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**PREPROJECTIVE REPRESENTATIONS OF VALUED QUIVERS
AND REDUCED WORDS IN THE WEYL GROUP OF A
KAC-MOODY ALGEBRA**

MARK KLEINER AND ALLEN PELLEY

Dedicated to the memory of Andrei Vladimirovich Roiter

ABSTRACT. This paper studies connections between the preprojective representations of a valued quiver, the (+)-admissible sequences of vertices, and the Weyl group by associating to each preprojective representation a canonical (+)-admissible sequence. A (+)-admissible sequence is the canonical sequence of some preprojective representation if and only if the product of simple reflections associated to the vertices of the sequence is a reduced word in the Weyl group. As a consequence, for any Coxeter element of the Weyl group associated to an indecomposable symmetrizable generalized Cartan matrix, the group is infinite if and only if the powers of the element are reduced words. The latter strengthens known results of Howlett, Fomin-Zelevinsky, and the authors.

INTRODUCTION

Let \mathcal{W} be a Coxeter group generated by reflections $\sigma_1, \dots, \sigma_n$ [3], and let c be any Coxeter element of \mathcal{W} , i.e., $c = \sigma_{x_n} \dots \sigma_{x_1}$ where x_1, \dots, x_n is any permutation of the numbers $1, \dots, n$. Recall that an indecomposable symmetrizable generalized Cartan matrix (see [9]) is an integral $n \times n$ matrix $A = (a_{ij})$ with $a_{ii} = 2$ and $a_{ij} \leq 0$ for $i \neq j$ that is not conjugate under a permutation matrix to a block-diagonal matrix $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ and such that there exist integers $d_i > 0$ satisfying $d_i a_{ij} = d_j a_{ji}$ for all i, j . We identify the root lattice Q associated with A with the free abelian group \mathbb{Z}^n by identifying the simple roots $\alpha_1, \dots, \alpha_n$ of Q with the standard basis vectors e_1, \dots, e_n of \mathbb{Z}^n . Then the simple reflections on Q identify with the reflections $\sigma_1, \dots, \sigma_n$ on \mathbb{Z}^n given by $\sigma_i(e_j) = e_j - a_{ij}e_i$ for all i, j , and the Weyl group $\mathcal{W}(A)$ is the subgroup of $GL(\mathbb{Z}^n)$ generated by $\sigma_1, \dots, \sigma_n$. Andrei Zelevinsky brought to our attention the following two results. Howlett proved that \mathcal{W} is infinite if and only if c has infinite order [8, Theorem 4.1]. Fomin and Zelevinsky proved the following. Let A be bipartite, i.e., the set $\{1, \dots, n\}$ is a disjoint union of nonempty subsets I, J and, for $h \neq l$, $a_{hl} = 0$ if either $h, l \in I$ or $h, l \in J$. If $c = \prod_{i \in I} \sigma_i \prod_{j \in J} \sigma_j$, then $\mathcal{W}(A)$ is infinite if and only if the powers of c are reduced words in the σ_h 's [7, Corollary 9.6]. Inspired by the latter, we proved [10, Theorem 4.8] that if A is symmetric and c is any Coxeter element, then $\mathcal{W}(A)$ is infinite if and only if the powers of c are reduced words, which strengthens the aforementioned

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results of Howlett and Fomin-Zelevinsky in the case when $\mathcal{W} = \mathcal{W}(A)$ and A is symmetric. In this paper we obtain a further strengthening (Theorem 2.8) by removing the additional assumption that A is symmetric.

The proof of the indicated result from [10] was based on the interplay between the category $\tilde{\mathcal{P}}$ of preprojective modules ((+)-irregular representations) over the path algebra of a finite connected quiver without oriented cycles, the set \mathfrak{S} of (+)-admissible sequences, and the Weyl group \mathcal{W} of the underlying (nonoriented) graph of the quiver, as defined by Bernstein, Gelfand, and Ponomarev [2]. We used the one-to-one correspondence between the finite connected graphs (with no loops but multiple edges allowed) and the indecomposable symmetric Cartan matrices, and relied on the study of $\tilde{\mathcal{P}}$ in terms of the combinatorics of \mathfrak{S} undertaken in [11]. Since there is a one-to-one correspondence between the connected valued graphs and the indecomposable symmetrizable Cartan matrices [4, p. 1], we replace graphs with valued graphs, representations of quivers with representations of valued quivers studied in [4], and rely on the extension of the results of [11] to representations of valued quivers provided in [12]. Our study of $\tilde{\mathcal{P}}$, \mathfrak{S} , and \mathcal{W} for a valued quiver is also of independent interest. We associate to any $M \in \tilde{\mathcal{P}}$ a canonical sequence $S_M \in \mathfrak{S}$ and consider, for each $S \in \mathfrak{S}$, the element $w(S) \in \mathcal{W}$ that is the composition of simple reflections associated to the vertices of S . Then if $M, N \in \tilde{\mathcal{P}}$ are indecomposable, we have $M \cong N$ if and only if $w(S_M) = w(S_N)$ (Theorem 2.2(b)). For all $S \in \mathfrak{S}$, there exists an $M \in \tilde{\mathcal{P}}$ satisfying $S \sim S_M$ if and only if the word $w(S) \in \mathcal{W}$ is reduced (Theorems 2.5 and 2.6). If the valued graph is not a Dynkin diagram of the type $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$, or G_2 , then for all $S \in \mathfrak{S}$, the word $w(S)$ is reduced and there exists an $M \in \tilde{\mathcal{P}}$ satisfying $S \sim S_M$ (Corollary 2.7). We note that the connection between Coxeter-sortable elements of \mathcal{W} and the canonical sequences in \mathfrak{S} associated to the modules in $\tilde{\mathcal{P}}$, as explained in [10, Section 5], holds for valued quivers. Nathan Reading introduced Coxeter-sortable elements for an arbitrary Coxeter group [15] and proved that if the group is finite, the set of Coxeter-sortable elements maps bijectively onto the set of clusters [6, 13] and onto the set of noncrossing partitions [14]. It follows that to each preprojective representation of a valued quiver correspond a cluster and a noncrossing partition, and we will study these clusters and noncrossing partitions in the future.

We leave it to the reader to dualize the results of this paper by replacing preprojective representations of valued quivers with preinjective ones, and (+)-admissible sequences with (-)-admissible sequences [4].

1. PRELIMINARIES

We recall some facts, definitions, and notation, using freely [1, 2, 4, 12]. A *graph* is a pair $\Gamma = (\Gamma_0, \Gamma_1)$, where Γ_0 is the set of vertices, and the set of edges Γ_1 consists of some two-element subsets of Γ_0 . A *valuation* \mathbf{b} of Γ is a set of integers $b_{ij} \geq 0$ for all pairs $i, j \in \Gamma_0$ where $b_{ii} = 0$; if $i \neq j$ then $b_{ij} \neq 0$ if and only if $\{i, j\} \in \Gamma_1$; and there exist integers $d_i > 0$ satisfying $d_i b_{ij} = d_j b_{ji}$ for all $i, j \in \Gamma_0$. The pair (Γ, \mathbf{b}) is a *valued graph*, which is called *connected* if Γ is connected.

An orientation, Λ , on Γ consists of two functions $s : \Gamma_1 \rightarrow \Gamma_0$ and $e : \Gamma_1 \rightarrow \Gamma_0$. If $a \in \Gamma_1$, then $s(a)$ and $e(a)$ are the vertices incident with a , called the starting point and the endpoint of a , respectively. The triple $(\Gamma, \mathbf{b}, \Lambda)$ is a *valued quiver*, and a is then called an arrow of the quiver. Given a sequence of arrows a_1, \dots, a_t , $t > 0$, satisfying $e(a_i) = s(a_{i+1})$, $0 < i < t$, one forms a path $p = a_t \dots a_1$ of length

t in $(\Gamma, \mathbf{b}, \Lambda)$ with $s(p) = s(a_1)$ and $e(p) = e(a_t)$. By definition, for all $x \in \Gamma_0$ there is a unique path e_x of length 0 with $s(e_x) = e(e_x) = x$. We say that p is a path from $s(p)$ to $e(p)$ and write $p : s(p) \rightarrow e(p)$. A path p of length > 0 is an oriented cycle if $s(p) = e(p)$. The set of vertices of any valued quiver without oriented cycles (no finiteness assumptions) becomes a partially ordered set (poset) if one puts $x \leq y$ whenever there is a path from x to y . If $(\Gamma, \mathbf{b}, \Lambda)$ has no oriented cycles, we denote this poset by (Γ_0, Λ) . Throughout, all orientations Λ, Θ , etc., are such that $(\Gamma, \mathbf{b}, \Lambda), (\Gamma, \mathbf{b}, \Theta)$, etc., have no oriented cycles.

To define representations of a valued quiver $(\Gamma, \mathbf{b}, \Lambda)$, one has to choose a *modulation* \mathfrak{B} of the valued graph (Γ, \mathbf{b}) , which by definition is a set of division rings $\mathbf{k}_i, i \in \Gamma_0$, together with a $\mathbf{k}_i - \mathbf{k}_j$ -bimodule ${}_i B_j$ and a $\mathbf{k}_j - \mathbf{k}_i$ -bimodule ${}_j B_i$ for each $\{i, j\} \in \Gamma_1$ such that

- (i) there are $\mathbf{k}_j - \mathbf{k}_i$ -bimodule isomorphisms

$${}_j B_i \cong \text{Hom}_{\mathbf{k}_i}({}_i B_j, \mathbf{k}_i) \cong \text{Hom}_{\mathbf{k}_j}({}_i B_j, \mathbf{k}_j)$$

and

- (ii) $\dim_{\mathbf{k}_i}({}_i B_j) = b_{ij}$.

Unless indicated otherwise, for the rest of the paper we denote by Γ an arbitrary finite connected valued graph with $|\Gamma_0| > 1$, where $|X|$ stands for the cardinality of a set X , and with a valuation \mathbf{b} and modulation \mathfrak{B} ; denote by (Γ, Λ) the corresponding valued quiver with orientation Λ ; and assume that $\dim_k \mathbf{k}_i < \infty$ for all i , where k is a common central subfield of the \mathbf{k}_i 's acting centrally on all bimodules ${}_i B_j$. Under the assumption, each ${}_i B_j$ is a finite dimensional k -space.

A (left) representation (V, f) of (Γ, Λ) is a set of finite dimensional left \mathbf{k}_i -spaces $V_i, i \in \Gamma_0$, together with \mathbf{k}_j -linear maps $f_a : {}_j B_i \otimes_{\mathbf{k}_i} V_i \rightarrow V_j$ for all arrows $a : i \rightarrow j$. Defining morphisms of representations in a natural way, one obtains the category $\text{Rep}(\Gamma, \Lambda)$ of representations of the valued quiver (Γ, Λ) .

Putting $\mathbf{k} = \prod_{i \in \Gamma_0} \mathbf{k}_i$ and viewing $B = \bigoplus_{i \rightarrow j} {}_j B_i$ as a \mathbf{k} - \mathbf{k} -bimodule where \mathbf{k} acts on ${}_j B_i$ from the left via the projection $\mathbf{k} \rightarrow \mathbf{k}_j$ and from the right via the projection $\mathbf{k} \rightarrow \mathbf{k}_i$, one forms the tensor ring $T(\mathbf{k}, B) = \bigoplus_{n=0}^{\infty} B^{(n)}$ where $B^{(n)} = B \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} B$ is the n -fold tensor product, and the multiplication is given by the isomorphisms $B^{(n)} \otimes B^{(m)} \rightarrow B^{(n+m)}$ [5, p. 386]. Since (Γ, Λ) has no oriented cycles, $T(\mathbf{k}, B)$ is a finite dimensional k -algebra and we denote it by $k(\Gamma, \Lambda)$. Let $e_i \in \mathbf{k}$ be the n -tuple that has 1 in the i th place and 0 elsewhere. A left $k(\Gamma, \Lambda)$ -module M is *finite dimensional* if $\dim_{\mathbf{k}_i} e_i M < \infty$ for all i , which is equivalent to $\dim_k M < \infty$. We let f.d. $k(\Gamma, \Lambda)$ denote the category of finite dimensional left $k(\Gamma, \Lambda)$ -modules. The categories $\text{Rep}(\Gamma, \Lambda)$ and f.d. $k(\Gamma, \Lambda)$ are equivalent [5, Proposition 10.1] and we view the equivalence as an identification. In this paper all $k(\Gamma, \Theta)$ -modules are finite dimensional, for any orientation Θ without oriented cycles.

If (Γ, Λ) is a valued quiver and $x \in \Gamma_0$, let $\sigma_x \Lambda$ be the orientation on Γ obtained by reversing the direction of each arrow incident with x and preserving the directions of the remaining arrows. There results a new valued quiver $(\Gamma, \sigma_x \Lambda)$ (with the same valuation \mathbf{b} and modulation \mathfrak{B}). A vertex x is a *sink* if no arrow starts at x . For a sink x , we recall the definition of the *reflection* functor $F_x^+ : \text{Rep}(\Gamma, \Lambda) \rightarrow \text{Rep}(\Gamma, \sigma_x \Lambda)$ [4, pp. 15-16].

Let $(V, f) \in \text{Rep}(\Gamma, \Lambda)$ and let $(W, g) = F_x^+(V, f)$. Then $W_y = V_y$ for all $y \neq x$, and $g_b = f_b$ for the arrows b of $(\Gamma, \sigma_x \Lambda)$ that do not start at x . Let $a_i : y_i \rightarrow x, i = 1, \dots, l$, be the arrows of (Γ, Λ) ending at x . Then the reversed arrows $a'_i : x \rightarrow y_i$

are the arrows of $(\Gamma, \sigma_x \Lambda)$ starting at x . Consider the exact sequence

$$0 \rightarrow \text{Ker } h \xrightarrow{j} \bigoplus_{i=1}^l {}_x B_{y_i} \otimes_{\mathbf{k}_{y_i}} V_{y_i} \xrightarrow{h} V_x$$

of \mathbf{k}_x -spaces where h is induced by the maps $f_{a_i} : {}_x B_{y_i} \otimes_{\mathbf{k}_{y_i}} V_{y_i} \rightarrow V_x$. Then $W_x = \text{Ker } h$ and each map $g_{a'_i} : {}_{y_i} B_x \otimes_{\mathbf{k}_x} W_x \rightarrow W_{y_i} = V_{y_i}$ is obtained from the map $W_x \rightarrow {}_x B_{y_i} \otimes_{\mathbf{k}_{y_i}} W_{y_i}$ induced by j using the isomorphisms below [4, pp. 14-15].

$$\begin{aligned} \text{Hom}_{\mathbf{k}_x}(W_x, {}_x B_{y_i} \otimes_{\mathbf{k}_{y_i}} W_{y_i}) &\cong \text{Hom}_{\mathbf{k}_x}(W_x, \text{Hom}_{\mathbf{k}_{y_i}}({}_{y_i} B_x, \mathbf{k}_{y_i}) \otimes_{\mathbf{k}_{y_i}} W_{y_i}) \cong \\ &\text{Hom}_{\mathbf{k}_x}(W_x, \text{Hom}_{\mathbf{k}_{y_i}}({}_{y_i} B_x, W_{y_i})) \cong \text{Hom}_{\mathbf{k}_{y_i}}({}_{y_i} B_x \otimes_{\mathbf{k}_x} W_x, W_{y_i}) \end{aligned}$$

A sequence of vertices $S = x_1, x_2, \dots, x_s$, $s \geq 0$, is called *(+)-admissible* on (Γ, Λ) if it either is empty, or satisfies the following conditions: x_1 is a sink in (Γ, Λ) , x_2 is a sink in $(\Gamma, \sigma_{x_1} \Lambda)$, and so on. We denote by \mathfrak{S} the set of *(+)-admissible* sequences on (Γ, Λ) .

The following is [12, Definition 1.1] and [10, Definition 2.1].

Definition 1.1. If $S = x_1, \dots, x_s$, $s \geq 0$, is in \mathfrak{S} , we write $\Lambda^S = \sigma_{x_s} \dots \sigma_{x_1} \Lambda$ and, in particular, $\Lambda^\emptyset = \Lambda$. The *support* of S , $\text{Supp } S$, is the set of distinct vertices among x_j , $1 \leq j \leq s$. The *length* of S is $\ell(S) = s$; the *multiplicity* of $v \in \Gamma_0$ in S , $m_S(v)$, is the (nonnegative) number of subscripts j satisfying $x_j = v$. A sequence $K \in \mathfrak{S}$ is *complete* if $m_K(v) = 1$ for all $v \in \Gamma_0$. If $S = x_1, \dots, x_s$ and $T = y_1, \dots, y_t$ are *(+)-admissible* on (Γ, Λ) and (Γ, Λ^S) , respectively, the concatenation of S and T is the sequence $ST = x_1, \dots, x_s, y_1, \dots, y_t$ in \mathfrak{S} . If K is complete, then $\Lambda^K = \Lambda$, so the concatenation K^m of $m > 0$ copies of K is in \mathfrak{S} .

We quote [12, Definition 1.2].

Definition 1.2. If $S = x_1, \dots, x_i, x_{i+1}, \dots, x_s$, $0 < i < s$, is in \mathfrak{S} and no edge of Γ connects x_i with x_{i+1} , then $T = x_1, \dots, x_{i+1}, x_i, \dots, x_s$ is in \mathfrak{S} and we set SrT . Let \sim be the equivalence relation that is a reflexive and transitive closure of the symmetric binary relation r .

If $S = x_1, \dots, x_s$ is in \mathfrak{S} , we set $F(S) = F_{x_s}^+ \dots F_{x_1}^+ : \text{Rep}(\Gamma, \Lambda) \rightarrow \text{Rep}(\Gamma, \Lambda^S)$. It follows from the analog of [2, Lemma 1.2, proof of part 3)] for representations of valued quivers that $S \sim T$ implies $F(S) = F(T)$. If $K \in \mathfrak{S}$ is complete, then $F(K) = \Phi^+$ is the *Coxeter* functor [4, p. 19]. We say that $S \in \mathfrak{S}$ *annihilates* $M \in \text{f.d. } k(\Gamma, \Lambda)$ if $F(S)(V, f) = 0$ where $(V, f) \in \text{Rep}(\Gamma, \Lambda)$ is identified with M . In light of this identification, we often write $F(S)M$ or $\Phi^+ M$.

A *source* is a vertex of a quiver at which no arrow ends. Replacing sinks with sources, one gets similar definitions of a *reflection* functor F_x^- , a *(-)-admissible* sequence, and the *Coxeter* functor Φ^- [4].

The definition of a preprojective representation [4, p. 22] is equivalent to the following (see [2, Note 2]).

Definition 1.3. A module $M \in \text{f.d. } k(\Gamma, \Lambda)$ is *preprojective* if there exists an $S \in \mathfrak{S}$ that annihilates it.

We quote [10, 11, 12] for the use in Section 2 and note that any result for ordinary quivers that deals with \mathfrak{S} but not with \mathcal{P} holds for valued quivers and no proof is needed, for *(+)-admissible* sequences depend on the graph Γ and orientation Λ , but not on the valuation \mathbf{b} or the modulation \mathfrak{B} .

The following is [11, Proposition 1.9].

Proposition 1.1. *Let $S \in \mathfrak{S}$ be nonempty.*

- (a) $S \sim S_1 \dots S_r$ where, for all i , $S_i \in \mathfrak{S}$ consists of distinct vertices and $\text{Supp } S_i = \text{Supp } S_i \dots S_r$. If $\text{Supp } S_i \neq \Gamma_0$ then $\text{Supp } S_{i+1} \subsetneq \text{Supp } S_i$.
- (b) If $T \sim T_1 \dots T_q$ in \mathfrak{S} is nonempty where, for all j , $T_j \in \mathfrak{S}$ consists of distinct vertices and $\text{Supp } T_j = \text{Supp } T_j \dots T_q$, then $S \sim T$ if and only if $r = q$ and $S_i \sim T_i$ on $(\Gamma, \Lambda^{S_1 \dots S_{i-1}})$, $i = 1, \dots, r$.

The sequence $S_1 \dots S_r$ of Proposition 1.1(a) is the *canonical form* of $S \in \mathfrak{S}$, and the integer r is the *size* of S .

The following is [11, Definition 2.1].

Definition 1.4. If $S, T \in \mathfrak{S}$, we say that S is a *subsequence* of T and write $S \preceq T$ if $T \sim SU$ for some (+)-admissible sequence U .

We quote [12, Proposition 1.5] and [10, Proposition 2.3].

Proposition 1.2. (a) *The relation \preceq is a preorder.*

Let $S, T \in \mathfrak{S}$.

- (b) *We have $S \preceq T$ and $T \preceq S$ if and only if $S \sim T$.*
- (c) *$S \preceq T$ if and only if for all $v \in \Gamma_0$, $m_S(v) \leq m_T(v)$.*

We quote [10, Definition 2.4 and Proposition 2.6].

Proposition 1.3. *The poset of equivalence classes in \mathfrak{S} is a lattice where the greatest lower bound \wedge and the least upper bound \vee are as follows.*

Let $S, T \in \mathfrak{S}$ be nonempty and let $S_1 \dots S_r, T_1 \dots T_q$ be their canonical forms, respectively, where $r \leq q$. Then:

- (a) *If $\text{Supp } S \cap \text{Supp } T = \emptyset$ then $S \wedge T = \emptyset$; and if $\text{Supp } S \cap \text{Supp } T \neq \emptyset$, the canonical form of $S \wedge T$ is $R_1 \dots R_s$ where $s \leq q, r$ is the largest integer satisfying $\text{Supp } R_i = \text{Supp } S_i \cap \text{Supp } T_i \neq \emptyset$ for $0 < i \leq s$.*
- (b) *The canonical form of $S \vee T$ is $R_1 \dots R_q$ where $\text{Supp } R_i = \text{Supp } S_i \cup \text{Supp } T_i$ for $0 < i \leq r$, and $\text{Supp } R_i = \text{Supp } T_i$ for $r < i \leq q$.*

For all $S \in \mathfrak{S}$, we have $S \wedge \emptyset = \emptyset$ and $S \vee \emptyset = S$.

Recall that a subset F of a poset (P, \leq) is a *filter* if $x \in F$ and $y \geq x$ imply $y \in F$. A filter F is *generated* by $X \subset P$ if $F = \langle X \rangle = \{y \in P \mid y \geq x \text{ for some } x \in X\}$. If $x \in P$ then $\langle \{x\} \rangle = \langle x \rangle$ is a *principal filter*.

The following two definitions quote [11, Definitions 1.5, 2.2, and 3.1].

Definition 1.5. The *hull* of a filter F of (Γ_0, Λ) is the smallest filter $H_\Lambda(F)$ of (Γ_0, Λ) containing F and each vertex of $\Gamma_0 \setminus F$ connected by an edge to a vertex in F . A nonempty $S \in \mathfrak{S}$ is *principal* if its canonical form $S_1 \dots S_r$ satisfies $\text{Supp } S_i = H_\Lambda(\text{Supp } S_{i+1})$ for $0 < i < r$ where $\text{Supp } S_r$ is a principal filter. The set \mathfrak{P} of principal sequences inherits the binary relations \sim and \preceq from \mathfrak{S} . By [11, Proposition 1.9(b)], a principal sequence is determined uniquely up to equivalence by r and $\text{Supp } S_r$, so we denote by $S_{r,x}$ the principal sequence of size r with $\text{Supp } S_r = \langle x \rangle$, $x \in \Gamma_0$.

Definition 1.6. If $S \in \mathfrak{S}$ annihilates a $k(\Gamma, \Lambda)$ -module M , but no proper subsequence of S annihilates M , we call S a *shortest* sequence annihilating M .

The following statement quotes [12, Proposition 2.1 and Theorems 2.2 and 2.6].

Theorem 1.4. *Let M be a preprojective $k(\Gamma, \Lambda)$ -module.*

- (a) *There exists a unique up to equivalence shortest sequence $S_M \in \mathfrak{S}$ annihilating M .*
- (b) *If M is indecomposable and N is an indecomposable preprojective $k(\Gamma, \Lambda)$ -module, then $S_N \sim S_M$ if and only if $N \cong M$.*
- (c) *If M is indecomposable, then $S_M \in \mathfrak{P}$.*
- (d) *If M is indecomposable and $S_M = x_1, \dots, x_s$, then $M \cong F_{x_1}^- \dots F_{x_{s-1}}^-(L_{x_s})$ where L_{x_s} is the simple projective $k(\Gamma, \sigma_{x_{s-1}} \dots \sigma_{x_1} \Lambda)$ -module associated with $x_s \in \Gamma_0$.*

2. REDUCED WORDS IN THE WEYL GROUP

For a graph Γ with a valuation \mathbf{b} we assume for the rest of the paper that $\Gamma_0 = \{1, \dots, n\}$. Then the matrix $A = (a_{ij})$ with $a_{ii} = 2$ and $a_{ij} = -b_{ij}$ for all $i \neq j$ is an indecomposable symmetrizable generalized $n \times n$ Cartan matrix, and $\mathcal{W} = \mathcal{W}(A)$ is the Weyl group of the valued graph (Γ, \mathbf{b}) [4]. As before, (Γ, Λ) denotes a quiver with a valuation \mathbf{b} and modulation \mathfrak{B} .

Definition 2.1. If $S = x_1, \dots, x_s$ is in \mathfrak{S} , we say that $w(S) = \sigma_{x_s} \dots \sigma_{x_1}$ is the word in the Weyl group \mathcal{W} associated to S . If no edge connects vertices i and j , then $\sigma_i \sigma_j = \sigma_j \sigma_i$ so that $S \sim T$ implies $w(S) = w(T)$.

Recall (see [3]) that the *length* of $w \in \mathcal{W}$ is the smallest integer $\ell(w) = l \geq 0$ such that w is the product of l simple reflections, and a word $w = \sigma_{y_t} \dots \sigma_{y_1}$ in \mathcal{W} is *reduced* if $\ell(w) = t$.

We recall definitions and facts from [1, 17]. Let \mathbb{N} be the set of nonnegative integers. The translation quiver $\mathbb{N}(\Gamma, \Lambda^{op})$ of the opposite quiver of (Γ, Λ) has $\mathbb{N} \times \Gamma_0$ as the set of vertices, and each arrow $a : u \rightarrow v$ of (Γ, Λ) gives rise to two series of arrows, $(n, a^\circ) : (n, v) \rightarrow (n, u)$ and $(n, a^\circ)' : (n, u) \rightarrow (n+1, v)$. Since $\mathbb{N}(\Gamma, \Lambda^{op})$ is a locally finite quiver without oriented cycles (we disregard the valuation and translation), $\mathbb{N} \times \Gamma_0$ is a poset.

If $X \in \text{f.d. } k(\Gamma, \Lambda)$ is indecomposable, let $[X]$ be the isomorphism class of X . If $Y \in \text{f.d. } k(\Gamma, \Lambda)$ is indecomposable, a path of length $m > 0$ from X to Y is a sequence of nonzero nonisomorphisms $X = A_0 \rightarrow \dots \rightarrow A_m = Y$, where $A_i \in \text{f.d. } k(\Gamma, \Lambda)$ is indecomposable for all i . By definition, there is a path of length 0 from X to X . Set $[X] \prec [Y]$ if there is a path of positive length from X to Y .

The preprojective component of (Γ, Λ) , $\tilde{\mathcal{P}}(\Gamma, \Lambda)$, is a locally finite connected quiver (we disregard the valuation and translation) whose set of vertices, $\tilde{\mathcal{P}}(\Gamma, \Lambda)_0$, is the set of isomorphism classes of indecomposable preprojective $k(\Gamma, \Lambda)$ -modules. If $X, Y \in \text{f.d. } k(\Gamma, \Lambda)$ are indecomposable, Y is preprojective, and $X = A_0 \rightarrow \dots \rightarrow A_m = Y$, $m > 0$, is a path from X to Y , then $[X] \neq [Y]$ and A_i is preprojective for all i . Hence the reflexive closure \preceq of the transitive binary relation \prec is a partial order on $\tilde{\mathcal{P}}(\Gamma, \Lambda)_0$. Moreover, $[X] \prec [Y]$ if and only if there is a finite sequence of irreducible morphisms $X = B_0 \rightarrow \dots \rightarrow B_n = Y$, where $n > 0$ and B_j is indecomposable preprojective for all j .

The following two statements extend [10, Proposition 3.7 and Theorem 4.3] to representations of valued quivers.

Proposition 2.1. *If M, N are indecomposable preprojective $k(\Gamma, \Lambda)$ -modules, then $[M] \preceq [N]$ in $\tilde{\mathcal{P}}(\Gamma, \Lambda)_0$ if and only if $S_M \preceq S_N$ in \mathfrak{P} .*

Proof. The necessity follows from [12, Corollary 2.9]. For the sufficiency, let $S_M \preceq S_N$. If $S_N \preceq S_M$, then $S_M \sim S_N$ by Proposition 1.2(b) whence $M \cong N$ and $[M] = [N]$ according to Theorem 1.4(b). If $S_N \not\preceq S_M$ then, by Theorem 1.4(a), $S_M \sim S_{p,u}$ and $S_N \sim S_{q,v}$ for some $p, q > 0$ and $u, v \in \Gamma_0$. By [11, Theorem 2.5(a)], which deals with \mathfrak{S} but not with $\tilde{\mathfrak{S}}$, $(p-1, u) < (q-1, v)$ in $\mathbb{N} \times \Gamma_0$ so there is a path $(p-1, u) \rightarrow (q-1, v)$ of positive length in $\mathbb{N}(\Gamma, \Lambda^{op})$. By [12, Proposition 2.8(d)], there is a path $[M] \rightarrow [N]$ of positive length in $\tilde{\mathcal{P}}(\Gamma, \Lambda)$, i.e., $[M] \preceq [N]$. \square

Theorem 2.2. *Let M be a preprojective $k(\Gamma, \Lambda)$ -module.*

- (a) *The word $w(S_M) \in \mathcal{W}$ is reduced.*
- (b) *If M is indecomposable and N is an indecomposable preprojective $k(\Gamma, \Lambda)$ -module, the following are equivalent.*
 - (i) $M \cong N$.
 - (ii) $S_M \sim S_N$.
 - (iii) $w(S_M) = w(S_N)$.

Proof. (a) If $M = 0$ the statement is trivial. If $M \neq 0$, let $S_M = x_1, \dots, x_s$ and proceed by induction on $s > 0$. The case $s = 1$ is clear, so suppose $s > 1$ and the statement holds for all sequences S_N of length $< s$. Since $s > 1$, $F_{x_1}^+ M \neq 0$ is a preprojective $k(\Gamma, \sigma_{x_1} \Lambda)$ -module, and $S_{F_{x_1}^+ M} = x_2, \dots, x_s$. By the induction hypothesis, the word $u = \sigma_{x_s} \dots \sigma_{x_2}$ in \mathcal{W} is reduced. Assume, to the contrary, that the word $u\sigma_{x_1} = \sigma_{x_s} \dots \sigma_{x_2} \sigma_{x_1}$ is not reduced. Then $\ell(u\sigma_{x_1}) < \ell(u)$ [3, Ch. IV, Proposition 1.5.4] and $\sigma_{x_s} \dots \sigma_{x_2}(e_{x_1}) < 0$ where e_{x_1} is the simple root associated to the vertex x_1 [9, Lemma 3.11]. By [4, Proposition 2.1], $F(S_{F_{x_1}^+ M})L_{x_1} = 0$ where L_{x_1} is the simple $k(\Gamma, \sigma_{x_1} \Lambda)$ -module associated to x_1 . Since x_1 is a source in $(\Gamma, \sigma_{x_1} \Lambda)$, then L_{x_1} is a simple injective and a preprojective $k(\Gamma, \sigma_{x_1} \Lambda)$ -module so that $[L_{x_1}]$ is a sink in $\tilde{\mathcal{P}}(\Gamma, \sigma_{x_1} \Lambda)$ and, hence, a maximal element of the poset $\tilde{\mathcal{P}}(\Gamma, \sigma_{x_1} \Lambda)_0$. On the other hand, $F(S_{F_{x_1}^+ M})L_{x_1} = 0$ implies $S_{L_{x_1}} \preceq S_{F_{x_1}^+ M}$ whence $S_{L_{x_1}} \preceq S_N$ for some indecomposable direct summand N of $F_{x_1}^+ M$ according to [10, Theorem 3.4(b)], whose proof works for representations of valued quivers. By Proposition 2.1, $[L_{x_1}] \preceq [N]$ in $\tilde{\mathcal{P}}(\Gamma, \sigma_{x_1} \Lambda)_0$. Since $[L_{x_1}]$ is a maximal element, $[L_{x_1}] = [N]$ whence $L_{x_1} \cong N$. The latter contradicts the well-known fact, contained in [4, Proposition 2.1], that the simple module associated to a vertex that is a source is not a direct summand of a module that belongs to the image of the positive reflection functor associated to the vertex. Thus $w(S_M)$ is a reduced word.

- (b) The proof is identical to that of [10, Theorem 4.3(b)]. \square

With the same proof, the following statement extends [10, Corollary 4.4] suggested by Zelevinsky.

Corollary 2.3. *Let $S = x_1, \dots, x_s$, $s > 0$, be in \mathfrak{S} , and set $M(S) = F_{x_1}^- \dots F_{x_{s-1}}^-(L_{x_s})$, where L_{x_s} is the simple projective $k(\Gamma, \sigma_{s-1} \dots \sigma_{x_1} \Lambda)$ -module associated to $x_s \in \Gamma_0$.*

- (a) *If the word $w(S) \in \mathcal{W}$ is reduced, $M(S)$ is an indecomposable module in $\tilde{\mathcal{P}}$.*
- (b) *If $M \in \tilde{\mathcal{P}}$ is indecomposable, then $M \cong M(S)$ for some sequence $S \in \mathfrak{S}$ where $\ell(S) > 0$ and the word $w(S)$ is reduced.*

Lemma 2.4. *Let $S = x_1, x_2, \dots, x_s$, $s > 1$, be in \mathfrak{S} , suppose that the full subgraph of Γ determined by $\text{Supp } S$ is connected, and set $T = x_2, \dots, x_s$. If $T \sim S_N$ for*

some indecomposable preprojective $k(\Gamma, \sigma_{x_1} \Lambda)$ -module N satisfying $M = F_{x_1}^- N \neq 0$, then M is indecomposable preprojective and $S \sim S_M$.

Proof. By [4, Proposition 2.1], M is indecomposable and $F_{x_1}^+ F_{x_1}^- N \cong N$. Hence S annihilates M and $S_M \preceq x_1 \vee S_M \sim x_1 U \preceq S = x_1 T$ for some U , so by Proposition 1.2(c), $U \preceq T$. On the other hand, $x_1 U$ annihilates M , so U annihilates N , giving $T \preceq U$, hence $U \sim T$ and $x_1 \vee S_M \sim S$. Since the full subgraph of Γ determined by $\text{Supp } S$ is connected, for some $x_l \in \text{Supp } S \setminus \{x_1\}$ there is an arrow $x_l \rightarrow x_1$ in (Γ, Λ) . This gives $x_l \in \text{Supp } S_M$ whence $x_1 \in \text{Supp } S_M$ because $\text{Supp } S_M$ is a filter of (Γ_0, Λ) by [11, Proposition 1.9(a)]. Hence $S_M \sim x_1 \vee S_M \sim S$. \square

Theorem 2.5. *If $S = x_1, \dots, x_s$, $s > 0$, is in \mathfrak{P} , the following are equivalent.*

- (a) *There exists an indecomposable preprojective $k(\Gamma, \Lambda)$ -module M satisfying $S \sim S_M$.*
- (b) *The word $w(S) \in \mathcal{W}$ is reduced.*
- (c) *For $0 < i < s$, $\sigma_{x_i} \dots \sigma_{x_{s-1}}(e_{x_s}) > 0$.*

Proof. (a) \implies (b) This is Theorem 2.2(a).

(b) \implies (c) This follows from Corollary 2.3(a).

(c) \implies (a) Proceed by induction on s . If $s = 1$ then $S \sim S_{L_{x_1}}$ where L_{x_1} is the simple projective $k(\Gamma, \Lambda)$ -module associated to $x_1 \in \Gamma_0$. Suppose $s > 1$ and the statement holds for all principal (+)-admissible sequences of length $< s$ on all valued quivers (Γ, Θ) without oriented cycles. By [10, Proposition 3.6], T is a principal (+)-admissible sequence on $(\Gamma, \sigma_{x_1} \Lambda)$, so by the induction hypothesis, $T \sim S_N$ for some indecomposable preprojective $k(\Gamma, \sigma_{x_1} \Lambda)$ -module N , and Theorem 1.4(d) says that $N \cong F_{x_2}^- \dots F_{x_{s-1}}^- (L_{x_s})$ where L_{x_s} is the simple projective $k(\Gamma, \sigma_{s-1} \dots \sigma_{x_1} \Lambda)$ -module associated to $x_s \in \Gamma_0$. Since $\sigma_{x_1} \dots \sigma_{x_{s-1}}(e_{x_s}) > 0$, [4, Proposition 2.1] implies that $M = F_{x_1}^- N \neq 0$. By [10, Remark 3.1], the full subgraph of Γ determined by $\text{Supp } S$ is connected. By Lemma 2.4, M is indecomposable and $S \sim S_M$. \square

The proof of the following statement is identical to that of [10, Theorem 4.6].

Theorem 2.6. *For all $S \in \mathfrak{S}$, the following are equivalent.*

- (a) *There exists a preprojective $k(\Gamma, \Lambda)$ -module M satisfying $S \sim S_M$.*
- (b) *The word $w(S) \in \mathcal{W}$ is reduced.*

Corollary 2.7. *If a valued graph (Γ, \mathbf{b}) is not a Dynkin diagram of the type $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$, or G_2 , let Λ be an orientation on Γ and $S \in \mathfrak{S}$.*

- (a) *The word $w(S) \in \mathcal{W}$ is reduced.*
- (b) *For any modulation \mathfrak{B} of (Γ, \mathbf{b}) , there exists a preprojective $k(\Gamma, \Lambda)$ -module M satisfying $S \sim S_M$.*

Proof. (a) By [16, Proposition, p. 224], there exists a modulation \mathfrak{B} of (Γ, \mathbf{b}) . For this modulation the algebra $k(\Gamma, \Lambda)$ is of infinite representation type by [4, Theorem, p. 3]. The poset (Γ_0, Λ) is finite, so by Theorem 2.2 there is an indecomposable preprojective $k(\Gamma, \Lambda)$ -module M with $S_M \sim K^i U$ for some i larger than the size of S . By Theorem 2.6, $w(S_M) \in \mathcal{W}$ is reduced. Since i is larger than the size of S , then $S_M \sim SV$ for some V , so $w(S) \in \mathcal{W}$ is also reduced.

(b) This is an immediate consequence of (a) and Theorem 2.6. \square

We characterize infinite Weyl groups in terms of reduced words.

Theorem 2.8. *Let $A = (a_{ij})$ be an indecomposable symmetrizable generalized $n \times n$ Cartan matrix, and let $c = \sigma_{v_n} \dots \sigma_{v_1}$ be a Coxeter element of the Weyl group $\mathcal{W}(A)$. Then $\mathcal{W}(A)$ is infinite if and only if for all $m \in \mathbb{Z}$, $\ell(c^m) = |m|n$.*

Proof. Since sufficiency is clear, we need only prove necessity. Consider the $n \times n$ matrix (b_{ij}) where $b_{ii} = 0$ for all i , and $b_{ij} = -a_{ij}$ for all $i \neq j$. Denote by $\Gamma = (\Gamma_0, \Gamma_1)$ the graph where $\Gamma_0 = \{1, \dots, n\}$ and, for $i \neq j$, we have $\{i, j\} \in \Gamma_1$ if and only if $b_{ij} \neq 0$. The collection of all b_{ij} defines a valuation \mathbf{b} of Γ , and $\mathcal{W}(A)$ coincides with the Weyl group \mathcal{W} of the valued graph (Γ, \mathbf{b}) . As noted in [4, p. 8], there is a unique orientation Λ on Γ without oriented cycles for which $K = v_1, \dots, v_n$ is a complete sequence in \mathfrak{S} . Then $c = w(K)$ and $c^m = w(K^m)$ for all $m > 0$. Since \mathcal{W} is infinite, [4, Propositions 1.2(a) and 1.5] imply that (Γ, \mathbf{b}) is not a Dynkin diagram of the type $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$, or G_2 . By Corollary 2.7(a), c^m is a reduced word. \square

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