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Covariant Quantum Fields on Noncommutative Spacetimes

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Abstract

A spinless covariant field φ on Minkowski spacetime $\mathcal{M}^{d+1}$ obeys the relation $U(a,\Lambda)\varphi(x)U(a,\Lambda)^{-1} = \varphi(\Lambda x + a)$ where (a,Λ) is an element of the Poincaré group $\mathscr{P}_+^\uparrow$ and $U : (a,\Lambda) \to U(a,\Lambda)$ is its unitary representation on quantum vector states. It expresses the fact that Poincaré transformations are being unitary implemented. It has a classical analogy where field covariance shows that Poincaré transformations are canonically implemented. Covariance is self-reproducing: products of covariant fields are covariant. We recall these properties and use them to formulate the notion of covariant quantum fields on noncommutative spacetimes. In this way all our earlier results on dressing, statistics, etc. for Moyal spacetimes are derived transparently. For the Voros algebra, covariance and the $*$-operation are in conflict so that there are no covariant Voros fields compatible with *, a result we found earlier. The notion of Drinfel'd twist underlying much of the preceding discussion is extended to discrete abelian and nonabelian groups such as the mapping class groups of topological geons. For twists involving nonabelian groups the emergent spacetimes are nonassociative.

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I. INTRODUCTION: POINCARÉ COVARIANCE ON COMMUTATIVE SPACETIMES

The Poincaré group $\mathscr{P}$ acts on Minkowski space $\mathcal{M}^{d+1}$ by transforming its coordinates (or coordinate functions), $x = (x_\mu)$ to $\Lambda x + a$

$$(a, \Lambda) \in \mathscr{P} : \quad (a, \Lambda)x = \Lambda x + a \quad . \tag{1}$$

If the spacetime algebra $\mathcal{A}_0(\mathcal{M}^{d+1})$ associated with $\mathcal{M}^{d+1}$ is commutative, and φ is a quantum relativistic scalar field on $\mathcal{M}^{d+1}$, we require that there exists a unitary representation

$$U : (a, \Lambda) \to U(a, \Lambda) \tag{2}$$

on the Hilbert space $\mathcal{H}$ of states vectors such that

$$U(a, \Lambda)\varphi(x)U(a, \Lambda)^{-1} = \varphi\big((a, \Lambda)x\big) \quad . \tag{3}$$

There are similar requirements on relativistic fields of all spins. They express the requirement that the spacetime transformations (1) can be unitarily implemented in quantum theory. It is analogous to the requirement in nonrelativistic quantum mechanics that infinitesimal spatial rotations are to be implemented by the (self-adjoint) angular momentum operators.

A field φ fulfilling (3) is said to be a "covariant field" and the condition in (3) is the covariance condition. We call it "primitive" as we later extend it to products of fields.

We can write (3) in the equivalent form

$$U(a, \Lambda)\varphi\big((a, \Lambda)^{-1}x\big)U(a, \Lambda) = \varphi(x) \tag{4}$$

Now in this form, covariance can be readly understood in terms of the coproduct on the Poincaré group. Thus

$$\varphi \in L(\mathcal{H}) \otimes S(\mathcal{M}^{d+1}) \tag{5}$$

where $L(\mathcal{H})$ are linear operators on $\mathcal{H}$ and $S(\mathcal{M}^{d+1})$ are distributions on $\mathcal{M}^{d+1}$. There is an action of $\mathscr{P}$ on both, that on $L(\mathcal{H})$ being the adjoint action $\mathrm{Ad}U(a, \Lambda)$ of $U(a, \Lambda)$,

$$\mathrm{Ad}U(a, \Lambda)\varphi = U(a, \Lambda)\varphi U(a, \Lambda)^{-1} \tag{6}$$

and that on $S(\mathcal{M}^{d+1})$ being

$$\alpha \to (a, \Lambda) \triangleright \alpha, \quad \big[(a, \Lambda)\alpha\big](x) = \alpha\big((a, \Lambda)^{-1}x\big), \quad \alpha \in S(\mathcal{M}^{d+1}) \quad . \tag{7}$$

We call the latter action as V.

Now the coproduct on $\mathscr{P}_+^\uparrow$ for commutative spacetimes is Δ_0, where

$$\Delta_0\big((a,\Lambda)\big) = (a,\Lambda)\otimes(a,\Lambda) \quad . \tag{8}$$

Then by (4)

$$(\mathrm{Ad}U\otimes V)\Delta_0\big((a,\Lambda)\big)\varphi = \varphi \quad . \tag{9}$$

We will have occasion to use both the versions (3) and (4,9) of covariance.

II. COVARIANCE FOR PRODUCTS: COMMUTATIVE SPACETIMES

We saw in the previous section that for a single field, covariance ties together spacetime transformations and its implementation on the quantum Hilbert space. Products of fields bring in new features which although present for commutative spacetimes, assume prominence on quantum spacetimes. We now briefly examine these features in the former case

A. *Tensor Products*

Consider

$$\varphi(x_1)\varphi(x_2)...\varphi(x_N) \quad . \tag{10}$$

This can be understood as the element $\varphi\otimes\varphi...\otimes\varphi$ belonging to $L(\mathcal{H})\otimes\big(S(\mathcal{M}^{d+1})\otimes S(\mathcal{M}^{d+1})\otimes...\otimes S(\mathcal{M}^{d+1})\big)$ evaluated at $x_1,x_2,...,x_N$

$$\varphi\otimes\varphi\otimes...\otimes\varphi\in L(\mathcal{H})\otimes\big(S(\mathcal{M}^{d+1})\big)^{\otimes N}, \quad (\varphi\otimes\varphi\otimes...\otimes\varphi)(x_1,x_2,...,x_N) = \varphi(x_1)\varphi(x_2)...\varphi(x_N) \quad . \tag{11}$$

Note that tensoring refers only to $S(\mathcal{M}^{d+1})$, there is no tensoring involving $L(\mathcal{H})$. There is only one Hilbert space $\mathcal{H}$ which for free particles is the Fock space and $U(a,\Lambda)$ acts by conjugation on the L.H.S. for all N.

But that is not the case for $S(\mathcal{M}^{d+1})^{\otimes N}$. The Poincaré group acts on it by the coproduct

$$(\underbrace{\mathbb{1}\otimes\mathbb{1}\otimes...\otimes\mathbb{1}\otimes\Delta_0}_{N-1})(\underbrace{\mathbb{1}\otimes\mathbb{1}\otimes...\otimes\mathbb{1}\otimes\Delta_0}_{N-2})...\Delta_0 \tag{12}$$

of (a,Λ). Thus

$$(12) \text{ on } (a,\Lambda) = (a,\Lambda)\otimes(a,\Lambda)\otimes ...\otimes(a,\Lambda) \tag{13}$$

and

$$\Big((12) \text{ on } (a,\Lambda)\triangleright\varphi^{\otimes N}\Big)(x_1,x_2,...,x_N) = \varphi^{\otimes N}\big((a,\Lambda)^{-1}x_1,(a,\Lambda)^{-1}x_2,...,(a,\Lambda)^{-1}x_N\big) \quad . \tag{14}$$

Covariance is now the demand

$$U(a,\Lambda)\Big(\varphi^{\otimes N}\big((a,\Lambda)^{-1}x_1,(a,\Lambda)^{-1}x_2,...,(a,\Lambda)^{-1}x_N\big)\Big)U(a,\Lambda)^{-1} = \varphi^{\otimes N}(x_1,x_2,...,x_N) \quad . \tag{15}$$

It is evidently fulfilled for the coproduct (12) if the primitive covariance (3,4) is fulfilled.

For free fields (or in and out-fields), covariance can be verified in a different manner. Thus for a free real scalar field φ of mass m, we have

$$\varphi = \int \mathrm{d}\mu(p)\Big(c_p^\dagger \mathrm{e}_p + c_p \mathrm{e}_{-p}\Big) = \varphi^{(-)}+\varphi^{(+)} \tag{16}$$
$$\mathrm{e}_p(x) = \mathrm{e}^{-ip\cdot x}, \quad |p_0| = (\vec{p}^2+m^2)^{\frac{1}{2}}, \quad \mathrm{d}\mu(p) = \tfrac{\mathrm{d}^d p}{2|p_0|}$$

where c_p, $c_p^\dagger$ are the standard annihilation and creation operators, and $\varphi^{(\mp)}$ refer to the annihilation and creation parts of φ.

Now $\varphi^{(\mp)}$ must separately fulfill the covariance requirement. Let us consider $\varphi^{(-)}$. We have that

$$\varphi^{(-)}(x_1)\varphi^{(-)}(x_2)...\varphi^{(-)}(x_N)|0\rangle = \int\prod_i \mathrm{d}\mu(p_i)c_{p_1}^\dagger c_{p_2}^\dagger ...c_{p_N}^\dagger|0\rangle \mathrm{e}_{p_1}(x_1)\mathrm{e}_{p_2}(x_2)...\mathrm{e}_{p_N}(x_N) \tag{17}$$

Let us first check translations. Let P_μ be the translation generators on the Hilbert space,

$$[P_\mu, c_p^\dagger] = p_\mu c_p^\dagger, \quad P_\mu|0\rangle = 0 \tag{18}$$

and let $\mathcal{P}_\mu = -i\partial_\mu$ be the translation generator on $S(\mathcal{M}^{d+1})$:

$$\mathcal{P}_\mu \mathrm{e}_p = -p_\mu \mathrm{e}_p \tag{19}$$

The coproduct Δ_0 gives for the Lie algebra element $\mathcal{P}_\mu$,

$$\Delta_0(\mathcal{P}_\mu) = \mathbb{1}\otimes\mathcal{P}_\mu + \mathcal{P}_\mu\otimes\mathbb{1} \tag{20}$$

[If $\underline{v}$ is the representation of the Lie algebra of $\mathscr{P}_+^\uparrow$ on functions, and $\hat{P}_\mu$ is the Lie algebra generator in the abstract group $\mathscr{P}_+^\uparrow$ so that $\underline{v}(P_\mu) = \mathcal{P}_\mu$, the L.H.S. here should strictly read $\underline{v}\big(\Delta_0(\hat{P}_\mu)\big)$. So we have simplified the notation in (20).]

It follows that

$$(\mathbb{1}\otimes\mathbb{1}\otimes...\otimes\mathbb{1}\otimes\Delta_0)...\Delta_0(\mathcal{P}_\mu)\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2}\otimes...\otimes\mathrm{e}_{p_N}=-\sum_i p_{i\mu}\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2}\otimes...\otimes\mathrm{e}_{p_N} \tag{21}$$

Covariance for translations is the requirement

$$P_\mu c^\dagger_{p_1}c^\dagger_{p_2}...c^\dagger_{p_N}|0\rangle\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2}\otimes...\otimes\mathrm{e}_{p_N}+c^\dagger_{p_1}c^\dagger_{p_2}...c^\dagger_{p_N}|0\rangle\big(-\sum_i p_{i\mu}\big)\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2}\otimes...\otimes\mathrm{e}_{p_N}=0 \tag{22}$$

which is clearly fulfilled.

Next consider Lorentz transformations. A Lorentz transformation Λ acts on e_p according to

$$(\Lambda\mathrm{e}_p)(x)=\mathrm{e}_p(\Lambda^{-1}x)=\mathrm{e}_{\Lambda p}(x) \tag{23}$$

or $\Lambda\mathrm{e}_p=\mathrm{e}_{\Lambda p}$.

For Lorentz transformations Λ, covariance is thus the identity

$$\int\prod_i \mathrm{d}\mu(p_i)c^\dagger_{\Lambda p_1}c^\dagger_{\Lambda p_2}...c^\dagger_{\Lambda p_N}|0\rangle\mathrm{e}_{\Lambda p_1}\otimes\mathrm{e}_{\Lambda p_2}...\otimes\mathrm{e}_{\Lambda p_N}=\int\prod_i \mathrm{d}\mu(p_i)c^\dagger_{p_1}c^\dagger_{p_2}...c^\dagger_{p_N}|0\rangle\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2}...\otimes\mathrm{e}_{p_N} \tag{24}$$

which is true because of the Lorentz invariance of the measure:

$$\mathrm{d}\mu(\Lambda^{-1}p_i)=\mathrm{d}\mu(p_i)\quad . \tag{25}$$

III. QUANTUM STATISTICS: THE SCHUR-WEYL DUALITY

The permutation group S_N and its irreducible representations govern statistics of N-particle state vectors on commutative spacetimes for $d\geq 3$. We consider only such d.

By axioms of quantum theory, the N-particle observables must commute with the action of S_N so that the action of observation does not affect particle identity. In particular the action of the symmetry group must commute with the action of S_N.

If that is the case, we can consistently work with irreducible representations of S_N.

In (17), (a,Λ) acts on $\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2}\otimes...\otimes\mathrm{e}_{p_N}$ via the coproduct (12). This action commutes with the action of S_N if S_N acts by permuting p_i. Thus we can work with irreducible representation of S_N.

In particular we can work with bosons and fermions by totally symmetrising or antisymmetrising $\otimes\mathrm{e}_{p_i}$. In the former case $c^\dagger_{p_i}$ can be taken to commute (their anticommutators do not contribute to (17)) and for the latter they anticommute.

The important point here is that the group algebras $\mathbb{C}\mathscr{P}$ and $\mathbb{C}S_N$ are commutants of each other in their action on N-particle states.

A. *The Double Commutant Theorem and the Schur-Weyl Duality*

A result of this sort is familiar to particle physicists in case the symmetry group is $U(k)$. Here $U(k)$ can be the k-flavour symmetry group. It acts on $\mathbb{C}^k$. Then to reduce the representation of $U(k)$ on $\mathbb{C}^{k\otimes N}$, we use the fact that $\mathbb{C}S_N$ commutes with $\mathbb{C}U(k)$. That lets us use Young tableaux methods.

It is in fact the case that $\mathbb{C}U(k)$ and $\mathbb{C}S_N$ exhaust the commutants of each other. This result and the Young tableaux methods are part of the contents of Schur-Weyl duality [1, 2].

So we are working with aspects of an infinite-dimensional analogue of this duality for a noncompact symmetry group $\mathscr{P}_+^\uparrow$ when we remark that $\mathbb{C}\mathscr{P}_+^\uparrow$ and $\mathbb{C}S_N$ mutually commute.

B. *A Presentation of S_N*

Let us imagine that S_N acts by transforming N objects numbered from 1 to N and let τ_{ij} denote the transformation of objects i and j. Then S_N has the presentation

$$S_N = \langle \tau_{i,i+1} : i \in [1,2,...,N-1], \tau_{i,i+1}^2 = \mathbb{1}, \tau_{i,i+1}\tau_{i+1,i+2}\tau_{i,i+1} = \tau_{i+1,i+2}\tau_{i,i+1}\tau_{i+1,i+2}\rangle \tag{26}$$

The N objects were introduced here for concreteness. The abstract S_N group is given just by (26).

We will have use of this presentation later.

C. *Multiplication Map and Self-Reproduction*

The multiplication map involves products of fields at the *same* point and hence the algebra of the underlying manifold. It is not the same as the tensor product which involves products of fields at *different* points.

There is a further property of φ, involving now the multiplication map, which is easily understood on commutative spacetimes. It has much importance for both commutative and noncommutative spacetimes. It is the property of self-reproduction. Let us first understand this property for $C^\infty(\mathcal{M})$, the set of smooth functions on a manifold $\mathcal{M}$. If $\alpha : p \to \alpha p,\ p \in \mathcal{M}$, is a diffeomorphism of $\mathcal{M}$, it acts on $f \in C^\infty(\mathcal{M})$ by pull-back:

$$(\alpha^* f)(p) = f(\alpha p) \quad . \tag{27}$$

But $C^\infty(\mathcal{M})$ has a further property, routinely used in differential geometry: $C^\infty(\mathcal{M})$ is closed under point-wise multiplication:

If $f_1, f_2 \in C^\infty(\mathcal{M})$, then

$$f_1 f_2 \in C^\infty(\mathcal{M}) \tag{28}$$

where

$$\big(f_1 f_2\big)(p) = f_1(p) f_2(p) \quad . \tag{29}$$

This property is very important for noncommutative geometry: the completion of this algebra under the supremum norm gives the commutative algebra of $C^0(\mathcal{M})$, a commutative C^*-algebra. By the Gel'fand-Naimark theorem [3, 4] it encodes the topology of $\mathcal{M}$.

Now by (27) and (28), we see that multiplication of functions preserves transformation under diffeos. This simple property gets generalised to covariant quantum field thus:

The pointwise product of covariant quantum fields is covariant.

That means in particular that

$$U(a,\Lambda)\varphi^2\big((a,\Lambda)^{-1}x\big)U(a,\Lambda)^{-1} = \varphi^2(x) \quad . \tag{30}$$

This result is obviously true modulo renormalization problems. It is at the basis of writing invariant interactions in quantum field theories on $\mathcal{A}_0(\mathcal{M}^{d+1})$.

Note that generally we require covariance of the product of any two covariant fields, distinct or the same.

D. *The $*$-covariance*

In quantum field theories on $\mathcal{A}_0(\mathcal{M}^{d+1})$, another routine requirement is that covariance and the $*$- or the adjoint operation be compatible. Thus if ψ is a covariant complex field,

$$U(a,\Lambda)\psi\big((a,\Lambda)^{-1}x\big)U(a,\Lambda)^{-1} = \psi(x) \quad , \tag{31}$$

we require that $\psi^\dagger$ is also a covariant complex field. That is fulfilled if $U(a,\Lambda)$ is unitary.

Thus $*$-covariance is linked to unitarity of time-evolution and the S-matrix and many more physical requirements.

E. *Summary: Covariance Requirements*

Here is a brief summary of our covariance requirements on quantum fields for commutative spacetimes (ignoring the possibility of parastatistics of order 2 or more): *A quantum field should be $*$- covariant with commutation or anti-commutation relations (symmetrisation postulates) compatible with $*$-covariance.*

IV. COVARIANCE ON THE MOYAL PLANE

The Moyal plane $\mathcal{A}_\theta(\mathcal{M}^{d+1})$ is the algebra of smooth functions on $\mathcal{M}^{d+1}$ with the product

$$m_\theta(\alpha\otimes\beta) = m_0\mathscr{F}_\theta(\alpha\otimes\beta), \quad \alpha,\beta\in\mathcal{A}_\theta(\mathcal{M}^{d+1}), \quad \mathscr{F}_\theta = \mathrm{e}^{\frac{i}{2}\partial_\mu\otimes\theta_{\mu\nu}\partial_\nu} \tag{32}$$

where m_0 is the point-wise product:

$$m_0(\gamma\otimes\delta)(x) = \gamma(x)\delta(x), \quad \gamma,\delta\in\mathcal{A}_0(\mathcal{M}^{d+1}) \quad . \tag{33}$$

The Poincaré group $\mathscr{P}$ acts on smooth functions α on $\mathcal{M}^{d+1}$ by pull-back as before:

$$\mathscr{P}\ni(a,\Lambda):\alpha\to(a,\Lambda)\alpha, \quad \big((a,\Lambda)\alpha\big)(x) = \alpha\big((a,\Lambda)^{-1}x\big) \tag{34}$$

It is by now well-known [5–7] that this action extends to the algebra $\mathcal{A}_\theta(\mathcal{M}^{d+1})$ compatibly with the product m_θ only if the coproduct on $\mathscr{P}$ is twisted. The twisted coproduct Δ_θ on $\mathscr{P}$ is

$$\Delta_\theta(g) = F_\theta^{-1}(g\otimes g)F_\theta, \quad F_\theta = \mathrm{e}^{-\frac{i}{2}\hat{P}_\mu\otimes\theta_{\mu\nu}\hat{P}_\nu} = \text{Drinfel}'\text{d twist} \tag{35}$$

Here $\hat{P}_\mu$ is as before the translation generator in $\mathscr{P}$ with representatives $\mathcal{P}_\mu = -i\partial_\mu$ and P_μ on functions and $L(\mathcal{H})$ respectively.

Equation (35) is the starting point for further considerations.

Let φ_θ be the twisted analogue of the field φ of section 2. Also let U_θ be the unitary operator implementing $\mathscr{P}$ in $L(\mathcal{H})$. Covariance then is the requirement

$$U_\theta(a,\Lambda)\varphi_\theta\big((a,\Lambda)^{-1}x\big)U_\theta(a,\Lambda)^{-1} = \varphi_\theta(x) \tag{36}$$

and its multifield generalisation, while compatibility with * or unitarity requires that $\varphi_\theta^\dagger$ is also covariant. There is also one further requirement, namely compatibility with symmetrisation postulate.

The analysis of these requirements becomes transparent on working with the mode expansion of φ_θ which is assumed to exist:

$$\varphi_\theta = \int \mathrm{d}\mu(p)\left[a_p^\dagger \mathrm{e}_p + a_p \mathrm{e}_{-p}\right] = \varphi_\theta^{(-)} + \varphi_\theta^{(+)}, \quad \mathrm{d}\mu(p) = \frac{\mathrm{d}^d p}{2|p_0|} \quad . \tag{37}$$

The expansion can refer to in- , out- or free fields.

We also assume the existence of vacuum $|0\rangle$:

$$a_p|0\rangle = 0, \ \forall p \quad . \tag{38}$$

A. *The Primitive Covariance of a Single Field*

We are here referring to (37). It requires that

$$U_\theta(a,\Lambda) a_p^\dagger U_\theta(a,\Lambda)^{-1} = a_{\Lambda p}^\dagger, \quad U_\theta(a,\Lambda) a_p U_\theta(a,\Lambda)^{-1} = a_{\Lambda p} \tag{39}$$

A particular consequence of (38,39) is that single particle states transform for all θ in the same manner or assuming that $U_\theta(a,\Lambda)|0\rangle = |0\rangle$:

$$U_\theta(a,\Lambda) a_p^\dagger |0\rangle = a_{\Lambda p}^\dagger |0\rangle \tag{40}$$

New physics can be expected only in multi-particle sectors.

B. *Covariance in Multi-Particle Sectors*

On the Moyal plane, multi-particle wave functions $\mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} \otimes ... \otimes \mathrm{e}_{p_N}$ transform under $\mathscr{P}$ with the twisted coproduct. This affects the properties of a_p, $a_p^\dagger$ in a $\theta_{\mu\nu}$-dependent manner.

Let us focus on the two-particle sector:

$$\int \prod_i \mathrm{d}\mu(p_i) a_{p_1}^\dagger a_{p_2}^\dagger |0\rangle \mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} \tag{41}$$

Since translations act in the usual way on $\mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2}$,

$$\Delta_\theta(\mathcal{P}_\mu)\mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} = (\mathbb{1} \otimes \mathcal{P}_\mu + \mathcal{P}_\mu \otimes \mathbb{1})\mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} = -(\sum_i p_{i\mu})\mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} \tag{42}$$

translational covariance requires the standard transformation of $a_{p_i}^\dagger$:

$$[P_\mu^\theta, a_p^\dagger] = p_\mu a_p^\dagger \quad , \tag{43}$$

P^θ_μ is the possibly θ dependent translation generator.

Lorentz transformations are more interesting. We have that

$$\Delta_\theta(\Lambda) \triangleright \mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} = \mathscr{F}_\theta^{-1}(\Lambda \otimes \Lambda)\mathscr{F}_\theta \mathrm{e}_{p_1} \otimes \mathrm{e}_{p_2} = \mathrm{e}^{\frac{i}{2}(\Lambda p_1)\wedge(\Lambda p_2)}\mathrm{e}^{-\frac{i}{2}p_1\wedge p_2}\mathrm{e}_{\Lambda p_1} \otimes \mathrm{e}_{\Lambda p_2} \quad . \tag{44}$$

(We do not consider the anti-unitary time-reversal in what follows.) Covariance thus requires that

$$\int \prod_i \mathrm{d}\mu(p_i) U_\theta(\Lambda) a^\dagger_{p_1} a^\dagger_{p_2}|0\rangle \mathrm{e}^{\frac{i}{2}(\Lambda p_1)\wedge(\Lambda p_2)}\mathrm{e}^{-\frac{i}{2}p_1\wedge p_2}\mathrm{e}_{\Lambda p_1} \otimes \mathrm{e}_{\Lambda p_2} = \int \prod_i \mathrm{d}\mu(p_i) a^\dagger_{p_1} a^\dagger_{p_2}|0\rangle \mathrm{e}_{\Lambda p_1} \otimes \mathrm{e}_{\Lambda p_2} \tag{45}$$

C. *The Dressing Transformation*

We can solve this requirement, as well as (43), by writing $a^\dagger_p$ in terms of the $c^\dagger_p$ and P_μ:

$$a^\dagger_p = c^\dagger_p \mathrm{e}^{\frac{i}{2}p\wedge P} \tag{46}$$

and setting

$$U_\theta(a,\Lambda) = U_0(a,\Lambda) = U(a,\Lambda) \quad . \tag{47}$$

The adjoint of (46) is

$$a_p = \mathrm{e}^{-\frac{i}{2}p\wedge P} c_p = c_p \mathrm{e}^{-\frac{i}{2}p\wedge P} \quad , \tag{48}$$

where the equality in the last step uses the anti-symmetry of $\theta_{\mu\nu}$.

As we can twist c_p on left *or* on right, we can write φ_θ as a twist applied to $\varphi_0 \equiv \varphi$:

$$\varphi_\theta = \varphi_0 \mathrm{e}^{-\frac{1}{2}\overleftarrow{\partial}\wedge P} \tag{49}$$

The transformation $\varphi_0 \to \varphi_\theta$ is an example of a dressing transformation. It was first introduced in the context of integrable models by Grosse [8] and by Faddeev and Zamalodichkov [9, 10].

It is important to note that (49) *is well-defined for a fully interacting Heisenberg field* φ_0 *if* P_μ *stands for the total four momentum of the interacting theory.* In that case φ_θ is the twisted Heisenberg field.

We can now check that

$$U(a,\Lambda)\varphi_\theta(x_1)\varphi_\theta(x_2)...\varphi_\theta(x_N)U(a,\Lambda)^{-1}|0\rangle = \varphi_\theta\big((a,\Lambda)x_1\big)\varphi_\theta\big((a,\Lambda)x_2\big)...\varphi_\theta\big((a,\Lambda)x_N\big)|0\rangle \tag{50}$$

with a similar equation for the vacuum $\langle 0|$ put on the left. Since vacuum is a cyclic vector, we can then be convinced that (49) fully solves the problem of constructing a covariant quantum field on the Moyal plane at the multi-field level as well.

A particular implication of (50) is that

$$U_\theta(a,\Lambda) = U(a,\Lambda) = U_0(a,\Lambda) \quad . \tag{51}$$

Its expression in terms of in-, out- or free fields looks the same as in the commutative case. It has no $\theta_{\mu\nu}$-dependence.

D. *Symmetrization and Covariance*

We will now show that the dressing transformations (46,48-49) are exactly what we need to be compatible with appropriate symmetrisation postulates.

At the level of the particle dynamics (functions on $\mathcal{M}^{d+1}$ and their tensor products), it is known that for the coproduct Δ_θ, symmetrisation and anti-symmetrisation should be based on the twisted flip operator

$$\tau_\theta = \mathscr{F}_\theta^{-1}\tau_0\mathscr{F}_\theta \tag{52}$$

$$\tau_0\alpha\otimes\beta := \beta\otimes\alpha \tag{53}$$

where α,β are single particle wave functions.

As defined, τ_0 and τ_θ act on two-particle wave functions and generate S_2 since

$$\tau_0^2 = \mathbb{1} \quad \Rightarrow \quad \tau_\theta^2 = \mathbb{1} \quad . \tag{54}$$

But soon we will generalise them to N-particles to get S_N.

Thus twisted bosons (fermions) have the two-particle plane wave states

$$\mathrm{e}_{p_1}\otimes_{S_\theta}\mathrm{e}_{p_2} = \frac{\mathbb{1}\pm\tau_\theta}{2}\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2} \quad . \tag{55}$$

Let us focus on S_θ:

$$\mathrm{e}_{p_1}\otimes_{S_\theta}\mathrm{e}_{p_2} = \frac{1}{2}\left[\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2} + \mathscr{F}_\theta^{-2}\mathrm{e}_{p_2}\otimes\mathrm{e}_{p_1}\right] \tag{56}$$

$$= \frac{1}{2}[\mathrm{e}_{p_1}\otimes\mathrm{e}_{p_2} + \mathrm{e}^{ip_2\wedge p_1}\mathrm{e}_{p_2}\otimes\mathrm{e}_{p_1}] \tag{57}$$

$$= \mathrm{e}^{ip_2\wedge p_1}\mathrm{e}_{p_2}\otimes_{S_\theta}\mathrm{e}_{p_1} \tag{58}$$

This gives

$$\int \prod_{i=1}^{2} \mathrm{d}\mu(p_i) a^\dagger_{p_1} a^\dagger_{p_2} |0\rangle \mathrm{e}_{p_1} \otimes_{S_\theta} \mathrm{e}_{p_2} \tag{59}$$

$$= \int \prod_{i=1}^{2} \mathrm{d}\mu(p_i) a^\dagger_{p_1} a^\dagger_{p_2} |0\rangle \mathrm{e}^{ip_2\wedge p_1} \mathrm{e}_{p_2} \otimes_{S_\theta} \mathrm{e}_{p_1} \tag{60}$$

$$= \int \prod_{i=1}^{2} \mathrm{d}\mu(p_i) \big(\mathrm{e}^{ip_1\wedge p_2} a^\dagger_{p_2} a^\dagger_{p_1}\big) |0\rangle \mathrm{e}_{p_1} \otimes_{S_\theta} \mathrm{e}_{p_2} \tag{61}$$

Thus we require that

$$a^\dagger_{p_1} a^\dagger_{p_2} = \mathrm{e}^{ip_1\wedge p_2} a^\dagger_{p_2} a^\dagger_{p_1} \tag{62}$$

which is fulfilled by (46).

We can extend this demonstration regarding the consistency of the twist to multinomials in $a^\dagger$'s and a's. The necessary tools are in [11]. We just note one point. In the N-particle sector, call $\mathscr{F}_\theta^{ij}$ the Drinfel'd twist (32) where in $\partial_\mu \otimes \partial_\nu$, ∂_μ acts on the ith and ∂_ν on the jth factor in the tensor product.

Define

$$\tau_\theta^{ij} = \mathscr{F}_\theta^{-1} \tau_0^{ij} \mathscr{F}_\theta = \mathscr{F}_\theta^{-2} \tau_0^{ij} \tag{63}$$

where τ_0^{ij} flips the entries of an N-fold tensor product by flipping the ith and j^{th} entries as in (53). Then

$$\left(\tau_0^{ij}\right)^2 = \mathbb{1} \tag{64}$$

which is obvious and

$$\tau_\theta^{i,i+1} \tau_\theta^{i+1,i+2} \tau_\theta^{i,i+1} = \tau_\theta^{i+1,i+2} \tau_\theta^{i,i+1} \tau_\theta^{i+1,i+2} \tag{65}$$

which is not obvious. It follows from (26) that $\tau_\theta^{i,i+1}$'s generate S_N in this sector.

One can check that the Poincaré group action with the twisted coproduct commutes with this action of S_N.

E. $*$-*Covariance*

Covariance requirements on the Moyal plane has led us to the dressed field (49). We now require it to be compatible with the $*$-operation. That is if $\varphi_0^* = \varphi_0$, we want that $\varphi_\theta^* = \varphi_\theta$. Now

$$\varphi_\theta^* = \mathrm{e}^{-\frac{1}{2}\partial\wedge P} \varphi_0 \tag{66}$$

where ∂_μ acts just on φ_0, P_ν acts on φ_0 and all that may follow. But since P_ν acting on φ_0 is $-i\partial_\nu\varphi_0$ and $\partial\wedge\partial=0$, we see that

$$\varphi^*_\theta = \varphi^*_0 \mathrm{e}^{-\frac{1}{2}\overleftarrow{\partial}\wedge P} \quad . \tag{67}$$

So the dressing transformations preserves $*$-covariance. The antisymmetry of θ plays a role in this process.

We can also understand these statements from (46). That gives

$$a_p = \mathrm{e}^{-\frac{i}{2}p\wedge P} c_p = c_p \mathrm{e}^{-\frac{i}{2}p\wedge P} \tag{68}$$

since $p\wedge p=0$. So *we can twist both creation and annihilation operators on the same side because θ is antisymmetric*. It is only because of this that we can get the twisted quantum Heisenberg field (49). The importance of its existence has been emphasised before.

V. MOYAL VS VOROS

The Voros plane $\mathcal{A}^V_\theta(\mathcal{M}^{d+1})$ is the algebra of functions on $\mathcal{M}^{d+1}$ with the star product

$$\alpha\star_V\beta = m_0(\mathscr{F}^V_\theta\,\alpha\otimes\beta), \quad \mathscr{F}^V_\theta = \mathrm{e}^{\frac{i}{2}\partial_\mu\otimes(\theta_{\mu\nu}-iS_{\mu\nu})\partial_\nu} \tag{69}$$

where $S_{\mu\nu}=S_{\nu\mu}$ defines a constant real *symmetric* matrix. The matrix θ fixes S, we will see how this happens for general d later.

But for $d=1$, this determination is easy to describe. For $d=1$, $\theta_{\mu\nu}=\hat\theta\epsilon_{\mu\nu}$, $\epsilon_{12}=-\epsilon_{21}=1$, $\epsilon_{11}=\epsilon_{22}=0$ and then $S_{\mu\nu}=\hat\theta\delta_{\mu\nu}$. So for $d=1$,

$$\mathscr{F}^V_\theta = \mathrm{e}^{\frac{i}{2}\partial_\mu\otimes\theta_{\mu\nu}\partial_\nu+\hat\theta\partial_\mu\cdot\partial_\mu} \tag{70}$$

where $\partial_\mu\cdot\partial_\mu$ is defined using the *Euclidean* scalar product:

$$\partial_\mu\cdot\partial_\mu := \sum_{i=0}^{1}\partial_\nu\partial_\nu \tag{71}$$

Let us first consider $d=1$.

On plane waves e_p ($\mathrm{e}_p(x)=\mathrm{e}^{-ip\cdot x}$), the Voros product is

$$\mathrm{e}_p\star_V\mathrm{e}_q = \mathrm{e}^{-\frac{1}{2}\hat\theta p\cdot q}\mathrm{e}^{-\frac{i}{2}p\wedge q}\mathrm{e}_{p+q} \tag{72}$$

where $p\cdot q$ is also defined using the Euclidean scalar product:

$$p\cdot q = \sum_{\nu=0}^{1} p_\nu q_\nu \quad . \tag{73}$$

It is well-known that $\mathcal{A}_\theta^{\mathcal{M}}$ and $\mathcal{A}_\theta^V$ are $*$-isomorphic algebras. Thus let

$$\mathrm{T} : \mathcal{A}_\theta^{\mathcal{M}} \to \mathcal{A}_\theta^V, \qquad \mathrm{T}\mathrm{e}_p = \mathrm{e}^{-\frac{1}{4}\hat{\theta}p^2}\mathrm{e}_p \tag{74}$$

Then a simple calculation shows that

$$\mathrm{T}(\mathrm{e}_p \star_{\mathcal{M}} \mathrm{e}_q) = (\mathrm{T}\mathrm{e}_p) \star_V (\mathrm{T}\mathrm{e}_q), \quad \mathrm{T}(\bar{\mathrm{e}}_p) = \overline{(\mathrm{T}\mathrm{e}_p)} \tag{75}$$

where bar denotes complex conjugation and $\star_{\mathcal{M}}$ denotes the Moyal product. (We denoted it previously as just $\star$.)

The $*$-isomorphism of $\mathcal{A}_\theta^{\mathcal{M},V}$ may suggest that quantum field theories are not sensitive to which algebra we use. But that is not the case. Thus we should require that the twisted (dressed) in- (out-) creation and annihilation operators on $\mathcal{A}_\theta^V$ are adjoints of each other for $*$-covariance. But this imposition spoils the possibility of constructing Heisenberg fields.

On the other hand, a naive construction of the dressed Heisenberg field is incompatible with the adjoint operation: such a dressing applied to a self-adjiont field is not self-adjoint.

These results have been discussed before [12, 13]. Here we recall the proofs.

Let us first assume that the Voros $\star$ also admits twisted creation-annihilatin operators and associated (in-, out-, or free-) field $\varphi_{\theta,V}$ as in (37):

$$\varphi_{\theta,V} = \int \mathrm{d}\mu(p) \left[a_{p,V}^\dagger \mathrm{e}_p + a_p \mathrm{e}_{-p}\right] := \varphi_{\theta,V}^{(-)} + \varphi_{\theta,V}^{(+)} \tag{76}$$

Primitive covariance gives as before

$$U(a,\Lambda) a_{p,V} U(a,\Lambda)^\dagger = a_{\Lambda p,V} \tag{77}$$

and

$$U(a,\Lambda) a_{p,V}^\dagger U(a,\Lambda)^\dagger = a_{\Lambda p,V}^\dagger \tag{78}$$

where we did not attach a θ to U.

In the two-particle sector, the coproduct by general principles is

$$\Delta_{\theta,V}(g) = \mathscr{F}_{\theta,V}^{-1}(g \otimes g)\mathscr{F}_{\theta,V} \tag{79}$$

As $\mathscr{F}_{\theta,V}$ is translationally invariant, the coproduct for P_μ is not affected by the twist. So we focus on Lorentz transformations.

For Lorentz transformations, (45) is modified to

$$\left(\int\prod \mathrm{d}\mu(p_i)U(\Lambda)a^\dagger_{p_1,V}a^\dagger_{p_2,V}|0\rangle\right)\mathrm{e}^{\frac{i}{2}(\Lambda p_1)\wedge(\Lambda p_2)-\frac{\hat\theta}{2}(\Lambda p_1)\cdot(\Lambda p_2)}\mathrm{e}^{-\frac{i}{2}p_1\wedge p_2-\frac{\hat\theta}{2}p_1\cdot p_2}\mathrm{e}_{\Lambda p_1}\otimes\mathrm{e}_{\Lambda p_2} \tag{80}$$

giving the dressing equation

$$a^\dagger_{p,V}=c^\dagger_p\mathrm{e}^{\frac{i}{2}p\wedge P-\frac{\hat\theta}{2}p\cdot P}\quad , \tag{81}$$

scalar products being Euclidean.

The adjoint of (81) is

$$a_{p,V}=\mathrm{e}^{-\frac{i}{2}p\wedge P-\frac{\hat\theta}{2}p\cdot P}c_p=\mathrm{e}^{\frac{\hat\theta}{2}p\cdot p}c_p\mathrm{e}^{-\frac{i}{2}p\wedge P-\frac{\hat\theta}{2}p\cdot P} \tag{82}$$

which is not what we get by dressing c_p on the right.

The result is that $\varphi_{\theta,V}$ is not the outcome of dressing $\varphi_{0,V}$ by a single twist. Its parts $\varphi^{(\mp)}_{\theta,V}$ get separate twists.

But then there is no way to dress a fully interacting Heisenberg field Φ_0 since Φ_0 cannot decomposed into positive and negative frequency parts.

Or else we can declare that the Voros Heisenberg field is

$$\Phi_{\theta,V}=\Phi_0\mathrm{e}^{\frac{1}{2}\overleftarrow{\partial}\wedge P+i\frac{\hat\theta}{2}\overleftarrow{\partial}\cdot P} \tag{83}$$

But then if $\Phi_0^\dagger=\Phi_0$, $\Phi^\dagger_{\theta,V}\neq\Phi_{\theta,V}$. Unitarity is spoilt.

It seems that the Voros plane is not suitable for quantum field theories.

If $d\neq 1$, say $d=3$, then by a change of coordinates, we can bring it to the form

$$\hat\theta_1\epsilon_{ab}+\hat\theta_2\epsilon_{a'b'}\quad (a,b\in[0,1],\ a',b'\in[2,3])\quad . \tag{84}$$

The preceding considerations then apply separately to $\hat\theta_1\epsilon_{ab}$ and $\hat\theta_2\epsilon_{a'b'}$.

VI. DISCRETE GROUPS

Covariance is a notion tied to symmetry group, and in our context especially to spacetime diffeomorphism groups.

A particularly interesting class of such symmetry groups are mapping class groups of manifolds. They are discrete and for spatial hypersurfaces supporting topological geons can be abelian and nonabelian. In this section we recall our discussion of covariant fields for such geon spatial slices from [14].

A. *Covariant Quantum Fields for Commutative Geons*

Let $\mathcal{P}$ be a prime three-manifold, and $\mathbb{R}^3\#\mathcal{P}$ the spatial slice where # denotes connected sum. Spacetime is then $(\mathbb{R}^3\#\mathcal{P})\#\mathbb{R}$.

Let D^∞/D_0^∞ be the mapping class group of $\mathbb{R}^3\#\mathcal{P}$ where D^∞ is the diffeo group which keeps a point p ("infinity") of $\mathbb{R}^3\#\mathcal{P}$ and a frame at p fixed and D_0^∞ its identity component. If φ_0 is a covariant quantum field, primitive covariance requires that

a) There is a unitary representation

$$U: \quad g^\infty \to U(g^\infty), \quad g^\infty \in D^\infty \tag{85}$$

such that

$$U(g^\infty)\varphi_0(p)U(g^\infty)^{-1} = \varphi(g^\infty p) \quad . \tag{86}$$

In addition, constraints in gravity theories require that

b) $\varphi_0(g_0^\infty p) = \varphi_0(p), \quad g_0^\infty \in D_0^\infty$.

Note that by b), (86) can be interpreted in terms of a unitary representation of D^∞/D_0^∞.

For the Poincaré group, the twists were all based on the abelian translation group. Likewise, for now we will base our considerations on twists on the maximal compact abelian subgroup

$$A = \times_{i=1}^k \mathbb{Z}_{n_i} \tag{87}$$

There is no loss of generality in assuming compactness as non-compact factors like $\mathbb{Z}$ do not enter the twist [14].

We now choose suitable basis of functions for $\mathbb{R}^3\#\mathcal{P}$ adapted to A.

Pick a Riemannian metric for $\mathbb{R}^3\#\mathcal{P}$. Its volume form defines a Hilbert space $\mathcal{H}$ of functions on $\mathbb{R}^3\#\mathcal{P}$.

Now the unitary irreducible represenation UIRR $m_i \in \mathbb{Z}/(n_i\mathbb{Z}) := \mathbb{Z}_{n_i}$ of $\mathbb{Z}_{n_i}$ is defined by

$$\mathbb{Z}_{n_i} \ni \xi = \mathrm{e}^{i\frac{2\pi}{n_i}} \to \xi^{m_i} \quad . \tag{88}$$

So the UIRR's of A are defined by

$$\vec{m} = (m_1, ..., m_k) \tag{89}$$

where m_i and $m_i + n_i$ are identified.

Since A acts on $\mathbb{R}^3\#\mathcal{P}$ and hence on $\mathcal{H}$, the latter has an orthonormal basis $\{f^{(\pm)}_{\vec{m}}\}$ which carries the UIRR $\vec{m}$ of A and have positive and negative frequencies $\pm|E_{\vec{m}}|$:

$$f^{(\pm)}_{\vec{m}}(g_0^\infty p) = f^{(\pm)}_{\vec{m}}(p), \quad g_0^\infty \in D_0^\infty \qquad , \tag{90}$$

$$f^{(\pm)}_{\vec{m}}(h^{-1}p) = f^{(\pm)}_{\vec{m}}(p)\chi_{\vec{m}}(h), \quad h \in D^\infty \qquad , \tag{91}$$

$$i\partial_0 f^{(\pm)}_{\vec{m}} = \pm|E_{\vec{m}}| f^{(\pm)}_{\vec{m}} \quad . \tag{92}$$

(We postulate that such $f^{(\pm)}_{\vec{m}}$ exist.)

Here $\chi_{\vec{m}}$ is the character in UIRR $\vec{m}$. Since $\bar{\chi}_{\vec{m}} = \chi_{-\vec{m}}$, we can assume that

$$\bar{f}^{(\pm)}_{\vec{m}} = f^{(\mp)}_{-\vec{m}} \tag{93}$$

If $g \in D^\infty$, we can write

$$f^{(\pm)}_{\vec{m}}(g^{-1}p) = \sum_{\vec{m}'} f^{(\pm)}_{\vec{m}'}(p)\mathscr{D}_{\vec{m}'\vec{m}}(g) \tag{94}$$

where $\mathscr{D}$ is a unitary representation of D^∞ which restricted to D_0^∞ becomes the trivial representation.

The untwisted quantum field (in, out or free) has the mode expansion

$$\varphi_0 = \sum_{\vec{m}} \left[c_{\vec{m}} f^{(+)}_{\vec{m}} + c^\dagger_{\vec{m}} f^{(-)}_{-\vec{m}}\right] \quad . \tag{95}$$

Then since $\bar{\mathscr{D}}_{\vec{m}'\vec{m}}\mathscr{D}_{\vec{n}'\vec{m}} = \delta_{\vec{m}'\vec{n}'}$, commutative covariance translates to the transformation law

$$U(g)c_{\vec{m}}U(g)^{-1} = c_{\vec{m}'}\bar{\mathscr{D}}_{\vec{m}',\vec{m}}(g) \quad , \tag{96}$$

$$U(g)c^\dagger_{\vec{m}}U(g)^{-1} = c^\dagger_{\vec{m}'}\mathscr{D}_{\vec{m}',\vec{m}}(g) \quad . \tag{97}$$

B. *Covariant Geon Fields for Abelian Twists*

This material (just as the preceding material) has been reported elsewhere [14]. So we will be brief.

Let $\mathbb{P}_{\vec{m}}$ be the projector in the group algebra $\mathbb{C}A$ to the UIRR $\vec{m}$. Then the Drinfel'd twist based on A is

$$F_\theta = \sum_{\vec{m}',\vec{m}} \mathrm{e}^{-\frac{i}{2}m_i\theta_{ij}m'_j}\mathbb{P}_{\vec{m}} \otimes \mathbb{P}_{\vec{m}'}, \quad \theta_{ij} = -\theta_{ji} = \frac{4\pi}{n_{ij}}, \quad n_{ij} \text{ divdes } n_i \text{ and } n_j \quad . \tag{98}$$

The mode expansion of the twisted field φ_θ is

$$\varphi_\theta = \sum_{\vec{m}} [a_{\vec{m}} f^{(+)}_{\vec{m}} + a^\dagger_{\vec{m}} f^{(-)}_{-\vec{m}}] \tag{99}$$

In [14], we show that the requirements of covariance for multiparticle states, twisted symmetrisation and self-reproduction are all compatible with the following expression for $a_{\vec{m}}$, $a^\dagger_{\vec{m}}$:

$$a_{\vec{m}} = \sum_{\vec{m}'} c_{\vec{m}} \mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} m'_j} \mathbb{P}_{m'_j} \quad , \tag{100}$$

$$a^\dagger_{\vec{m}} = \sum_{\vec{m}'} c^\dagger_{\vec{m}} \mathrm{e}^{-\frac{i}{2} m_i \theta_{ij} m'_j} \mathbb{P}_{m'_j} \quad . \tag{101}$$

C. *Non-abelian Twsts*

Twists such as F_θ based on abelian groups A lead to associative spacetimes. They can be generalised to twists based on nonabelian group algebras. They lead to nonassociative spacetimes [14, 15].

A brief examination of covariant quantum fields for such twists is contained in [14]. It requires more elaboration. In particular not only does spacetime become nonassociative, the coproduct on the symmetry group also loses coassociativity: the symmetry algebra becomes quasi-Hopf [15]. The implications of nonassociativity and quasi-Hopf algebras for quantum field theory and phenomenology remain unexplored.

VII. FINAL REMARKS

Many papers have been written regarding quantum fields on the Moyal and similar algebras [16–18] and on geon spacetimes as well. (See [14] for references.) In much of this work, quantum fields were constructed using the dressing transformation. This paper systematically clarifies the conceptual basis behind this transformation: it is just covariance. The latter in essence means that symmetry transformations on spacetime and associated structures like suitable symmetrisation postulates of particle wave functions are implementable in the quantum Hilbert space. In classical theory, the analogous requirement would be the canonical implementability of symmetry transformations.

From this point of view, it is clear that covariance and dressing are sensible ideas to construct quantum fields on spacetimes based on Drinfel'd twists.

Noncommutative spacetimes lead to theories which are acausual and violate Poincaré invariance in scattering processes. They violate CPT as well and can lead to Pauli-forbidden transitions [17, 18]. But all these seem to be controlled by Planck-scales, and not susceptible to tests by current experiments. It remains a challenge to locate potential signals of Planck scale spacetime effects at presenty accessible energy scales.

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