $f = \sum_{j \in K} f_j$ for some $K \subset J$. Also, f commutes with q due to (8.4). Thus we obtain

$$0 = q^\perp f n - n f q = f q^\perp n - n q f = f n - n f = f n f^\perp - f^\perp n f,$$

hence $f n f^\perp = f^\perp n f = 0$. From the definition of n one can see that $f n f^\perp = 0$ only if

$$q^\perp f = 0 \quad \text{or} \quad q f^\perp = 0. \tag{8.6}$$

Similarly, $f^\perp n f = 0$ implies

$$q^\perp f^\perp = 0 \quad \text{or} \quad q f = 0. \tag{8.7}$$

Since $q, f \notin \{0, \mathrm{id}\}$, the relations (8.6)–(8.7) contradict each other. This completes the proof. □

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