Syracuse University
SUrface

Mathematics Faculty Scholarship | Department of Mathematics

4-5-2005

Local Theory of Almost Split Sequences for Comodules

William Chin
DePaul University, wchin@condor.depaul.edu

Mark Kleiner
Syracuse University, mkleiner@syr.edu

Declan Quinn
Syracuse University, dpquinn@mail.syr.edu

Follow this and additional works at: http://surface.syr.edu/mat

Part of the Mathematics Commons

Recommended Citation
Chin, William; Kleiner, Mark; and Quinn, Declan, "Local Theory of Almost Split Sequences for Comodules" (2005). *Mathematics Faculty Scholarship*. Paper 73.
http://surface.syr.edu/mat/73

This Article is brought to you for free and open access by the Department of Mathematics at SUrface. It has been accepted for inclusion in Mathematics Faculty Scholarship by an authorized administrator of SUrface. For more information, please contact surface@syr.edu.

arXiv:math/0504081v1 [math.RT] 5 Apr 2005

Local Theory of Almost Split Sequences for Comodules

William Chin, Mark Kleiner, and Declan Quinn

Abstract. We show that almost split sequences in the category of comodules over a coalgebra Γ with finite-dimensional right-hand term are direct limits of almost split sequences over finite dimensional subcoalgebras. In previous work we showed that such almost split sequences exist if the right hand term has a quasifinitely copresented linear dual. Conversely, taking limits of almost split sequences over finte-dimensional comodule categories, we then show that, for countable-dimensional coalgebras, certain exact sequences exist which satisfy a condition weaker than being almost split, which we call "finitely almost split". Under additional assumptions, these sequences are shown to be almost split in the appropriate category.

1. Introduction

In [CKQ] we approached the problem of proving the existence of almost split sequences for comodules following the approach used in [ARS]. In this paper we focus on the local theory and we study how almost split sequences relate to almost split sequences over finite dimensional subcoalgebras. Rather than the more functorial approach of [CKQ] and [Tak], we lean here toward using limits to obtain results from ones known in the finite-dimensional case. The results make use of techniques using duality and idempotents in the dual algebra. This approach is amply represented in other recent work on coalgebras, e.g. [DT, Si].

A fundamental, but surprisingly difficult, result in [CKQ] is the fact that the functor $*$ is a duality on quasifinite injectives. We begin by giving an easy proof of this result using the characterization of the cohom functor as the direct limit of duals of Hom's.

We show in 3.2 that the almost split sequences with finite-dimensional right-hand terms are direct limits of almost split sequences over some finite-dimensional subcoalgebras. This follows from our result in 3.1 which shows that the transpose of a comodule can be expressed as a direct limit of finite-dimensional transposes.

In section 4, we attempt to construct almost split sequences over a coalgebra Γ from such sequences over finite-dimensional subcoalgebras. We show that (4.3 Theorem 2), for countable-dimensional coalgebras, starting with a finite-dimensional comodule on the right, certain exact sequences exist which satisfy a condition similar to, but weaker than, being almost split. This type of sequence, which we call "right finitely almost split", was investigated by E. Green and E. Marcos [GM] in a different setting. The assumption on dimension seems to be a mild one, as any indecomposable coalgebra with a locally countable Ext-quiver (i.e., $\dim_{\mathrm{k}}\mathrm{Ext}^1_{-\Gamma}(S,T)$ is

countable for all simple comodules S, T) has countable dimension. The dimension assumption arises mainly from the need to construct a direct system of almost split sequences; to do this (in the proofs of Theorems 2 and 3) we need to work with a chain of finite-dimensional subcoalgebras whose union is Γ, a condition equivalent to Γ being of countable dimension. Under additional assumptions these sequences are shown to be almost split in the quasifinite comodule category in Theorem 2. This result is predicted by [CKQ, Corollary 4.3] along with our results in Section 1. In addition, the result from [CKQ] says that the sequences so obtained are almost split in the comodule category.

The situation in Theorem 2 dualizes. We start with a finite-dimensional comodule on the left and construct a left finitely almost split sequence in the category of prorational modules, which consists of inverse limits of finite-dimensional comodules. These modules form a dual category to the category of comodules and are developed in 4.2. As for the right-hand variant, these sequence are almost split under appropriate hypotheses. This result can predicted from the result (4.3, Theorem 4), that dualizes [CKQ, Corollary 4.3] by working in the prorational category.

The Mittag-Leffler condition concerns the exactness of the inverse limit. It is the main tool for showing that the sequences in Section 4 might be almost split. The relevant special case is discussed briefly in 4.1.

We close by presenting some examples of almost split sequences over path coalgebras. Our examples include almost split sequences, starting with finite-dimensional comodules on either the left or right, that may not be in the finite-dimensional comodule category. One of the examples exhibits how the direct limit of almost split sequences may fail to be almost split. More examples of almost split sequences and AR quivers for comodules appear in the recent article [KS].

Notation Let Γ denote a coalgebra over the fixed base field k. Set the following
$\mathcal{M}^\Gamma$ the category of right Γ-comodules.
$\mathcal{M}^\Gamma_f$ the category of finite-dimensional right Γ -comodules
$\mathcal{M}^\Gamma_q$ the category of quasifinite right Γ -comodules
$\mathcal{M}^\Gamma_{qc}$ the category of quasifinitely copresented right Γ -comodules
$\mathcal{I}^\Gamma$ the category of quasifinite injective left Γ-comodules
$R =$DΓ the dual algebra $Hom_k(\Gamma$,k)
$\mathcal{M}_R$ the category of right R-modules
$\mathcal{D}^\Gamma$ the category of duals of left Γ -comodules
$\mathcal{D}^\Gamma_q$ the category of duals of quasifinite left Γ -comodules

$h_{-\Gamma}(_,_)$ the cohom functor
$\square$ the cotensor product (over Γ)
D the linear dual $\mathrm{Hom}_k(_,\mathrm{k})$
$(-)^*$ the functor $h_{-\Gamma}(_,\Gamma)$.

We will use the obvious left-handed variants of these notations. We shall generally follow the conventions of [CKQ] an the books [Mo, Sw]. The socle of a comodule $M \in \mathcal{M}^\Gamma$ is denoted by soc(M). If $M = \Gamma$ then soc(M)=corad(Γ), the coradical. The *coefficient space* of M in Γ is denoted by cf(M).

M is said to be *quasifinite* if $\mathrm{Hom}_{-\Gamma}(F, M)$ is finite-dimensional for all $F \in \mathcal{M}_f^\Gamma$. Equivalently, the simple summands of $\mathrm{soc}(M)$ have finite (but perhaps unbounded) multiplicities [Ch].

An *injective copresentation* of M is an exact sequence $0 \to M \to I_0 \to I_1$, where I_0 and I_1 are injectives. The comodule M is said to be *quasifinitely copresented* if I_0 and I_1 can be chosen to be quasifinite.

Assume for the moment that Γ is a finite-dimensional, so that $\mathcal{M}_f^\Gamma \approx \mathrm{mod}\, R$. Let tr denote the usual transpose on $\mathrm{mod}\, R$, as in the representation theory of finite-dimensional algebras (defined using projective resolutions, see [ARS, §IV]). It is easy to see from duality that for finite-dimensional comodules that DTrD= tr, so we can say Dtr = TrD and trD=DTr.

2. Duality for quasifinite injective comodules

We define the contravariant functor $*$ as in [CKQ] as $h_{-\Gamma}(-, \Gamma) : \mathcal{M}_q^\Gamma \to \mathcal{M}^\Gamma$ (also the version on the left as well)

With $R = \mathrm{D}\Gamma$, we have the right and left hit actions of R on Γ (more generally on any Γ, Γ-bicomodule), usually denoted by the symbols $\leftharpoonup, \rightharpoonup$ as in e.g. [Mo, Sw]. Here we will omit these symbols and simply use juxtapostion, e.g., $e\Gamma = e \rightharpoonup \Gamma$, $e \in R$. Notice that Γe is an injective right comodule.

Theorem 1 (CKQ). *$*$ restricts to a duality on the category of quasifinite injective comodules $\mathcal{I}^\Gamma$.*

We first collect some facts concerning duality and injectives.

Lemma 1. *Let $e = e^2 \in R$. Then:*
(a) $\mathrm{D}(\Gamma e) \cong eR$ *as right R-modules.*
(b) If Γ is of finite dimension, then $D(eR) \cong \Gamma e$ as left R-modules
(c) $\mathrm{Hom}_{-\Gamma}(\Lambda, \Gamma e) = \mathrm{Hom}_{-\Gamma}(\Lambda, \Lambda e)$ *for every subcoalgebra $\Lambda \subset \Gamma$*
(d) $(\Gamma e)^* \cong e\Gamma$ *as left Γ-comodules.*

Proof. The statement in (a) is essentially the definition of the right hit action as dual to the left multiplication by R. We leave the details to the reader. The proof of (b) follows from (a) and the duality D for finite dimensional (co)modules.

For (c), observe that the image of any comodule map on the left-hand term has its image in $\Delta^{-1}(\Gamma e \otimes \Lambda) \cong \Gamma e \Box \Lambda$. The result follows immediately from the additivity of the cotensor product, as Γe is a summand of Γ.

Lastly we prove (d). Let Λ denote a finite-dimensional subcoalgebra of Γ. Then we have

$$\begin{aligned} Hom_{-\Gamma}(\Lambda, \Gamma e) &= \mathrm{Hom}_{-\Gamma}(\Lambda, \Lambda e) \\ &\cong \mathrm{Hom}_R(\mathrm{D}(\Lambda e), \mathrm{D}\Lambda) \\ &\cong \mathrm{Hom}_R(e(\mathrm{D}\Lambda), \mathrm{D}\Lambda) \\ &\cong (\mathrm{D}\Lambda)e \end{aligned} \tag{2.1}$$

by applying the part (c), the duality D, and part (a). Note too that we consider e to act on $\mathrm{D}\Lambda$ via the ring map $\mathrm{D}\Gamma \to \mathrm{D}\Lambda$ given by restriction. Now we obtain

$$\begin{aligned}(\Gamma e)^* &= \varinjlim \mathrm{D}Hom_{-\Gamma}(\Lambda, \Gamma e)\\ &\cong \varinjlim \mathrm{D}((D\Lambda)e)\\ &\cong \varinjlim e\Lambda\\ &\cong e\Gamma\end{aligned}$$

where the direct limits are over the finite-dimensional subcoalgebras Λ, using the definition of $*$, (1) above, and part (b). This completes the proof of the Lemma. □

Proof. (of the Theorem) Let I be a quasifinite injective comodule. Then I is the direct sum of indecomposable injectives, all of the form Γe, where is a primitive idempotent in R. Part (d) of the Lemma and its right-handed counterpart yield $I^{**} \cong I$, an isomorphism that is easily seen to be natural. □

3. Direct Limits

3.1. The Transpose. Let $M \in \mathcal{M}^{\Gamma}_{qc}$. The transpose $\mathrm{Tr}\, M \in {}^{\Gamma}\mathcal{M}$ is defined in [CKQ] to be by $0 \to \mathrm{Tr}\, M \to I_0^* \to I_1^*$ where $0 \to M \to I_0 \to I_1$is a minimal quasifinite injective copresentation of M. Actually, it is easy to see that that we can use any quasifinite injective copresentation of M to define $\mathrm{Tr}\, M$, i.e., we may omit the minimality requirement from the definition. For a finite-dimensional subcoalgebra $\Lambda \subset \Gamma$, $\mathrm{Tr}_{\Lambda}\, M$ denotes the transpose of $M \in \mathcal{M}^{\Lambda}_{qc}$

Lemma 2. *Let $M \in \mathcal{M}^{\Gamma}_{f}$ be a finite-dimensional quasifinitely copresented comodule. Then for every finite-dimensional subcoalgebra $\Lambda \subset \Gamma$ containing* $\mathrm{cf}(M)$

$$\mathrm{Tr}_{\Lambda}\, M \cong \Lambda \Box\, \mathrm{Tr}\, M$$

Proof. Let Λ be as in the statement. Let

$$0 \to M \to I_0 \Box \Lambda \to I_1 \Box \Lambda$$

be an injective copresentation of M. We obtain an injective copresentation

$$0 \to M \to I_0 \Box \Lambda \to I_1 \Box \Lambda$$

for M in $\mathcal{M}^{\Lambda}$. The defining copresentation for the transpose in $\mathcal{M}^{\Lambda}$ using "$*$ in $\mathcal{M}^{\Lambda}$" is

$$0 \to \mathrm{Tr}_{\Lambda}\, M \to h_{-\Lambda}(I_1 \Box \Lambda, \Lambda) \to h_{-\Lambda}(I_0 \Box \Lambda, \Lambda).$$

On the other hand, the defining copresentation

$$0 \to \mathrm{Tr}\, M \to h_{-\Gamma}(I_1, \Gamma) \to h_{-\Gamma}(I_0, \Gamma)$$

for M in $\mathcal{M}^{\Gamma}$ can be cotensored with Λ, yielding

$$0 \to \Lambda \Box\, \mathrm{Tr}\, M \to \Lambda \Box h_{-\Gamma}(I_1, \Lambda') \to \Lambda \Box h_{-\Gamma}(I_0, \Lambda)$$

In view of [Tak, 1.14(b)], which says that $\Lambda \Box h_{-\Gamma}(M, \Gamma) \cong h_{-\Lambda}(M, \Lambda)$, the result is established. □

Assume that Λ is a finite-dimensional coalgebra. We see next that the transpose commutes with certain direct limits: Let $\mathrm{tr}_{D\Lambda}$ and Tr_Λ denote the transposes in the module category ${}_{D\Lambda}\mathcal{M}$ and the comodule category $\mathcal{M}^\Lambda$, respectively. Since cotensoring commutes with direct limits and is left exact, it follows from the preceding lemma that

PROPOSITION 1. *Suppose that $M \in \mathcal{M}^\Gamma_{qc}$ finite-dimensional, indecomposable and not injective. Then*

$$\varinjlim_{\mathcal{L}} \mathrm{Tr}_\Lambda M = \mathrm{Tr}\, M$$

for the direct system $\mathcal{L}$ of finite-dimensional subcoalgebras of Γ.

REMARK 1. *The preceding result gives a way of constructing indecomposables which are unions of chains of indecomposable subcomodules. See Example 3 at the end of this paper for an example of such an infinite-dimensional indecomposable.*

We show next that the almost split sequences (e.g. those whose existence is provided by [CKQ]) are direct limits of almost split sequences over finite-dimensional subcoalgebras.

3.2. Almost Split Sequences as Direct Limits.

PROPOSITION 2. *Suppose that $C \in \mathcal{M}^\Gamma_{qc}$ is finite-dimensional. Let*

$$d : 0 \to A \to B \to C \to 0$$

be an almost split sequence in $\mathcal{M}^\Gamma$. Then d is the direct limit of almost split sequences $(d\Box\Lambda)_{\Lambda\in\mathcal{L}}$ for some direct system $\mathcal{L}$ of finite-dimensional subcoalgebras of Γ.

PROOF. Let Λ be any finite-dimensional coalgebra of Γ containing both the coefficient space of C, and a finite-dimensional subspace of B mapping onto C. It is clear that the sequence $d\Box\Lambda : B\Box\Lambda \to C\Box\Lambda = C \to 0$ is exact (onto C) for any finite-dimensional subcoalgebra $\Lambda \subseteq \Gamma$, provided $\Lambda \supseteq \Lambda'$. It is trivial to check that $B\Box\Lambda \to C \to 0$ is right almost split in $\mathcal{M}^\Lambda$.

The direct system $\mathcal{L}$ of subcoalgebras is taken to be the finite-dimensional subcoalgebras Λ as just described, ordered by inclusion. The Lemma ensures that $A\Box\Lambda = \mathrm{Tr}_\Lambda D(C)$, and $\mathrm{Tr}_\Lambda D(C)$ is indecomposable by the finite-dimensional theory. This is the left-hand term of the right almost split sequence $d\Box\Lambda$, so by the standard result of Auslander (see [CKQ], 4.3 Proof) $\delta\Box\Lambda$ is in fact an almost split sequence in $\mathcal{M}^\Lambda$. The maps between the sequences is the obvious one using inclusions. $\Box$

4. Finitely Almost Split Sequences

4.1. The Mittag-Leffler Condition. For abelian groups, the direct limit (or filtered colimit) is an exact functor but (inverse) limits are left exact but not always right exact. The Mittag-Leffler condition [Gro] (see also [Ha, p.119]) guarantees that inverse limits of certain inverse systems of exact sequences are exact. It has the following special case:

PROPOSITION 3. *Let*

$$d_i : 0 \to M_i \to E_i \to N_i \to 0$$

be an inverse system of exact sequences of vector spaces where the index set has a countable cofinal subset. If each M_i is finite-dimensional, then

$$\lim_{\leftarrow} d_i = d : 0 \to M \to E \to N \to 0$$

is exact.

The reader may consult the references just mentioned, or [Je, Proposition 2.3] for a simple proof.

4.2. Prorational Modules. Define $\mathrm{D} : {}^{\Gamma}\mathcal{M} \to \mathcal{D}^{\Gamma}$ to be the functor D with range being the full subcategory of $\mathcal{M}_R$ having objects $\mathrm{D}M$, $M \in {}^{\Gamma}\mathcal{M}$. We let $\mathcal{D}^{\Gamma}_q$ denote the full subcategory of duals of quasifinite comodules. Note that the objects in $\mathcal{D}^{\Gamma}$ are precisely the inverse limits of finite-dimensional rational right R-modules. Accordingly the finite-dimensional objects in $\mathcal{D}^{\Gamma}$ are precisely the finite-dimensional rational right R-modules. We refer to the objects of $\mathcal{D}^{\Gamma}$ as *prorational* right R-modules. We shall freely use the variant of D (denoted by the same symbol) with opposite categories, e.g. $\mathrm{D} : \mathcal{M}^{\Gamma} \to {}^{\Gamma}\mathcal{D}$.

Definition 1. *We say that* $\mathrm{D}M \in \mathcal{D}^{\Gamma}$ *is* ***coquasifinite*** *if* $\mathrm{Hom}_R(\mathrm{D}M, F)$ *is finite-dimensional for all finite-dimensional* $F \in \mathcal{D}^{\Gamma}$.

The dual notion "quasifinite" was introduced in [Tak]. The objects of $\mathcal{D}^{\Gamma}_q$ are coquasifinite prorational right R-modules.

We show below that $\mathcal{D}^{\Gamma}$ (resp. $\mathcal{D}^{\Gamma}_q$) is the dual category of the category of (resp. quasifinite) comodules.

If $\rho : M \to M \otimes \Gamma$ is the structure map of $M \in \mathcal{M}^{\Gamma}$, then by restricting to $\mathrm{D}M \otimes R \subset \mathrm{D}(M \otimes R)$ and abusing notation, we obtain the map

$$\mathrm{D}\rho : \mathrm{D}M \otimes R \to \mathrm{D}M$$

It is straightforward to check that this coincides with the rational right R-module structure on $\mathrm{D}M$ which arises from the left $\Gamma-$comodule structure on $\mathrm{DD}M$, which in turn comes from ρ.

Note that the objects in $\mathcal{D}^{\Gamma}$ are precisely the inverse limits of finite-dimensional rational right R-modules. Accordingly the finite-dimensional objects in $\mathcal{D}^{\Gamma}$ are precisely the finite-dimensional rational right R-modules (i.e. left comodules). We refer to the objects of $\mathcal{D}^{\Gamma}$ as *prorational modules.*

In the following lemma, a finite-dimensional cotensor product of comodules is seen to be dual to the tensor product.

Lemma 3. *Let* $M \in \mathcal{M}^{\Gamma}$ *and* $N \in {}^{\Gamma}\mathcal{M}$ *and assume that* $M\Box N$ *is finite-dimensionsal. Then* $\mathrm{D}(M\Box N)$ *is isomorphic to* $\mathrm{D}M \otimes_R \mathrm{D}N$.

Proof. Let $\rho : M \to M \otimes \Gamma$ and $\lambda : N \longrightarrow \Gamma \otimes N$ be the structure maps of M and N respectively. Then as is noted above, $\mathrm{D}M$ is a left comodule and is a (rational) right R-module. Similarly $\mathrm{D}N$ is a left R-module.

The cotensor $M\Box N$ is defined by the usual equalizer

$$M\Box N \to M \otimes N \underset{1\otimes\lambda}{\overset{\rho\otimes 1}{\rightrightarrows}} M \otimes \Gamma \otimes N.$$

Dualizing, we have the coequalizer

$$\mathrm{D}(M \otimes \Gamma \otimes N) \underset{\mathrm{D}(1\otimes\lambda)}{\overset{\mathrm{D}(\rho\otimes 1)}{\rightrightarrows}} \mathrm{D}(M \otimes N) \xrightarrow{p} \mathrm{D}(M\Box_{\Gamma}N) \longrightarrow 0.$$

By hypothesis $M\Box_\Gamma N$ is finite-dimensional, so the density of $\mathrm{D}(M)\otimes\mathrm{D}(N)$ in $\mathrm{D}(M\otimes N)$ implies that the restriction $\mathrm{D}M\otimes\mathrm{D}N\longrightarrow\mathrm{D}(M\Box N)$ is onto. The kernel of this map is $\ker p\cap(\mathrm{D}M\otimes\mathrm{D}N)$. Thus we have the coequalizer

$$\mathrm{D}M\otimes\mathrm{D}\Gamma\otimes DN\underset{1\otimes\mathrm{D}\lambda}{\overset{\mathrm{D}\rho\otimes 1}{\rightrightarrows}}\mathrm{D}M\otimes DN\xrightarrow{p}\mathrm{D}(M\Box_\Gamma N).$$

This finishes the proof of the lemma. □

Let Λ denote a finite-dimensional subcoalgebra of Γ. The finite-dimensional algebra $\mathrm{D}\Lambda$ is isomorphic to $\mathrm{D}\Gamma/\Lambda^\perp$, where $\Lambda^\perp$ is the ideal of functionals in $R=\mathrm{D}\Gamma$ vanishing on Λ. The following is now immediate. Let $\mathcal{L}$ denote the direct system of finite-dimensional subcoalgebras of Γ. Lemma 2 immediately yields

Proposition 4. *Let M_Λ denote $M\Box\Lambda$, $M\in\mathcal{M}_q^\Gamma$. Then*
(a) $\mathrm{D}(M_\Lambda)\cong\mathrm{D}M\otimes_R\mathrm{D}\Lambda\cong\mathrm{D}M/M\Lambda^\perp$.
(b) $\mathrm{D}M\cong\varprojlim_{\mathcal{L}}\mathrm{D}M_\Lambda$.

Proposition 5. *(a)* D*:* ${}^\Gamma\mathcal{M}\to\mathcal{D}^\Gamma$ *is a duality*
(b) D restricts to a duality ${}^\Gamma\mathcal{M}_q\to\mathcal{D}_q^\Gamma$
(c) $\mathrm{D}M$ *is coquasifinite for all* $M\in{}^\Gamma\mathcal{M}_q$.

Proof. A functor D':$\mathcal{D}^\Gamma\to{}^\Gamma\mathcal{M}$ can be defined by setting $\mathrm{D}'\mathrm{D}M=M$, giving a correspondence on objects. Suppose $f\in\mathrm{Hom}_R(\mathrm{D}N,\mathrm{D}M)$. We put $\mathrm{D}'f=\varinjlim\mathrm{D}f_i$ where $f_i=f\otimes R_i$. One can check that the functor D' is an inverse duality for D. We are done with (a). Part (c) follows from the duality, and part (b) is clear. □

4.3. Finitely Almost Split Sequences.

Definition 2. *We say that a nonsplit exact sequence of objects* $0\to A\to B\to C\to 0$ *in* ${}_R\mathcal{M}$ *is* ***right finitely almost split*** *if*

1. C is indecomposable

2. every morphism $X\to C$ *with* $X\in\mathcal{M}_f^\Gamma$ *, which is not a split epimorphism, lifts to B.*

Dually, the sequence is said to be ***left finitely almost split*** *if A is indecomposable if every morphism* $A\to X$ *with* $X\in\mathcal{M}_f^\Gamma$*, which is not a split monomorphism, extends to B.*

These definitions are a coalgebraic version of the definition of finitely almost split sequences given for "local nests of quivers" given in [GM], where they are called "special sequences". Further extending their work we have

Theorem 2. *Let* Γ *be a countable-dimensional coalgebra. If* $C\in\mathcal{M}_f^\Gamma$ *is nonprojective and indecomposable, then there exists a right finitely almost split sequence*

$$d:0\to A\to B\to C\to 0$$

in $\mathcal{M}_q^\Gamma$*. If* $\mathrm{D}C$ *is quasifinitely copresented, then d can be chosen to be an almost split sequence in* $\mathcal{M}_q^\Gamma$.

Proof. We can write $\Gamma=\cup\Gamma_i$ as the ascending union of a chain of finite-dimensional subcoalgebras (not necessarily the coradical filtration) Γ_i where we may assume that $C\Box\Gamma_0=C$.

It is easy to check that C is not projective in $\mathcal{M}^{\Lambda_i}$ for all i. For, let $p : B \to C$ be a surjection in $\mathcal{M}^{\Gamma}$; then $B\Box\Lambda_i \to C\Box\Lambda_i = C$ is a surjection in $\mathcal{M}^{\Lambda_i}$. Were C projective in $\mathcal{M}^{\Lambda_i}$, then a splitting map $C \to B\Box\Lambda_i$ in $\mathcal{M}^{\Lambda_i}$ would also split p.

Let tr_i and Tr_i denote the transposes in the module category ${}_{\mathrm{D}\Lambda_i}\mathcal{M}$ and in $\mathcal{M}^{\Lambda_i}$, respectively, (as mentioned in the introduction). By results for finite-dimensional algebras, there exists an almost split sequence

$$d_i : 0 \to A_i \to B_i \to C \to 0$$

in $\mathcal{M}_f^{\Gamma_i}$ with $A_i{=}\mathrm{Dtr}_i\, C{=}\mathrm{Tr}_i\mathrm{D}C$, $i \in \mathbb{N}$. We obtain exact sequences indexed by $\mathbb{N}$, along with maps $d_i \to d_{i+1}$ obtained by assigning the identity map on C, and then a little diagram chasing using the fact that d_{i+1} is almost split. Furthermore, the maps $A_i \to A_{i+1}$ and $B_i \to B_{i+1}$are monomorphisms. This can be seen as follows. Suppose that $A_i \to A_{i+1}$ has kernel $K_i \neq 0$. Then we obtain an exact sequence

$$d_i' : 0 \to A_i' \to B_i' \to C \to 0$$

where $A_i' = A_i/K$ and $B_i' = B_i/K$, with epimorphisms $d_i \to d_i'$ (identity map on C). It follows easily from the fact that d_i is almost split that d_i' is split. But this immediately implies that d_i splits. Thus we obtain a direct system of exact sequences $(d_i)_{i\in\mathbb{N}}$ with monomorphic connecting maps $d_i \to d_{i+1}$, each being the identity on C.

The direct limit is an exact sequence

$$d : 0{\to}A{\to}B{\to}C{\to}0.$$

since the direct limit is an exact functor.

We show that d is right finitely almost split. The sequence is not split, for otherwise a splitting map $C \to B$ would have its image in B_i for some i (since C is finite-dimensional). This would then be a splitting of d_i, a contradiction. Let $X \in \mathcal{M}_f^{\Gamma}$, so that $X \in \mathcal{M}^{\Gamma_i}$ for some i. Then any morphism $X \to C$, which is not a split epimorphism, lifts to B_i. Composing with the inclusion map $B_i \to B$, we get the required lifting. This shows that d is right finitely almost split.

Now assume as in the statement that $\mathrm{D}C$ is quasifinitely copresented. To show that d is right almost split in $\mathcal{M}_q^{\Gamma}$, it suffices to show that the sequence $\mathrm{Hom}(X, d)$

$$0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C) \to 0$$

is exact for all $X \in \mathcal{M}_q^{\Gamma}$. Let $X_i = X\Box\Gamma_i$, which is a finite-dimensional comodule (since X is quasifinite). Note that $\mathrm{Hom}(X_i, d)$ is an inverse system of short exact sequences. By the Mittag-Leffler condition (Proposition 3), it suffices to show $\mathrm{Hom}(X_i, A)$ is finite-dimensional for all i. Thus it suffices to know that A is quasifinite. First, we have $A = \varinjlim A_i{=}\varinjlim \mathrm{Tr}_i\mathrm{D}C$ $=\mathrm{Tr}\mathrm{D}C$ (3.1 Lemma 2). Secondly, the rudimentary property of Tr [CKQ, Lemma 3.2] states in fact that $\mathrm{Tr}\mathrm{D}C$ is quasifinite (because $\mathrm{D}C$ is quasifinitely copresented).

By duality, $\mathrm{D}C$ in indecomposable and noninjective. By [CKQ, 3.2] and 3.1 Lemma 2, $A = \mathrm{Tr}\mathrm{D}C$ is also indecomposable noninjective. By a result of Auslander (see [CKQ, 4.3]), d is an almost split sequence.

This completes the proof of the Theorem. □

Remark 2. *The second statement of the theorem holds ([CKQ, Corollary 4.3]) without the assumption on dimension, and the almost sequences obtained are almost split in all of* $\mathcal{M}^{\Gamma}$

The argument of the theorem dualizes.

THEOREM 3. *Let Γ be a countable-dimensional coalgebra. Let $A \in \mathcal{M}_f^\Gamma$. If A is noninjective and indecomposable, then there exists a left finitely almost split sequence*

$$d : 0 \to A \to B \to C \to 0$$

in $\mathcal{D}_q^\Gamma$. If A is quasifinitely copresented, then d can chosen to be an almost split sequence in $\mathcal{D}_q^\Gamma$.

PROOF. Since the proof is dual to the one above we only give sample details. Assume as in the statement that A is quasifinitely copresented and we have a left finitely almost split sequence

$$d : 0 \to A \to B \to C \to 0$$

in $\mathcal{D}_q^\Gamma$. Here d is obtained as the inverse limit of sequences $d_i : 0 \to A \to B_i \to C_i \to 0$, that are almost split in $\mathcal{D}_q^{\Gamma_i}$. To show that d is left almost split in $\mathcal{M}_q^\Gamma$, it suffices to show that the sequence $\mathrm{Hom}(d, Y)$ is exact for all $Y \in \mathcal{D}_q^\Gamma$. We have $Y = \varprojlim Y_i$ where the Y_i are finite dimensional rational R-modules. According to the Mittag-Leffler condition, it suffices to show $\mathrm{Hom}(A, Y_i)$ is finite-dimensional for all i. Thus we want to know that A is coquasifinite. First, we have

$$C = \varprojlim C_i = \varprojlim \mathrm{D}\,\mathrm{Tr}_i\, A = \mathrm{D} \varinjlim CA = \mathrm{D}\,\mathrm{Tr}\, A$$

Secondly [CKQ, Lemma 3.2] states that $\mathrm{Tr}\, A$ is quasifinite. Therefore $C = \mathrm{D}\,\mathrm{Tr}\, A$ is coquasifinite.

By [CKQ, 3.2] and 3.1 Lemma 2, $\mathrm{Tr}\, A$ is an indecomposable noninjective comodule. Hence $C = \mathrm{D}\,\mathrm{Tr}\, A$ is indecomposable and nonprojective in $\mathcal{D}_q^\Gamma$. □

By dualizing [CKQ], Corollary 4.3(b), we obtain a result extending the second statement of the Thereom above. We leave the details to the reader.

THEOREM 4. *Let $A \in \mathcal{M}_f^\Gamma$. If A is quasifinitely copresented, indecomposable and not injective, then exists an almost split sequence $0 \to A \to B \to C \to 0$ in $\mathcal{D}^\Gamma$ with $C = \mathrm{D}\,\mathrm{Tr}\, A$.*

REMARK 3. *With A in the hypothesis, DA is coquasifinitely presented. This means that there exists a projective resolution $DI_1 \to \mathrm{D}I_0 \to \mathrm{D}A \to 0$ in $\mathcal{D}^\Gamma$, with the DI_i coquasifinite. If I is a finitely cogenerated injective comodule (see [Ch]), then $\mathrm{D}I$ is a projective as an R-module. But if I is only assumed to be quasifinite injective, we do not know this to be the case.*

REMARK 4. *The construction of almost split sequences as limits gives a way of showing that the category of finite-dimensional comodules does not have almost split sequences. Let C be a finite-dimensional comodule such that $\mathrm{Tr}\,\mathrm{D}C$ is infinite-dimensional. Then there is no almost split sequence in $\mathcal{M}_f^\Gamma$ ending in C. Dually, if $\mathrm{Tr}\, A$ is infinite-dimensional, then there is no almost split sequence starting with A in $\mathcal{M}_{fd}^\Gamma$. An example of this type is given below. On the other hand, if $\mathcal{M}_f^\Gamma$ has almost split sequences (e.g. if Γ is right semiperfect, see [CKQ]), then the Auslander-Reitan quiver exists. Examples of AR-quivers for $\mathcal{M}_f^\Gamma$ for various path coalgebras are given in [NS].*

5. Examples

The coalgebras in the next two examples are neither right nor left semiperfect, and finite-dimensional indecomposables have finite-dimensional transposes. The almost split sequences are almost split in the category of finite-dimensional comodules. In these examples, the almost split sequence obtained is the same as the one obtained over a finite-dimensional subcoalgebra (i.e. over a finite subquiver).

In contrast, the third example shows that a simple comodule can have an infinite-dimensional transpose. So the category of finite-dimensional comodules does not have almost split sequences.

1. Let Q be the quiver of type $\mathbb{A}_\infty^\infty$ with vertices labeled by the integers and arrows $a_i : i \to i+1$, $i \in \mathbb{Z}$. We write $S(i)$ for the simple in $\mathcal{M}^{kQ}$ corresponding to each vertex i and denote its injective hull by $I(i)$

By the theory of Nakayama algebras and Dynkin quivers (see [ARS] and [Ga]), the isomorphism classes of finite-dimensional indecomposable comodules are given by the representations $V_{ij} = (V, f)_{ij}$ (for all $i \leq j$) defined by $V_t =$k for $i \leq t \leq j$, zero otherwise; the linear maps are $f_t : V_t \to V_{t+1}$ given by $f_t = 1$ for $i \leq t < j$ and zero otherwise. We compute injective envelope of V_{ij} to be

$$I(V_{ij}) = I(j)$$

and we find that $I(V_{ij})/V_{ij} \cong I(i-1)$. Thus

$$0 \to \mathrm{Tr}(V_{ij}) \to I(i-1)^* \to I(j-1)^*$$

is a copresentation yielding $\mathrm{DTr}(V_{ij}) = V_{i-1,j-1}$. The almost split sequences are

$$0 \to V_{i,j} \to V_{i-1,j} \oplus V_{i,j-1} \to V_{i-1,j-1} \to 0$$

with irreducible maps $V_{i,j} \to V_{i,j+1}$ and $V_{i-1,j} \to V_{i,j}$ being the obvious monomorphism into the first summand and epimorphism onto the second summand. The map on the right is given by natural epimorphism and monomorphism with alternate signs.

2. Similarly, let Q denote the quiver with one vertex and one loop. Then there is a unique finite-dimensional indecomposable right comodule V_n of dimension $n \geq 0$. It is straightforward to see that $\mathrm{DTr}\, V_n = V_n$ and that the almost split sequences are given by

$$d_n : 0 \to V_n \to V_{n-1} \oplus V_{n+1} \to V_n \to 0$$

(just as in [ARS, p. 141]). Following the ideas in this article. one might hope to take the limit of these sequences to get and almost split sequence for the infinite-dimensional indecomposable $V = \varinjlim V_n \cong kQ$. Unfortunately, the sequence so obtained is split.

3. Let Q be the quiver of type $\mathbb{D}_\infty$ with vertices labeled by positive integers and two special vertices $0, 0\prime$. The arrows are defined to be $a_i : i \to i+1$, $i \in \mathbb{Z}^+$, and $a_0 : 0 \to 1$, $a_{0'} : 0' \to 1$.

We compute the transpose of the simple right noninjective comodule $S(1)$.

$I(1)$ is given by the representation $V = (V, f)$ defined by $V_t =$ k for $t = 1, 0, 0'$ and zero otherwise, with linear maps are $f_t : V_{t+1} \to V_t$ given by $f_t = 1$ for $t = 0, 0'$. Thus $I(1)/S(1)$ is $S(0) \oplus S(0)'$, the direct sum of two simple injectives.

Next observe that $I(S(0)\oplus S(0)')^* = I(0)^*\oplus I(0')^*$ is given by the representation $V = (V, f)$ with by $V_t = \mathrm{k}$ for $t = 0, 0'$, $V_t = \mathrm{k}^2$ for $t > 0$, and zero otherwise. We leave the maps to the reader.

$I(1)^*$is given by the representation $V = (V, f)$ with by $V_t = 0$ for $t = 0, 0'$, $V_t = \mathrm{k}$ for $t > 0$, and zero otherwise.

Finally, $0 \to \mathrm{Tr}(S(1)) \to I(S(0) \oplus S(0)')^* \to I(1)^*$ is a copresentation yielding $\mathrm{Tr}(1) = V$ where the representation $V = (V, f)$ of Q^{op} is defined by $V_t = \mathrm{k}$ for all vertices t, with linear maps are $f_t : V_{t+1} \to V_t$ given by $f_t = 1$ for all t. We know that V is an infinite-dimensional indecomposable left comodule; thus $\mathcal{M}_f^{\mathrm{k}Q}$ does not have an almost split sequence *starting* at $S(1)$. The almost split sequence in $\mathcal{D}_q^{kQ}$ given by the Theorem 4 is of the form

$$0 \to S(1) \to B \to \mathrm{D}V \to 0$$

.

References

[ARS] M. Auslander, I. Reiten, S. O. Smalø,. Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, 1997.

[Ch] W. Chin, A Brief introduction to coalgebra representation theory, in: Hopf Algebras (Conf. Proceedings, DePaul Univ.) Lecture Notes in Pure and Applied Math., vol. 237, 109-131, Marcel Dekker Inc., New York 2004.

[CKQ] W. Chin, M. Kleiner and D. Quinn, Almost split sequences for comodules, J. Alg. 249 (2002) no. 1, 1-19.

[DT] J. Cuadra and J. Gomez-Torrecillas, Idempotents and Morita-Takeuchi theory, Comm. Algebra 30(2002), 2405-2426.

[Ga] P. Gabriel, Unzerlegbare darstellungen I, Manuscripta Math. 6(1972), 71-103.

[Gr] J.A. Green, Locally finite representations, J. of Algebra, 41 (1976), 137-171.

[GM] E. Green and E. Marcos, Graded quotients of path algebras: a local theory. J. Pure Appl. Algebra 93 (1994), no. 2, 195–226.

[Gro] J. Dieudonné and A. Grothendieck, Eléments de Géometrie Algébrique III, Publ. Math. IHES 11(1961).

[Ha] R. Hartshorne, Algebraic Geometry, Springer GTM, 1977.

[Je] C.U. Jensen; Les foncteurs derivés de $\varprojlim$ et leurs applications en théorie des modules, Springer Lecture Notes in Math 254, 1972.

[KS] J. Kosakowska and D. Simson, Hereditary coalgebras and representations of species, J. Algebra (to appear).

[Mo] S. Montgomery, Hopf Algebras and their Actions on Rings, AMS, CBMS No. 82, 1993.

[Si] D. Simson, Coalgebras, comodules, psuedocompact algebras and tame comodule type, Colloq. Math. 90(2001), 101-150.

[SN] D. Simson and S. Nowak, Locally finite Dynkin quivers and hereditary coalgebras whose left comodules are direct sums of finite dimensional comodules, Comm. Alg., 30 (2002) 455-476.

[Sw] M. Sweedler, Hopf Algebras, Benjamin, New York 1969

[Tak] M. Takeuchi, Morita Theorems for categories of comodules, J. Fac. Sci. Univ. Tokyo 24 (1977), 629-644.

DePaul University, Chicago, Illinois 60614
E-mail address: wchin@condor.depaul..edu
URL: http://www.depaul.edu/~wchin

Syracuse University, Syracuse, NY 13244
E-mail address: mkleiner@mail.syr.edu

Syracuse University, Syracuse, NY 13244
E-mail address: dpquinn@mail.syr.edu