1-1-2008

Testing for Random Effects and Spatial Lag Dependence in Panel Data Models

Badi H. Baltagi
Syracuse University, bbaltagi@maxwell.syr.edu

Long Liu
Syracuse University

ISSN: 1525-3066

Center for Policy Research
Working Paper No. 102

TESTING FOR RANDOM EFFECTS AND SPATIAL LAG DEPENDENCE IN PANEL DATA MODELS

Badi H. Baltagi and Long Liu

Center for Policy Research
Maxwell School of Citizenship and Public Affairs
Syracuse University
426 Eggers Hall
Syracuse, New York 13244-1020
(315) 443-3114 | Fax (315) 443-1081
e-mail: ctrpol@syr.edu

March 2008

$5.00

CENTER FOR POLICY RESEARCH – Spring 2008

Timothy Smeeding, Director
Professor of Economics & Public Administration

Associate Directors

Margaret Austin
Associate Director
Budget and Administration

Douglas Wolf
Professor of Public Administration
Associate Director, Aging Studies Program

John Yinger
Professor of Economics and Public Administration
Associate Director, Metropolitan Studies Program

SENIOR RESEARCH ASSOCIATES

Badi Baltagi..Economics
Kalena Cortes……………………………….Education
Thomas Dennison ....Public Administration
William Duncombe ....Public Administration
Gary Engelhardt ....Economics
Deborah Freund ....Public Administration
Madonna Harrington Meyer ....Sociology
Christine Himes.......................................Sociology
William C. Horrace ....Economics
Duke Kao ....Economics
Eric Kingson ....Social Work
Thomas Kniesner ....Economics
Jeffrey Kubik ....Economics
Andrew London.......................................Sociology
Len Lopoo ....Public Administration
Amy Lutz……………………………………Sociology
Jerry Miner ....Economics
Jan Ondrich...Economics
John Palmer ....Public Administration
Lori Ploutz-Snyder........................Exercise Science
David Popp............................Public Administration
Christopher Rohlfs ....Economics
Stuart Rosenthal ....Economics
Ross Rubenstein....................Public Administration
Perry Singleton……………………………Economics
Margaret Usdansky ....Sociology
Michael Wasylenko ....Economics
Janet Wilmoth ....Sociology

GRADUATE ASSOCIATES

Amy Agulay……………………Public Administration
Javier Baez ....Economics
Sonali Ballal ....Public Administration
Jesse Bricker ....Economics
Maria Brown....................................Social Science
Il Hwan Chung…………………Public Administration
Mike Eriksen ....Economics
Qu Feng ....Economics
Katie Fitzpatrick.....................................Economics
Chantell Frazier………………………........Sociology
Alexandre Genest ....Public Administration
Julie Anna Golebiewski..........................Economics
Nadia Greenhalgh-Stanley ....Economics
Sung Hyo Hong...................................... Economics
Neelakshi Medhi .... Social Science
Larry Miller .... Public Administration
Phuong Nguyen .... Public Administration
Wendy Parker .... Sociology
Shawn Rohlin... Economics
Carrie Roseamelia .... Sociology
Jeff Thompson .... Economics
Coady Wing .... Public Administration
Ryan Yeung .... Public Administration
Can Zhou .... Economics

STAFF

Kelly Bogart..…..……………Administrative Secretary
Martha Bonney……Publications/Events Coordinator
Karen Cimilluca..……………Administrative Secretary
Roseann DiMarzo…Receptionist/Office Coordinator
Kitty Nasto……...……………Administrative Secretary
Candi Patterson......................Computer Consultant
Mary Santy……..……………Administrative Secretary

Abstract

This paper derives a joint Lagrange Multiplier (LM) test which simultaneously tests for the absence of spatial lag dependence and random individual effects in a panel data regression model. It turns out that this LM statistic is the sum of two standard LM statistics. The first one tests for the absence of spatial lag dependence ignoring the random individual effects, and the second one tests for the absence of random individual effects ignoring the spatial lag dependence. This paper also derives two conditional LM tests. The first one tests for the absence of random individual effects without ignoring the possible presence of spatial lag dependence. The second one tests for the absence of spatial lag dependence without ignoring the possible presence of random individual effects.

JEL codes: C12 and C23
Keywords: Panel Data; Spatial Lag Dependence; Lagrange Multiplier Tests; Random Effects.

Testing For Random Effects and Spatial Lag Dependence in Panel Data Models

Badi H. Baltagi*, Long Liu†
Syracuse University

March 25, 2008

Abstract

This paper derives a joint Lagrange Multiplier (LM) test which simultaneously tests for the absence of spatial lag dependence and random individual effects in a panel data regression model. It turns out that this LM statistic is the sum of two standard LM statistics. The first one tests for the absence of spatial lag dependence ignoring the random individual effects, and the second one tests for the absence of random individual effects ignoring the spatial lag dependence. This paper also derives two conditional LM tests. The first one tests for the absence of random individual effects without ignoring the possible presence of spatial lag dependence. The second one tests for the absence of spatial lag dependence without ignoring the possible presence of random individual effects.

Key Words: *Panel Data; Spatial Lag Dependence; Lagrange Multiplier Tests; Random Effects.*

1 Introduction

Spatial models deal with correlation across spatial units usually in a cross-section setting, see Anselin (1988a). Panel data models allow the researcher to control for heterogeneity across these units, see Baltagi (2005). Spatial panel models can control for both heterogeneity and spatial correlation, see Baltagi, Song and Koh (2003). Testing for spatial dependence has been extensively studied by Anselin (1988a, 1988b, 2001) and Anselin and Bera (1998), to mention a few. Baltagi, Song and Koh (2003) considered the problem of jointly testing for random region effects in the panel as well as spatial correlation across these regions. However, the last study allowed for spatial correlation only in the remainder error term. This paper generalizes the Baltagi, Song and Koh (2003) to allow for spatial lag dependence of the autoregressive kind in the dependent variable rather than the error term. In fact, this paper derives a joint LM test which simultaneously tests for the absence of spatial lag dependence and random individual effects in a panel data regression model. It

*Address correspondence to: Badi H. Baltagi, Center for Policy Research, 426 Eggers Hall, Syracuse University, Syracuse, NY 13244-1020; e-mail: bbaltagi@maxwell.syr.edu.

†Long Liu: Economics Department, 110 Eggers Hall, Syracuse University, Syracuse, NY 13244-1020; e-mail: loliu@maxwell.syr.edu.

turns out that this LM statistic is the sum of two standard LM statistics. The first LM, tests for the absence of spatial lag dependence ignoring the random individual effects. This is the standard LM test derived in Anselin (1988b) for cross-section data. The second LM, tests for the absence of random individual effects ignoring the spatial lag dependence. This is the standard LM test derived in Breusch and Pagan (1980) for panel data. This paper also derives two conditional LM tests. The first one tests for the absence of random individual effects without ignoring the possible presence of spatial lag dependence. The second one tests for the absence of spatial lag dependence without ignoring the possible presence of random individual effects. This should provide useful diagnostics for applied researchers working in this area.

2 The model and test statistics

Consider a panel data regression model with spatial lag dependence:

$$y_t = \rho W y_t + X_t \beta + u_t, \quad i = 1, \ldots, N; \quad t = 1, ..., T \tag{1}$$

where $y_t' = (y_{t1}, \ldots, y_{tN})$ is a vector of observations on the dependent variables for N regions or households at time $t = 1, ..., T$. ρ is a scalar spatial autoregressive coefficient and W is a known $N \times N$ spatial weight matrix whose diagonal elements are zero. W also satisfies the condition that $(I_N - \rho W)$ is non-singular for all $|\rho| < 1$. I_N is an identity matrix of dimension N. X_t is an $N \times k$ matrix of observations on k explanatory variables at time t. $u_t' = (u_{t1}, \ldots, u_{tN})$ is a vector of disturbances following an error component model:

$$u_t = \mu + \nu_t \tag{2}$$

where $\mu' = (\mu_1, \ldots, \mu_N)$ and μ_i is i.i.d. over i and is assumed to be $N(0, \sigma_\mu^2)$. $\nu_t' = (\nu_{t1}, \ldots, \nu_{tN})$ and ν_{ti} is i.i.d. over t and i and is assumed to be $N(0, \sigma_\nu^2)$. The $\{\mu_i\}$ process is also independent of the $\{\nu_{it}\}$ process.

Equation (1) can be rewritten in matrix notation as

$$y = \rho \left(I_T \otimes W\right) y + X\beta + u, \quad i = 1, \ldots, N; \quad t = 1, ..., T \tag{3}$$

where y is of dimension $NT \times 1$, X is $NT \times k$, β is $k \times 1$ and u is $NT \times 1$. The observations are ordered with t being the slow running index and i the fast running index, i.e., $y' = (y_{11}, \ldots, y_{1N}, \ldots, y_{T1}, \ldots, y_{TN})$. X is assumed to be of full column rank and its elements are assumed to be asymptotically bounded in absolute value. Equation (2) can also be written in vector form as

$$u = \left(\iota_T \otimes I_N\right) \mu + \nu, \tag{4}$$

where $\nu' = (\nu'_1, \ldots, \nu'_T)$, ι_T is a vector of ones of dimension T, I_N is an identity matrix of dimension N, and $\otimes$ denotes the Kronecker product. Under these assumptions, the variance-covariance matrix for u can be written as

$$\Omega = \sigma_\mu^2 (J_T \otimes I_N) + \sigma_\nu^2 (I_T \otimes I_N), \tag{5}$$

where J_T is a matrix of ones of dimension T.

Under the normality assumption, the log-likelihood function of equation (1) is given by

$$L = -\frac{NT}{2} \ln 2\pi - \frac{1}{2} \ln |\Omega| + T \ln |A| - \frac{1}{2} [(I_T \otimes A) y - X\beta]' \Omega^{-1} [(I_T \otimes A) y - X\beta] \tag{6}$$

where $A = I_N - \rho W$. Ord (1975) shows that $\ln |I_N - \rho W| = \sum_{i=1}^N \ln (1 - \rho \omega_i)$, where ω_i's are the eigenvalues of W. Using the notation in Baltagi (2005), we can write $\Omega = \sigma_\nu^2 \Sigma$, where $\Sigma = Q + \phi^{-2} P, P = \bar{J}_T \otimes I_N, \bar{J}_T = \iota_T \iota'_T / T, Q = I_{TN} - P, \phi^2 = \sigma_\nu^2 / \sigma_1^2$ and $\sigma_1^2 = T\sigma_\mu^2 + \sigma_\nu^2$. From which it follows that $\ln |\Omega| = NT \ln \sigma_\nu^2 + N \ln \phi^2$. The log-likelihood function in (6) can be rewritten as

$$L = -\frac{NT}{2} \ln 2\pi - \frac{1}{2} \left[NT \ln \sigma_\nu^2 + N \ln \phi^2\right] + T \sum_{i=1}^N \ln (1 - \rho\omega_i) - \frac{1}{2\sigma_\nu^2} [(I_T \otimes A) y - X\beta]' \Sigma^{-1} [(I_T \otimes A) y - X\beta] \tag{7}$$

and one can estimate this model using maximum likelihood, see Anselin (1988a).

This paper derives a joint LM test for the absence of spatial lag dependence as well as random effects. The null hypothesis is $H_0^a : \rho = \sigma_\mu^2 = 0$, and the alternative H_1^a is that at least one component is not zero. This generalizes the LM test derived in Anselin (1988b) for the absence of spatial lag dependence $H_0^b : \rho = 0$ (assuming no random effects, i.e., $\sigma_\mu^2 = 0$), and the Breusch and Pagan (1980) LM test for the absence of random effects $H_0^c : \sigma_\mu^2 = 0$ (assuming no spatial lag dependence, i.e., $\rho = 0$). We also derive two conditional LM tests, one for $H_0^d : \rho = 0$ (assuming the possible existence of random effects, i.e., $\sigma_\mu^2 \geqslant 0$), and the other one for $H_0^e : \sigma_\mu^2 = 0$ (assuming the possible existence of spatial lag dependence, i.e., ρ may be different from zero). All the proofs are given in the Appendix to the paper.

2.1 Joint LM test for $H_0^a : \rho = \sigma_\mu^2 = 0$

The joint LM test statistic for testing $H_0^a : \rho = \sigma_\mu^2 = 0$ is given by

$$LM_J = \frac{R^2}{B} + \frac{NT}{2(T-1)} G^2 = LM_\rho + LM_\mu \tag{8}$$

where $B = T \cdot tr\left[W^2 + W'W\right] + \widetilde{\sigma}_\nu^{-2} \widetilde{\beta}' X' \left(I_T \otimes W'\right) M \left(I_T \otimes W\right) X \widetilde{\beta}$, $M = I - X\left(X'X\right)^{-1} X'$, $G = T\frac{\tilde{u}' P \tilde{u}}{\tilde{u}'\tilde{u}} - 1$, $R = NT\frac{\tilde{u}'(I_T \otimes W) y}{\tilde{u}'\tilde{u}}$. $LM_\rho = R^2/B$, and $LM_\mu = NTG^2/2\left(T-1\right)$. $\widetilde{\beta}$ is the restricted MLE under H_0^a which yields OLS, $\tilde{u}$ denotes the OLS residuals, and $\widetilde{\sigma}_\nu^2 = \tilde{u}'\tilde{u}/NT$. R is a generalization of a similar term defined in Anselin (1988b) for the LM test of no spatial dependence in the cross-section case. In fact, R can be interpreted as NT times the regression coefficient of $\left(I_T \otimes W\right) y$ on $\tilde{u}$. Here, the joint LM test LM_J is the sum of two LM test statistics: The first is $LM_\rho = R^2/B$,which is the LM test statistic for testing $H_0^b : \rho = 0$ assuming there is no random region effects, i.e., assuming $\sigma_\mu^2 = 0$, see Anselin (1988a). LM_ρ is asymptotically distributed as χ_1^2 under H_0^b. The second is $LM_\mu = \frac{NT}{2(T-1)} G^2$,which is the LM test statistic for testing $H_0^c : \sigma_\mu^2 = 0$ assuming there is no spatial lag dependence, i.e., assuming that $\rho = 0$, see Breusch and Pagan (1980). Since LM_ρ and LM_μ are asymptotically independent, LM_J is asymptotically distributed as χ_2^2 under H_0^a. It is important to point out that the asymptotic distribution of our test statistics are not explicitly derived in the paper but that they are likely to hold under a similar set of primitive assumptions developed by Kelejian and Prucha (2001).

2.2 Conditional LM Test for $H_0^d : \rho = 0$ (assuming $\sigma_\mu^2 \geqslant 0$)

When one uses LM_ρ defined in (8) to test $H_0^b : \rho = 0$, one implicitly assumes that the random region effects do not exist. This may lead to incorrect inference especially when σ_μ^2 is large. To overcome this problem, we derive a conditional LM test for no spatial lag dependence assuming the possible existence of random region effects. The null hypothesis is $H_0^d : \rho = 0$ (assuming $\sigma_\mu^2 \geqslant 0$), and the conditional LM test statistic is given by

$$LM_{\rho/\mu} = R_1^2/B_1, \tag{9}$$

where $R_1 = \hat{\sigma}_1^{-2} \hat{u}' \left(\bar{J}_T \otimes W\right) y + \hat{\sigma}_\nu^{-2} \hat{u}' \left(E_T \otimes W\right) y$,

$$B_1 = T \cdot tr\left[W^2 + W'W\right] + \hat{\sigma}_1^{-2} \widehat{\beta}' X' \left(\bar{J}_T \otimes W'W\right) X\widehat{\beta} + \hat{\sigma}_\nu^{-2} \widehat{\beta}' X' \left(E_T \otimes W'W\right) X\widehat{\beta}$$
$$- \left[\hat{\sigma}_1^{-2} X' \left(\bar{J}_T \otimes W\right) X\widehat{\beta} + \hat{\sigma}_\nu^{-2} X' \left(E_T \otimes W\right) X\widehat{\beta}\right]' \left[X'\widehat{\Omega}^{-1} X\right]^{-1} \left[\hat{\sigma}_1^{-2} X' \left(\bar{J}_T \otimes W\right) X\widehat{\beta} + \hat{\sigma}_\nu^{-2} X' \left(E_T \otimes W\right) X\widehat{\beta}\right]$$

and $(\widehat{\beta}, \hat{\sigma}_1^2, \hat{\sigma}_\nu^2)$ denote the restricted MLE under H_0^d. These are in fact the MLE under a random effects panel data model with no spatial lag dependence. $\hat{u}$ denotes the corresponding restricted MLE residuals under the null hypothesis H_0^d. This LM statistic is asymptotically distributed as χ_1^2 under H_0^d. $E_T = I_T - \bar{J}_T$, $\hat{\sigma}_1^2 = \hat{u}' P \hat{u}/N$, and $\hat{\sigma}_\nu^2 = \hat{u}' Q \hat{u}/N(T-1)$. Note that $LM_{\rho/\mu}$ in (9) is of the same form as LM_ρ in (8). However, R_1 and B_1 are now different from R and B, and they are based on different restricted ML residuals, namely

$\hat{u}$, those of a random effects panel data model with no spatial lag dependence, see Baltagi (2005), rather than the OLS residuals $\tilde{u}$.

2.3 Conditional LM Test for $H_0^e : \sigma_\mu^2 = 0$ (assuming ρ may or may not be zero)

Similarly, if one uses LM_μ defined in (8) to test $H_0^c : \sigma_\mu^2 = 0$, one implicitly assumes that the spatial lag dependence does not exist. This may lead to incorrect inference especially when ρ is large. To overcome this problem, we derive a conditional LM test for no random region effects given the existence of spatial lag dependence. The null hypothesis is $H_0^e : \sigma_\mu^2 = 0$ (assuming ρ may not be zero), and the conditional LM test statistic is given by

$$LM_{\mu/\rho} = \frac{NT}{2\,(T-1)} G_1^2, \tag{10}$$

where $G_1 = T\frac{\bar{u}' P \bar{u}}{\bar{u}'\bar{u}} - 1$ and $\bar{u}$ denotes the restricted maximum likelihood residuals under the null hypothesis H_0^e, i.e., under a spatial lag dependence panel data model with no random effects. Note that $LM_{\mu/\rho}$ in (10) is of the same form as LM_μ in (8). However, G_1 differs from G in that they are based on different restricted ML residuals. The former is based on $\overline{u}_t = y_t - \overline{\rho} W y_t + X_t \overline{\beta}$, where $\overline{\rho}$ and $\overline{\beta}$ are the MLE of ρ and β in a spatial lag panel data model with no random effects, while the latter is based on OLS residuals $\tilde{u}$.

REFERENCES

Anselin, L. (1988a) Spatial Econometrics: Methods and Models. Kluwer Academic Publishers, Dordrecht.

Anselin, L. (1988b) Lagrange Multiplier Test Diagnostics for Spatial Dependence and Spatial Heterogeneity, Geographical Analysis, 20, 1–17.

Anselin, L. (2001) Rao's score tests in spatial econometrics. Journal of Statistical Planning and Inference, 97, 113-139.

Anselin, L., Bera, A.K. (1998) Spatial Dependence in Linear Regression Models with an Introduction to Spatial Econometrics. In: Ullah, A., Giles, D.E.A. (Eds.), Handbook of Applied Economic Statistics. Marcel Dekker, New York.

Baltagi, B. (2005), Econometric Analysis of Panel Data, New York, Wiley.

Baltagi, B.H., S.H. Song and W.Koh (2003) Testing Panel Data Regression Models with Spatial Error Correlation, Journal of Econometrics 117, 123–150.

Breusch, T.S. and A.R. Pagan (1980) The Lagrange multiplier test and its applications to model specification in econometrics, Review of Economic Studies 47, 239–253.

Kelejian, H.H. and I.R. Prucha (2001), On the asymptotic distribution of the Moran I test with applications. Journal of Econometrics 104, 219-257.

Ord, J. (1975), Estimation Methods for Models of Spatial Interaction, Journal of the American Statistical Association, 70, 120-126.

3 Appendix

3.1 The first-order and second-order derivatives

From the log-likelihood function given in (6), one can obtain the score equations as follows:

$$
\begin{aligned}
\frac{\partial L}{\partial \rho} &= -T \cdot tr\left[A^{-1}W\right] + u'\Omega^{-1}\left(I_T \otimes W\right) y \\
\frac{\partial L}{\partial \sigma_\mu^2} &= -\frac{1}{2} NT\sigma_1^{-2} + \frac{1}{2} T\sigma_1^{-4}\left[u'Pu\right] \\
\frac{\partial L}{\partial \sigma_\nu^2} &= -\frac{1}{2}\left[N\sigma_1^{-2} + N\left(T-1\right)\sigma_\nu^{-2}\right] + \frac{1}{2}\left[u'\left(\sigma_1^{-4}P + \sigma_\nu^{-4}Q\right)u\right] \\
\frac{\partial L}{\partial \beta} &= X'\Omega^{-1}u
\end{aligned}
$$

where $\sigma_1^2 = T\sigma_\mu^2 + \sigma_\nu^2$, $P = \bar{J}_T \otimes I_N$, $\bar{J}_T = \iota_T \iota_T'/T$, with ι_T denoting a vector of ones of dimension T.

The second-order derivatives are given by

$$\begin{aligned}
\frac{\partial^2 L}{\partial\rho\partial\rho} &= -T\cdot tr\left[\left(WA^{-1}\right)^2\right] - y'\left(I_T\otimes W'\right)\Omega^{-1}\left(I_T\otimes W\right)y \\
\frac{\partial^2 L}{\partial\rho\partial\sigma_\mu^2} &= -T\sigma_1^{-4}u'P\left(I_T\otimes W\right)y \\
\frac{\partial^2 L}{\partial\rho\partial\sigma_\nu^2} &= -u'\left(\sigma_1^{-4}P+\sigma_\nu^{-4}Q\right)\left(I_T\otimes W\right)y \\
\frac{\partial^2 L}{\partial\rho\partial\beta} &= -X'\Omega^{-1}\left(I_T\otimes W\right)y = -X'\left(\sigma_1^{-2}P+\sigma_\nu^{-2}Q\right)\left(I_T\otimes W\right)y \\
\frac{\partial^2 L}{\partial\sigma_\mu^2\partial\sigma_\mu^2} &= \frac{1}{2}NT^2\sigma_1^{-4} - T^2\sigma_1^{-6}\left[u'Pu\right] \\
\frac{\partial^2 L}{\partial\sigma_\mu^2\partial\sigma_\nu^2} &= \frac{1}{2}NT\sigma_1^{-4} - T\sigma_1^{-6}\left[u'Pu\right] \\
\frac{\partial^2 L}{\partial\sigma_\mu^2\partial\beta} &= -TX'\Omega^{-1}P\Omega^{-1}u = -\sigma_1^{-4}TX'Pu \\
\frac{\partial^2 L}{\partial\sigma_\nu^2\partial\sigma_\nu^2} &= \frac{1}{2}\left[N\sigma_1^{-4}+N\left(T-1\right)\sigma_\nu^{-4}\right] - \left[u'\left(\sigma_1^{-6}P+\sigma_\nu^{-6}Q\right)u\right] \\
\frac{\partial^2 L}{\partial\sigma_\nu^2\partial\beta} &= -X'\Omega^{-1}\Omega^{-1}u = -X'\left(\sigma_1^{-4}P+\sigma_\nu^{-4}Q\right)u \\
\frac{\partial^2 L}{\partial\beta\partial\beta'} &= -X'\Omega^{-1}X = -X'\left(\sigma_1^{-2}P+\sigma_\nu^{-2}Q\right)X
\end{aligned}$$

3.2 Joint Test

Under the null hypothesis $H_0^a:\rho=\sigma_\mu^2=0$, equation (1) becomes a regression model with no spatial lag dependence or random region effects. The variance-covariance matrix reduces to $\sigma_\nu^2 I_{NT}$ and the restricted MLE of β is $\tilde{\beta}_{OLS}$, so that $\tilde{u}=y-X\tilde{\beta}_{OLS}$ are the OLS residuals and $\tilde{\sigma}_\nu^2=\tilde{u}'\tilde{u}/NT$. This is clear from the score equations evaluated under $H_0^a:\rho=\sigma_\mu^2=0:$

$$\begin{aligned}
\frac{\partial L}{\partial\rho}|_{H_0^a} &= -T\cdot tr\left[W\right]+\tilde{\sigma}_\nu^{-2}\tilde{u}'\left(I_T\otimes W\right)y = \tilde{\sigma}_\nu^{-2}\tilde{u}'\left(I_T\otimes W\right)y \\
\frac{\partial L}{\partial\sigma_\mu^2}|_{H_0^a} &= -\frac{1}{2}NT\tilde{\sigma}_\nu^{-2}+\frac{1}{2}T\tilde{\sigma}_\nu^{-4}\tilde{u}'P\tilde{u} = \frac{1}{2}NT\left(T\frac{\tilde{u}'P\tilde{u}}{\tilde{u}'\tilde{u}}-1\right)\tilde{\sigma}_\nu^{-2} \\
\frac{\partial L}{\partial\sigma_\nu^2}|_{H_0^a} &= -\frac{1}{2}NT\tilde{\sigma}_\nu^{-2}+\frac{1}{2}\tilde{\sigma}_\nu^{-4}\tilde{u}'\tilde{u}=0 \\
\frac{\partial L}{\partial\beta}|_{H_0^a} &= \tilde{\sigma}_\nu^{-2}X'\tilde{u}=0
\end{aligned}$$

Therefore, the score with respect to $\theta'=(\rho,\sigma_\mu^2,\sigma_\nu^2,\beta')$, evaluated under the null hypothesis $H_0^a:\rho=\sigma_\mu^2=0$ is given by

$$\tilde{D}_\theta = \begin{pmatrix} \tilde{D}_\rho \\ \tilde{D}_{\sigma_\mu^2} \\ \tilde{D}_{\sigma_\nu^2} \\ \tilde{D}_\beta \end{pmatrix} = \begin{pmatrix} \tilde{\sigma}_\nu^{-2} \tilde{u}' \left(I_T \otimes W \right) y \\ \frac{NT}{2\tilde{\sigma}_\nu^2} \left(T \frac{\tilde{u}' P \tilde{u}}{\tilde{u}' \tilde{u}} - 1 \right) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} R \\ \frac{NT}{2\tilde{\sigma}_\nu^2} G \\ 0 \\ 0 \end{pmatrix}$$

where R is a generalization of a similar term defined in Anselin (1988b) for the LM test of no spatial dependence in the cross-section case. In fact, R can be interpreted as NT times the regression coefficient of $(I_T \otimes W)\, y$ on $\tilde{u}$.

Under H_0^a, the elements of the information matrix $\tilde{J}$ are given by:

$$\begin{aligned}
E\left[-\frac{\partial^2 L}{\partial \rho \partial \rho} \right] |_{H_0^a} &= T \cdot tr \left[W^2 \right] + \widetilde{\sigma}_\nu^{-2} E \left[y' \left(I_T \otimes W'W \right) y \right] \\
&= T \cdot tr \left[W^2 \right] + \widetilde{\sigma}_\nu^{-2} E \left[u' \left(I_T \otimes W'W \right) u \right] + \widetilde{\sigma}_\nu^{-2} \widetilde{\beta}' X' \left(I_T \otimes W'W \right) X \widetilde{\beta} \\
&= T \cdot tr \left[W^2 + W'W \right] + \widetilde{\sigma}_\nu^{-2} \widetilde{\beta}' X' \left(I_T \otimes W'W \right) X \widetilde{\beta} \\
E\left[-\frac{\partial^2 L}{\partial \rho \partial \sigma_\mu^2} \right] |_{H_0^a} &= T \widetilde{\sigma}_\nu^{-4} E \left[u' P \left(I_T \otimes W \right) y \right] = T \widetilde{\sigma}_\nu^{-4} E \left[tr \left(uu' \left(\bar{J}_T \otimes W \right) \right) \right] = T \widetilde{\sigma}_\nu^{-2} tr \left(\bar{J}_T \otimes W \right) = 0 \\
E\left[-\frac{\partial^2 L}{\partial \rho \partial \sigma_\nu^2} \right] |_{H_0^a} &= \widetilde{\sigma}_\nu^{-4} E \left[u' \left(I_T \otimes W \right) y \right] = \widetilde{\sigma}_\nu^{-4} E \left[tr \left(uu' \left(I_T \otimes W \right) \right) \right] = \widetilde{\sigma}_\nu^{-2} tr \left(I_T \otimes W \right) = 0 \\
E\left[-\frac{\partial^2 L}{\partial \rho \partial \beta} \right] |_{H_0^a} &= \widetilde{\sigma}_\nu^{-2} E \left[X' \left(I_T \otimes W \right) y \right] = \widetilde{\sigma}_\nu^{-2} X' \left(I_T \otimes W \right) X \widetilde{\beta}
\end{aligned}$$

$$\begin{aligned}
E\left[-\frac{\partial^2 L}{\partial \sigma_\mu^2 \partial \sigma_\mu^2} \right] |_{H_0^a} &= -\frac{1}{2} N T^2 \widetilde{\sigma}_\nu^{-4} + T^2 \widetilde{\sigma}_\nu^{-6} E \left[u' P u \right] = \frac{1}{2} N T^2 \widetilde{\sigma}_\nu^{-4} \\
E\left[-\frac{\partial^2 L}{\partial \sigma_\mu^2 \partial \sigma_\nu^2} \right] |_{H_0^a} &= -\frac{1}{2} N T \widetilde{\sigma}_\nu^{-4} + T \widetilde{\sigma}_\nu^{-6} E \left[u' P u \right] = \frac{1}{2} N T \widetilde{\sigma}_\nu^{-4} \\
E\left[-\frac{\partial^2 L}{\partial \sigma_\mu^2 \partial \beta} \right] |_{H_0^a} &= T \widetilde{\sigma}_\nu^{-4} E \left[X' P u \right] = 0 \\
E\left[-\frac{\partial^2 L}{\partial \sigma_\nu^2 \partial \sigma_\nu^2} \right] |_{H_0^a} &= -\frac{1}{2} N T \widetilde{\sigma}_\nu^{-4} + \widetilde{\sigma}_\nu^{-6} E \left[u' u \right] = \frac{1}{2} N T \widetilde{\sigma}_\nu^{-4} \\
E\left[-\frac{\partial^2 L}{\partial \sigma_\nu^2 \partial \beta} \right] |_{H_0^a} &= \widetilde{\sigma}_\nu^{-4} E \left[X' u \right] = 0 \\
E\left[-\frac{\partial^2 L}{\partial \beta \partial \beta'} \right] |_{H_0^a} &= \widetilde{\sigma}_\nu^{-2} X' X
\end{aligned}$$

Hence, the information matrix $\tilde{J}$ evaluated under H_0^a can be written as

$$\tilde{J} = \begin{pmatrix} \tilde{J}_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{pmatrix}$$

where $\tilde{J}_{11} = \begin{pmatrix} T \cdot tr\left[W^2 + W'W\right] + \widetilde{\sigma}_\nu^{-2} \widetilde{\beta}' X' \left(I_T \otimes W'W\right) X\widetilde{\beta} & 0 \\ 0 & \frac{1}{2}NT^2\widetilde{\sigma}_\nu^{-4} \end{pmatrix}$,

$\tilde{J}_{12} = \tilde{J}'_{21} = \begin{pmatrix} 0 & \left(\widetilde{\sigma}_\nu^{-2} X' \left(I_T \otimes W\right) X\widetilde{\beta}\right)' \\ \frac{1}{2}NT\widetilde{\sigma}_\nu^{-4} & 0 \end{pmatrix}$, and $\tilde{J}_{22} = \begin{pmatrix} \frac{1}{2}NT\widetilde{\sigma}_\nu^{-4} & 0 \\ 0 & \widetilde{\sigma}_\nu^{-2} X'X \end{pmatrix}$.

Using partitioned inversion, we know that the upper 2×2 block of the inverse matrix $\tilde{J}^{-1}$ is given by $\tilde{J}^{11} = \left(\tilde{J}_{11} - \tilde{J}_{12}\tilde{J}_{22}^{-1}\tilde{J}_{21}\right)^{-1}$.

This can be easily derived as:

$\tilde{J}^{11} = \begin{pmatrix} B^{-1} & 0 \\ 0 & \frac{2\sigma_\nu^4}{NT(T-1)} \end{pmatrix}$

where $B = T \cdot tr\left[W^2 + W'W\right] + \widetilde{\sigma}_\nu^{-2}\widetilde{\beta}' X' \left(I_T \otimes W'\right) M \left(I_T \otimes W\right) X\widetilde{\beta}$, and $M = I - X\left(X'X\right)^{-1} X'$. See Anselin and Bera (1998) for a similar B term in the cross-section case.

Therefore, the joint LM statistic for H_0^a is given by

$LM_J = \tilde{D}'_\theta \tilde{J}^{-1} \tilde{D}_\theta = \begin{pmatrix} \tilde{D}_\rho & \tilde{D}_{\sigma_\mu^2} \end{pmatrix} \tilde{J}^{11} \begin{pmatrix} \tilde{D}_\rho \\ \tilde{D}_{\sigma_\mu^2} \end{pmatrix} = \begin{pmatrix} R & \frac{NT}{2\tilde{\sigma}_\nu^2} G \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & \frac{2\sigma_\nu^4}{NT(T-1)} \end{pmatrix} \begin{pmatrix} R \\ \frac{NT}{2\tilde{\sigma}_\nu^2} G \end{pmatrix}$

$= \frac{R^2}{B} + \frac{NT}{2(T-1)} G^2.$

3.3 Conditional LM Test for $H_0^d : \rho = 0$ (assuming $\sigma_\mu^2 \geqslant 0$)

This section derives the conditional LM test for no spatial lag dependence given the existence of random region effects. The null hypothesis is $H_0^d : \rho = 0$ (assuming $\sigma_\mu^2 \geqslant 0$). Under the null, the score equations are given by

$$\begin{aligned}
\frac{\partial L}{\partial \rho}|_{H_0^d} &= -T \cdot tr\left[W\right] + \hat{u}'\left(\hat{\sigma}_1^{-2}P + \hat{\sigma}_\nu^{-2}Q\right)\left(I_T \otimes W\right) y = \hat{\sigma}_1^{-2}\hat{u}'\left(\bar{J}_T \otimes W\right) y + \hat{\sigma}_\nu^{-2}\hat{u}'\left(E_T \otimes W\right) y \\
\frac{\partial L}{\partial \sigma_\mu^2}|_{H_0^d} &= -\frac{1}{2}NT\hat{\sigma}_1^{-2} + \frac{1}{2}T\hat{\sigma}_1^{-4}\left[\hat{u}'P\hat{u}\right] = 0 \\
\frac{\partial L}{\partial \sigma_\nu^2}|_{H_0^d} &= -\frac{1}{2}\left[N\hat{\sigma}_1^{-2} + N\left(T-1\right)\hat{\sigma}_\nu^{-2}\right] + \frac{1}{2}\left[\hat{u}'\left(\hat{\sigma}_1^{-4}P + \hat{\sigma}_\nu^{-4}Q\right)\hat{u}\right] = 0 \\
\frac{\partial L}{\partial \beta}|_{H_0^d} &= X'\left(\hat{\sigma}_1^{-2}P + \hat{\sigma}_\nu^{-2}Q\right)\hat{u} = 0
\end{aligned}$$

using $tr\left[W\right] = 0$. Under the null hypothesis H_0^d, there is no spatial lag dependence and the variance-covariance matrix $\Omega = \sigma_\mu^2 J_T \otimes I_N + \sigma_\nu^2 I_{NT}$. It is the familiar form of the one-way error component model, see Baltagi (2005). The restricted MLE of β, σ_ν^2, and σ_ν^2, are those based on MLE of a random effects panel data model with no spatial lag dependence. These are denoted by $\widehat{\beta}, \widehat{\sigma}_\nu^2$, and $\widehat{\sigma}_\mu^2$, respectively. The corresponding restricted MLE residuals are denoted by $\hat{u}$. In fact, $\hat{\sigma}_1^2 = \hat{u}'P\hat{u}/N$, and $\hat{\sigma}_\nu^2 = \hat{u}'Q\hat{u}/N(T-1)$. Therefore, the score with respect to $\theta' = (\rho, \sigma_\mu^2, \sigma_\nu^2, \beta')$, evaluated under the null hypothesis H_0^d, is given by

$$\hat{D}_\theta = \begin{pmatrix} \hat{D}_\rho \\ \hat{D}_{\sigma_\mu^2} \\ \hat{D}_{\sigma_\nu^2} \\ \hat{D}_\beta \end{pmatrix} = \begin{pmatrix} R_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where $R_1 = \hat{\sigma}_1^{-2}\hat{u}'\left(\bar{J}_T \otimes W\right)y + \hat{\sigma}_\nu^{-2}\hat{u}'\left(E_T \otimes W\right)y$. Under H_0^d, the elements of the information matrix $\widehat{J}$ are given by:

$$\begin{aligned}
E\left[-\frac{\partial^2 L}{\partial\rho\partial\rho}\right]|_{H_0^d} &= T \cdot tr\left[W^2\right] + y'\left(I_T \otimes W'\right)\left(\hat{\sigma}_1^{-2}P + \hat{\sigma}_\nu^{-2}Q\right)\left(I_T \otimes W\right)y \\
&= T \cdot tr\left[W^2\right] + \hat{\sigma}_1^{-2}E\left[u'\left(\bar{J}_T \otimes W'W\right)u\right] + \hat{\sigma}_\nu^{-2}E\left[u'\left(E_T \otimes W'W\right)u\right] \\
&\quad + \hat{\sigma}_1^{-2}\widehat{\beta}'X'\left(\bar{J}_T \otimes W'W\right)X\widehat{\beta} + \hat{\sigma}_\nu^{-2}\widehat{\beta}'X'\left(E_T \otimes W'W\right)X\widehat{\beta} \\
&= T \cdot tr\left[W^2\right] + tr\left(\bar{J}_T \otimes W'W\right) + tr\left(E_T \otimes W'W\right) \\
&\quad + \hat{\sigma}_1^{-2}\widehat{\beta}'X'\left(\bar{J}_T \otimes W'W\right)X\widehat{\beta} + \hat{\sigma}_\nu^{-2}\widehat{\beta}'X'\left(E_T \otimes W'W\right)X\widehat{\beta} \\
&= T \cdot tr\left[W^2 + W'W\right] + \hat{\sigma}_1^{-2}\widehat{\beta}'X'\left(\bar{J}_T \otimes W'W\right)X\widehat{\beta} + \hat{\sigma}_\nu^{-2}\widehat{\beta}'X'\left(E_T \otimes W'W\right)X\widehat{\beta} \\
E\left[-\frac{\partial^2 L}{\partial\rho\partial\sigma_\mu^2}\right]|_{H_0^d} &= T\widehat{\sigma}_1^{-4}E\left[u'P\left(I_T \otimes W\right)y\right] = T\widehat{\sigma}_1^{-4}E\left[tr\left(uu'\left(\bar{J}_T \otimes W\right)\right)\right] = 0 \\
E\left[-\frac{\partial^2 L}{\partial\rho\partial\sigma_\nu^2}\right]|_{H_0^d} &= E\left[u'\left(\widehat{\sigma}_1^{-4}P + \widehat{\sigma}_\nu^{-4}Q\right)\left(I_T \otimes W\right)y\right] = \widehat{\sigma}_1^{-2}tr\left(\bar{J}_T \otimes W\right) + \widehat{\sigma}_\nu^{-2}tr\left(E_T \otimes W\right) = 0 \\
E\left[-\frac{\partial^2 L}{\partial\rho\partial\beta}\right]|_{H_0^d} &= E\left[X'\left(\widehat{\sigma}_1^{-2}P + \widehat{\sigma}_\nu^{-2}Q\right)\left(I_T \otimes W\right)y\right] = \widehat{\sigma}_1^{-2}X'\left(\bar{J}_T \otimes W\right)X\widehat{\beta} + \widehat{\sigma}_\nu^{-2}X'\left(E_T \otimes W\right)X\widehat{\beta}
\end{aligned}$$

$$\begin{aligned}
E\left[-\frac{\partial^2 L}{\partial\sigma_\mu^2\partial\sigma_\mu^2}\right]|_{H_0^d} &= -\frac{1}{2}NT^2\widehat{\sigma}_1^{-4} + T^2\widehat{\sigma}_1^{-6}E\left[u'Pu\right] = \frac{1}{2}NT^2\widehat{\sigma}_1^{-4} \\
E\left[-\frac{\partial^2 L}{\partial\sigma_\mu^2\partial\sigma_\nu^2}\right]|_{H_0^d} &= -\frac{1}{2}NT\widehat{\sigma}_1^{-4} + T\widehat{\sigma}_1^{-6}E\left[u'Pu\right] = \frac{1}{2}NT\widehat{\sigma}_1^{-4} \\
E\left[-\frac{\partial^2 L}{\partial\sigma_\mu^2\partial\beta}\right]|_{H_0^d} &= T\widehat{\sigma}_1^{-4}E\left[X'Pu\right] = 0 \\
E\left[-\frac{\partial^2 L}{\partial\sigma_\nu^2\partial\sigma_\nu^2}\right]|_{H_0^d} &= -\frac{1}{2}\left[N\widehat{\sigma}_1^{-4} + N\left(T-1\right)\widehat{\sigma}_\nu^{-4}\right] + E\left[u'\left(\widehat{\sigma}_1^{-6}P + \widehat{\sigma}_\nu^{-6}Q\right)u\right] = \frac{1}{2}\left[N\widehat{\sigma}_1^{-4} + N\left(T-1\right)\widehat{\sigma}_\nu^{-4}\right] \\
E\left[-\frac{\partial^2 L}{\partial\sigma_\nu^2\partial\beta}\right]|_{H_0^d} &= E\left[X'\left(\widehat{\sigma}_1^{-4}P + \widehat{\sigma}_\nu^{-4}Q\right)u\right] = 0 \\
E\left[-\frac{\partial^2 L}{\partial\beta\partial\beta'}\right]|_{H_0^d} &= X'\left(\widehat{\sigma}_1^{-2}P + \widehat{\sigma}_\nu^{-2}Q\right)X
\end{aligned}$$

Therefore, the information matrix $\hat{J}$ evaluated under H_0^d can be written as

$$\hat{J} = \begin{pmatrix} \hat{J}_{11} & \hat{J}_{12} \\ \hat{J}_{21} & \hat{J}_{22} \end{pmatrix}$$

where $\hat{J}_{11} = \left(T \cdot tr\left[W^2 + W'W\right] + \hat{\sigma}_1^{-2}\widehat{\beta}' X' \left(\bar{J}_T \otimes W'W\right) X\widehat{\beta} + \hat{\sigma}_\nu^{-2}\widehat{\beta}' X' \left(E_T \otimes W'W\right) X\widehat{\beta}\right)$,

$\hat{J}_{12} = \hat{J}_{21}' = \left(0 \quad 0 \quad \left(\widehat{\sigma}_1^{-2} X' \left(\bar{J}_T \otimes W\right) X\widehat{\beta} + \widehat{\sigma}_\nu^{-2} X' \left(E_T \otimes W\right) X\widehat{\beta}\right)'\right)$,

and $\hat{J}_{22} = \begin{pmatrix} \frac{1}{2}NT^2\widehat{\sigma}_1^{-4} & \frac{1}{2}NT\widehat{\sigma}_1^{-4} & 0 \\ \frac{1}{2}NT\widehat{\sigma}_1^{-4} & \frac{1}{2}\left[N\widehat{\sigma}_1^{-4} + N(T-1)\widehat{\sigma}_\nu^{-4}\right] & 0 \\ 0 & 0 & X'\left(\widehat{\sigma}_1^{-2}P + \widehat{\sigma}_\nu^{-2}Q\right)X \end{pmatrix}$.

Using partitioned inversion, we know that the upper 1×1 element of the inverse matrix $\widehat{J}^{-1}$ is given by $\widehat{J}^{11} = \left(\hat{J}_{11} - \hat{J}_{12}\hat{J}_{22}^{-1}\hat{J}_{21}\right)^{-1}$. Here

$$\begin{aligned}
&\hat{J}_{11} - \hat{J}_{12}\hat{J}_{22}^{-1}\hat{J}_{21} \\
&= \left(T \cdot tr\left[W^2 + W'W\right] + \hat{\sigma}_1^{-2}\widehat{\beta}' X' \left(\bar{J}_T \otimes W'W\right) X\widehat{\beta} + \hat{\sigma}_\nu^{-2}\widehat{\beta}' X' \left(E_T \otimes W'W\right) X\widehat{\beta}\right) \\
&- \left(0 \quad 0 \quad \left(\widehat{\sigma}_1^{-2} X' \left(\bar{J}_T \otimes W\right) X\widehat{\beta} + \widehat{\sigma}_\nu^{-2} X' \left(E_T \otimes W\right) X\widehat{\beta}\right)'\right) \\
&\begin{pmatrix} \frac{2\widehat{\sigma}_\nu^4}{NT^2(T-1)} + \frac{2\widehat{\sigma}_1^4}{NT^2} & \frac{-2\widehat{\sigma}_\nu^4}{NT(T-1)} & 0 \\ \frac{-2\widehat{\sigma}_\nu^4}{NT(T-1)} & \frac{2\widehat{\sigma}_\nu^4}{N(T-1)} & 0 \\ 0 & 0 & \left[X'\left(\widehat{\sigma}_1^{-2}P + \widehat{\sigma}_\nu^{-2}Q\right)X\right]^{-1} \end{pmatrix} \\
&\begin{pmatrix} 0 \\ 0 \\ \widehat{\sigma}_1^{-2} X' \left(\bar{J}_T \otimes W\right) X\widehat{\beta} + \widehat{\sigma}_\nu^{-2} X' \left(E_T \otimes W\right) X\widehat{\beta} \end{pmatrix} \\
&= \left(T \cdot tr\left[W^2 + W'W\right] + \hat{\sigma}_1^{-2}\widehat{\beta}' X' \left(\bar{J}_T \otimes W'W\right) X\widehat{\beta} + \hat{\sigma}_\nu^{-2}\widehat{\beta}' X' \left(E_T \otimes W'W\right) X\widehat{\beta}\right) \\
&- \left(\left[\widehat{\sigma}_1^{-2} X' \left(\bar{J}_T \otimes W\right) X\widehat{\beta} + \widehat{\sigma}_\nu^{-2} X' \left(E_T \otimes W\right) X\widehat{\beta}\right]' \left[X'\widehat{\Omega}^{-1}X\right]^{-1} \left[\widehat{\sigma}_1^{-2} X' \left(\bar{J}_T \otimes W\right) X\widehat{\beta} + \widehat{\sigma}_\nu^{-2} X' \left(E_T \otimes W\right) X\widehat{\beta}\right]\right) \\
&= \widehat{B}_1.
\end{aligned}$$

Therefore, the LM statistic for H_0^d is given by $LM_{\rho/\mu} = \hat{D}'\hat{J}^{-1}\hat{D} = R_1 B_1^{-1} R_1 = R_1^2/B_1$.

This is of the same form as LM_ρ for testing $H_0^b : \rho = 0$ (assuming no random effects, i.e., $\sigma_\mu^2 = 0$). However, R_1 and B_1 are now different from R and B. In fact, they are based on different restricted ML residuals, namely $\hat{u}$, those of a random effects panel data model with no spatial lag dependence, see Baltagi (2005), rather than the OLS residuals $\tilde{u}$.

3.4 Conditional LM Test for $H_0^e : \sigma_\mu^2 = 0$ (assuming ρ may or may not be zero)

This section derives the conditional LM test for no random region effects given the existence of spatial lag dependence. The null hypothesis is $H_0^e : \sigma_\mu^2 = 0$ (assuming that ρ may not be zero). Under the null, the score equations are given by

$$
\begin{aligned}
\frac{\partial L}{\partial \sigma_\mu^2}|_{H_0^e} &= -\frac{1}{2}NT\bar{\sigma}_\nu^{-2} + \frac{1}{2}T\bar{\sigma}_\nu^{-4}\left[\bar{u}'P\bar{u}\right] \\
\frac{\partial L}{\partial \beta}|_{H_0^e} &= \bar{\sigma}_\nu^{-2}X'\bar{u} = 0 \\
\frac{\partial L}{\partial \sigma_\nu^2}|_{H_0^e} &= -\frac{1}{2}NT\bar{\sigma}_\nu^{-2} + \frac{1}{2}\bar{\sigma}_\nu^{-4}\bar{u}'\bar{u} = 0 \\
\frac{\partial L}{\partial \rho}|_{H_0^e} &= -T \cdot tr\left[A^{-1}W\right] + \bar{\sigma}_\nu^{-2}\bar{u}'\left(I_T \otimes W\right)y = 0
\end{aligned}
$$

Under the null hypothesis H_0^e, the variance-covariance matrix reduces to $\sigma_\nu^2 I_{NT}$ and the restricted MLE of β and ρ are in fact the MLE of a spatial lag model with no random effects, see Anselin (1988a). These are denoted by $\overline{\beta}$ and $\overline{\rho}$. Here, $\bar{\sigma}_\nu^2 = \bar{u}'\bar{u}/NT$, with $\bar{u} = y - \overline{\rho}\left(I_T \otimes W\right)y - X\overline{\beta}$. Therefore, the score with respect to $\theta' = (\sigma_\mu^2, \beta', \sigma_\nu^2, \rho)$, evaluated under the null hypothesis H_0^d, is given by

$$
\bar{D}_\theta = \begin{pmatrix} \bar{D}_{\sigma_\mu^2} \\ \bar{D}_\beta \\ \bar{D}_{\sigma_\nu^2} \\ \bar{D}_\rho \end{pmatrix} = \begin{pmatrix} \frac{NT}{2\bar{\sigma}_\nu^2}\left(T\frac{\bar{u}'P\bar{u}}{\bar{u}'\bar{u}} - 1\right) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{NT}{2\bar{\sigma}_\nu^2}G_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

where $G_1 = \left(T\frac{\bar{u}'P\bar{u}}{\bar{u}'\bar{u}} - 1\right)$. Under H_0^e, the elements of the information matrix J are given by:

$$\begin{aligned}
E\left[-\frac{\partial^2 L}{\partial\sigma_\mu^2\partial\sigma_\mu^2}\right]|_{H_0^e} &= -\frac{1}{2}NT^2\overline{\sigma}_\nu^{-4} + T^2\overline{\sigma}_\nu^{-6}E\left[u'Pu\right] = \frac{1}{2}NT^2\overline{\sigma}_\nu^{-4} \\
E\left[-\frac{\partial^2 L}{\partial\sigma_\mu^2\partial\beta}\right]|_{H_0^e} &= T\overline{\sigma}_\nu^{-4}E\left[X'Pu\right] = 0 \\
E\left[-\frac{\partial^2 L}{\partial\sigma_\mu^2\partial\sigma_\nu^2}\right]|_{H_0^e} &= -\frac{1}{2}NT\overline{\sigma}_\nu^{-4} + T\overline{\sigma}_\nu^{-6}E\left[u'Pu\right] = \frac{1}{2}NT\overline{\sigma}_\nu^{-4} \\
E\left[-\frac{\partial^2 L}{\partial\sigma_\mu^2\partial\rho}\right]|_{H_0^e} &= T\overline{\sigma}_\nu^{-4}E\left[u'P\left(I_T\otimes W\right)y\right] = T\overline{\sigma}_\nu^{-4}E\left[tr\left(uu'\left(\bar{J}_T\otimes WA^{-1}\right)\right)\right] = T\overline{\sigma}_\nu^{-2}tr\left(WA^{-1}\right) \\
E\left[-\frac{\partial^2 L}{\partial\beta\partial\beta'}\right]|_{H_0^e} &= \overline{\sigma}_\nu^{-2}X'X \\
E\left[-\frac{\partial^2 L}{\partial\beta\partial\sigma_\nu^2}\right]|_{H_0^e} &= \overline{\sigma}_\nu^{-4}E\left[X'u\right] = 0 \\
E\left[-\frac{\partial^2 L}{\partial\beta\partial\rho}\right]|_{H_0^e} &= \overline{\sigma}_\nu^{-2}E\left[X'\left(I_T\otimes W\right)y\right] = \overline{\sigma}_\nu^{-2}X'\left(I_T\otimes WA^{-1}\right)X\overline{\beta} \\
E\left[-\frac{\partial^2 L}{\partial\sigma_\nu^2\partial\sigma_\nu^2}\right]|_{H_0^e} &= -\frac{1}{2}NT\overline{\sigma}_\nu^{-4} + \overline{\sigma}_\nu^{-6}E\left[u'u\right] = \frac{1}{2}NT\overline{\sigma}_\nu^{-4} \\
E\left[-\frac{\partial^2 L}{\partial\sigma_\nu^2\partial\rho}\right]|_{H_0^e} &= \overline{\sigma}_\nu^{-4}E\left[u'\left(I_T\otimes W\right)y\right] = \overline{\sigma}_\nu^{-4}E\left[tr\left(uu'\left(I_T\otimes WA^{-1}\right)\right)\right] = T\overline{\sigma}_\nu^{-2}tr\left(WA^{-1}\right) \\
E\left[-\frac{\partial^2 L}{\partial\rho\partial\rho}\right]|_{H_0^e} &= T\cdot tr\left[\left(WA^{-1}\right)^2\right] + \overline{\sigma}_\nu^{-2}E\left[y'\left(I_T\otimes W'W\right)y\right] \\
&= T\cdot tr\left[\left(WA^{-1}\right)^2 + \left(WA^{-1}\right)'\left(WA^{-1}\right)\right] + \overline{\sigma}_\nu^{-2}\overline{\beta}'X'\left(I_T\otimes\left(WA^{-1}\right)'\left(WA^{-1}\right)\right)X\overline{\beta}
\end{aligned}$$

Therefore, the information matrix evaluated under H_0^e can be written as:

$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$

with $J_{11} = \begin{pmatrix} \frac{1}{2}NT^2\overline{\sigma}_\nu^{-4} & 0 \\ 0 & \overline{\sigma}_\nu^{-2}X'X \end{pmatrix}$,

$J_{12} = J_{21}' = \begin{pmatrix} \frac{1}{2}NT\overline{\sigma}_\nu^{-4} & T\overline{\sigma}_\nu^{-2}tr\left(WA^{-1}\right) \\ 0 & \left(\overline{\sigma}_\nu^{-2}X'\left(I_T\otimes WA^{-1}\right)X\overline{\beta}\right)' \end{pmatrix}$,

and $J_{22} = \begin{pmatrix} \frac{1}{2}NT\overline{\sigma}_\nu^{-4} & T\overline{\sigma}_\nu^{-2}tr\left(WA^{-1}\right) \\ T\overline{\sigma}_\nu^{-2}tr\left(WA^{-1}\right) & J_{\rho\rho} \end{pmatrix}$,

where $J_{\rho\rho} = T\cdot tr\left[\left(WA^{-1}\right)^2 + \left(WA^{-1}\right)'\left(WA^{-1}\right)\right] + \overline{\sigma}_\nu^{-2}\beta'X'\left(I_T\otimes\left(WA^{-1}\right)'\left(WA^{-1}\right)\right)X\overline{\beta}$. Using partitioned inversion, we know that the upper 2×2 block of the inverse matrix J^{-1} is given by $J^{11} = \left(J_{11} - J_{12}J_{22}^{-1}J_{21}\right)^{-1}$. After some tedious algebra, this can be derived as:

$$J^{11} = \begin{pmatrix} \frac{2\overline{\sigma}_\nu^4}{NT(T-1)} & 0 \\ 0 & \{\overline{\sigma}_\nu^{-2} X'X - \frac{1}{H}\frac{1}{2}NT\overline{\sigma}_\nu^{-4}\left[\overline{\sigma}_\nu^{-2}X'\left(I_T \otimes WA^{-1}\right)X\overline{\beta}\right]'\left[\overline{\sigma}_\nu^{-2}X'\left(I_T \otimes WA^{-1}\right)X\overline{\beta}\right]\}^{-1} \end{pmatrix}.$$

where $H = \frac{1}{2}NT\overline{\sigma}_\nu^{-4}\left(T \cdot tr\left[\left(WA^{-1}\right)^2 + \left(WA^{-1}\right)'\left(WA^{-1}\right)\right] + \overline{\sigma}_\nu^{-2}\overline{\beta}'X'\left(I_T \otimes \left(WA^{-1}\right)'\left(WA^{-1}\right)\right)X\overline{\beta}\right) - \left[T\overline{\sigma}_\nu^{-2}tr\left(WA^{-1}\right)\right]^2.$

We only need the first element of J^{11}. Therefore, the LM statistic for H_0^e is given by

$LM_{\mu/\rho} = \bar{D}'J^{-1}\bar{D} = \bar{D}_{\sigma_\mu^2}(\frac{2\overline{\sigma}_\nu^4}{NT(T-1)})\bar{D}_{\sigma_\mu^2} = \left[\frac{NT}{2\overline{\sigma}_\nu^2}G_1\right]^2 \frac{2\overline{\sigma}_\nu^4}{NT(T-1)} = \frac{NT}{2(T-1)}G_1^2.$

This is of the same form as LM_μ for testing $H_0^c : \sigma_\mu^2 = 0$ (assuming no spatial lag dependence, i.e., $\rho = 0$). However, $G_1 = \left(T\frac{\bar{u}'P\bar{u}}{\bar{u}'\bar{u}} - 1\right)$ is based on different restricted ML residuals, $\overline{u}_t = y_t - \overline{\rho}Wy_t + X_t\overline{\beta}$, based on the MLE of a spatial lag model with no random effects, see Anselin (1988a), rather than the OLS residuals $\tilde{u}$.