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Logic is to the quantum as geometry is to gravity*

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Abstract

I will propose that the reality to which the quantum formalism implicitly refers is a kind of generalized history, the word history having here the same meaning as in the phrase sum-over-histories. This proposal confers a certain independence on the concept of event, and it modifies the rules of inference concerning events in order to resolve a contradiction between the idea of reality as a single history and the principle that events of zero measure cannot happen (the Kochen-Specker paradox being a classic expression of this contradiction). The so-called measurement problem is then solved if macroscopic events satisfy classical rules of inference, and this can in principle be decided by a calculation. The resulting conception of reality involves neither multiple worlds nor external observers. It is therefore suitable for quantum gravity in general and causal sets in particular.

1. Quantum gravity and quantal reality

Why, in our attempts to unify our theories of gravity and the quantum, has progress been so slow? One reason, no doubt is that it's simply a very hard problem. Another is that we lack clear guidance from experiments or astronomical observations. But I believe that a third thing holding us back is that we haven't learned how to think clearly about the quantum world in itself, without reference to "observers" and other external agents.

Because of this we don't really know how to think about the Planckian regime where quantum gravity is expected to be most relevant. We don't know how to think about the

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vacuum on small scales, or about the inside of a black hole, or about the early universe. Nor do we have a way to pose questions about relativistic causality in such situations. This is particularly troubling for the causal set program [1], within which a condition of "Bell causality" has been defined in the classical case, and has led there to a natural family of dynamical laws (those of the CSG or "classical sequential growth" models) [2]. If we possessed an analogous concept of "quantal Bell causality", we could set about deriving a dynamics of quantal sequential growth. But without an observer-free notion of reality, how does one give meaning to superluminal causation or its absence in a causal set?

It's not that individual physicists have no notion of what the quantal world is like, of course. We all employ intuitive pictures in our work, and for example, I imagine that very few people think of a rotons in a superfluid in terms of selfadjoint operators. But what we lack is a coherent descriptive framework. We lack, in other words, an answer to the question, What is a quantal reality?

My main purposes in this talk are first to propose an answer to this question (or really a family possible answers), and second to explain how, on the basis of this answer, the so called measurement problem can be posed and plausibly solved. My proposal belongs to the histories-based way of thinking about dynamics, which in a quantal context corresponds to path-integral formulations. More specifically it rests on three or four basic ideas, that of *event*, that of *preclusion*, and that of *anhomomorphic inference* concerning *coevents*, whose meaning I will try to clarify in what follows.

2. Histories and events (the kinematic input)

In classical physics, it was easy to say what a possible reality was, although the form of the answer was not static, but changed as our knowledge of nature grew. Electromagnetic theory, for example, conceived reality as a background Minkowski spacetime inhabited by a Faraday field F_{ab} together with a collection of particle worldlines, each with a given charge and mass, while reality for General Relativity was a 4-geometry together with possible matter fields, thus a diffeomorphism-equivalence class of Lorentzian metrics and other fields. Of course, we were far from knowing all the details of the actual reality, but we could say exactly what in principle it would have taken to describe reality fully if we did have the details. Thus we could survey all the kinematically possible realities, and then

go on to state the dynamical laws (equations of motion or field equations) that further circumscribed these possibilities.

Another example comes from Brownian motion, which in important ways stands closer to quantum mechanics than deterministic classical theories do. Here, if we imagine that nothing exists but one Brownian particle, then a possible reality is just a single worldline (continuous but not differentiable), and the dynamical law is a set of transition probabilities, or more correctly a probability-measure on the space of all worldlines.

Such a possible reality — a spacetime, a field, a worldline, etc. — is what is meant by the word "history" in the title of this section; and according to the view I am proposing, such histories furnish the raw material from which reality is constructed.* However, unlike in classical physics, we will not (in general) identify quantal reality with a single history, instead we will have certain sorts of "logical combinations" of histories which will be described by *coevents*. In the simplest case a coevent will correspond merely to a set of histories. Although, without further preparation, I cannot yet make precise what quantal reality will be, let me stress at the outset what it will *not* be, namely a wave-function or state-vector. Nor will the Schrödinger equation enter the story at all. Such objects can have a technical role to play, but at no stage will they enter the basic interpretive framework.

As already indicated the concepts of event and coevent will be fundamental to this framework. In order to define them, we need first to introduce the *history space* Ω whose elements are the individual histories. An *event* is then a subset of Ω. When Ω contains an infinite number of histories, not every subset will be an event because some sort of "measurability" condition will be required, but for present purposes I will always assume that $|\Omega| < \infty$. In that case, one can equate the word "event" to the phrase "subset of Ω".

Notice that this definition of the word "event" parallels its use in everyday speech, where a sentence like "It rained all day yesterday" denotes in effect a large number of

* The word history thus denotes the same thing it does when people call the path integral a "sum-over-histories". To avoid confusion with other uses of the word, one might say *proto-history* or perhaps "kinematical" or "bare" history. Given this distinction, one might then refer to quantal reality as a "quantal history".

more detailed specifications of the weather, all lumped together under the heading "rain". (This usage of "event" also follows the customary terminology in probability theory, where however Ω is often called the "sample space" rather than the history space.) On the other hand, one should not confuse event in this sense with the word "event" used to denote a point of spacetime. The latter may also be regarded as a type of event, perhaps, but that would only be a very special case of what I mean by event herein, and it would be relevant only in connection with quantum gravity per se.

Let us write $\mathfrak{A}$ for the space of all events. Structurally, $\mathfrak{A}$ is a Boolean algebra, meaning that union, intersection, complementation and symmetric-difference are defined for it. In logical terms these correspond respectively to the connectives *or*, *and*, *not*, and *xor*.

Finally, we need to define *coevent*. We have defined the dual object, an event $E \in \mathfrak{A}$, as a subset of Ω, but we can also think of an event as a *question*, in our previous example the question "Did it rain all day yesterday"? A coevent can then be thought of as something that answers all conceivable questions. More formally, a *coevent* will be a map $\phi : \mathfrak{A} \to \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the two-element set $\{0, 1\}$, the intended meaning being that $\phi(E) = 1$ if and only if the event E actually happens. Thus, 1 represents the answer "yes" (or "true") and 0 the answer "no" (or "false"). I will take the point of view that such a coevent is a full description of reality, quantal or classical. After all, what more could one ask for in the way of description than an answer to every question that one might ask about the world? For now we place no conditions on ϕ other than that it takes events to members of $\mathbb{Z}_2$. (Notice that a coevent is in some sense a "higher order object". If one thought of events as "predicates" then a coevent would be a "predicate of predicates". [†])

[†] A function from a Boolean algebra to $\mathbb{Z}_2$ is sometimes called a "truth valuation", but that terminology would be misleading here. It would suggest too strongly that the event-algebra belongs to some *a priori* formal language or logical scheme, with reality being merely a "model" of that logic. But in the context of physics, it would seem that the events come first and the descriptive language second. Moreover, one of the main points of this paper philosophically is (as its title indicates) that the rules of logical inference are part of physics and can never be written down fully without some knowledge of the dynamics.

3. Preclusion and the quantal measure (the dynamical input)

I've said that anhomomorphic logic grows out of the path integral, but in order to understand what this means, you must think of the path integral as something more than just a propagator from one wave function ψ to another. What, in fact, does the path integral really compute, if we try to understand it on its own terms?

Let us for a (very brief) moment adopt an "operational" point of view which only cares about the probabilities of instrumental "pointer events". Such events can be idealized as "position measurements" (the positions of the pointers), and it has been known for a long time that the joint probabilities for successive position measurements can be computed directly from the path integral without reference to any wave function, except possibly as a shortcut to specifying initial conditions. The probabilities in question here are those furnished by the standard evolve−collapse−evolve algorithm to be found in any textbook of quantum mechanics. When one abstracts from the mathematical machinery used to compute them, what remains is a probability measure μ on a space of "instrument events", and it is this measure, rather than any wave function, that has direct operational meaning! How one may compute μ directly from the path integral is described in more detail in reference [3], but for us here the important points are three. First that μ refers not to "measurements" per se, but merely to certain macroscopic happenings, and second that it is natural, when μ is expressed as a path-integral, to regard these macroscopic happenings as being *events* in precisely the sense defined above. (The histories in this case would specify the trajectories of the molecules comprising the "pointer", and an event would be, as always, a set of histories.) The crucial observation then is that the path integral computation of μ makes sense for *any* set of histories — any event — and therefore need not be tied to some undefined notion of "measurement".

This event-function $\mu : \mathfrak{A} \to \mathbb{R}^+$ I will call the *quantal measure*, and I will take the point of view that it is the answer to my question above about what the path integral really computes when we try to understand it on its own terms. From this histories vantage point, the textbook rules for computing probabilities are not fundamental principles, but rather rules of thumb whose practical success for certain macroscopic events needs to be explained on the basis of a deeper understanding of what the quantal measure is telling us about microscopic reality. Or to put the point another way, quantum mechanics formulated via the path integral presents itself to us as a generalized probability theory with μ appearing

as a generalized probability measure [4] [5][6]. Our first task then is to interpret this generalized measure. (Much more could be said about the formal properties of μ and their relation to the hierarchy of [7] and to the *decoherence functional* that was first defined within the "decoherent histories" interpretation of quantum mechanics [8]. I will, however, limit myself to these two references and to a reference to an experiment currently testing the "3-slit sum rule" that expresses in great generality the quadratic nature of the Born rule [9].)

One's first thought might be to interpret $\mu(E)$ as an ordinary probability attaching to the event E, but this idea founders at once on the failure of μ to be additive on disjoint events. Since such non-additivity expresses the physical phenomenon of *interference* that lies at the heart of quantum mechanics, the impasse seems to me to be definitive: some other concept than probability in the sense of relative frequency seems to be called for. Moreover, μ fails to be bounded above by 1, whence some events E would have to be "more than certain", were we to take $\mu(E)$ as a probability in the ordinary sense. There is however, one special case in which normalization and additivity become irrelevant, namely for events E such that $\mu(E) = 0$. In such a case, one could conclude on almost any interpretation that the event E should never happen. (Classically, $\mu(E)$ can never vanish exactly except in trivial cases, but quantally it can, thanks precisely to interference!) Such an event (of μ-measure 0), I will denote as *precluded*, and I propose to interpret μ in terms of the following *preclusion postulate*: If $\mu(E) = 0$ then E does not happen.

With respect to a given coevent ϕ, the "not happening" of E is expressed, as we have seen, by the equation $\phi(E) = 0$, and the preclusion postulate becomes thereby a rule limiting the coevents that are dynamically possible:

$$\mu(E) = 0 \quad \Rightarrow \quad \phi(E) = 0$$

A coevent that fulfills this condition, I will call *preclusive*.

By isolating in this manner what one might call the purely logical implications of the generalized measure μ , one may hope to bring out those aspects which are peculiarly quantal, as opposed to aspects pertaining to probability more generally. Of course it will be necessary at some stage to recover not only the "logical" but also the properly probabilistic predictions one obtains from the standard quantum apparatus. Whether

or not the preclusion rule above will suffice for this is not entirely clear, but if it does, one will have cleared up some of the confusion surrounding even the classical probability concept. If on the other hand, one needed something more than the strict preclusion rule, one could simply extend it to embrace the case of "approximate preclusion", where $\mu(E)$ is not exactly zero but still small enough to be treated as if it vanished. In this way, the difficulties of classical probability would not have grown any better, but (hopefully) they would not have grown any worse either [10]. By basing the probability concept on approximate preclusion, one would in effect be adopting the interpretation sometimes known as Cournot's principle, according to which the assertion that an event of sufficiently small measure will not happen exhausts the scientific meaning of probability. (See [11] for a concise statement of this idea.) [♭] Cournot's principle is not free of problems, of course, but neither is any other account of probability, as far as I know. In any case, it seems prudent to leave probability aside at first, and concentrate on the purely logical questions raised by the preclusion principle. Considering that the latter seem to require a radical revision of some basic logical presuppositions, questions of probability might appear in a very different light, once a more adequate picture of quantal reality is in place.

To summarize the burden of this section then, the idea is that the whole dynamical content of the quantal formalism reduces to the preclusion rule stated above (possibly supplemented by its generalization to the case of approximate preclusion).

4. The three-slit paradox and its cognates

Viewed through the lens of the path-integral, quantum theory appears as a generalized theory of stochastic processes characterized by the quantal measure μ, and this makes feasible a "histories based" way of thinking about the dynamics that seems more suited to the needs of quantum gravity than alternative accounts inspired by either the S-matrix or the Schrödinger equation. For such an approach to succeed, however, one needs to free the path integral from its *conceptual* dependence on objects like the wave function.

[♭] Predictions about frequencies follow when one construes multiple repetitions of some experiment as a single, combined experiment grouping all the repetitions together into a single sample space. The event that the overall frequencies come out wrong will then possess a tiny measure.

That is, one needs a *free-standing* histories-based formulation of quantum theory. A priori, such a formulation need not base itself on the path integral, but as things stand, no other alternative has so far offered itself. In practice then we can (for now at least) vindicate the histories-based viewpoint (also called the "spacetime" viewpoint) only by clarifying the physical meaning of the quantal measure μ.

I have proposed above that the dynamical implications of μ are mediated by what it tells us about the precluded events, the sets of histories of zero measure in Ω. Perhaps there is more to it than this, but even if preclusion is not the full story, it is hard to see how — without entirely abandoning the attempt to interpret μ as some sort of generalized probability measure — one could avoid the implication that events of measure zero do not happen. If this is correct, then acceptance of the preclusion principle is a minimal requirement for re-conceiving quantum mechanics along lines suggested by the path-integral formalism.

The problem then is that, thanks to interference, there are far too many sets of measure zero, so many in fact that events which are in reality able to occur seem to be ruled out as a logical consequence of the preclusion of other events that overlap them. (Remember that event = subset of Ω.) Here I'm referring to the numerous "logical paradoxes" of quantum theory, including the Kochen-Specker paradox,[⋆] the GHZ paradox, the Hardy paradox, and the "three-slit paradox" that I'll focus on in a moment. Each of these can be realized in terms of sets of particle trajectories together with appropriate combinations of slits or Stern-Gerlach-like devices (as in [13] or [14] for example), so that the relevant quantal measure can be discerned. What then makes all these paradoxes paradoxical is that all or part of the history space Ω is covered by precluded events. In the Kochen-Specker setup, these overlapping preclusions cover the whole of Ω, implying, according to our customary way of reasoning, that nothing at all can happen (cf. [15]). The other examples are similar, but not quite as dramatic.

[⋆] This comes in two versions, the original version referring to a single spin-1 system, and the version of Allen Stairs [12] referring to an entangled pair of such systems. The latter seems to have been the first example of an obstruction to locally causal theories based purely on logic (as opposed to probability-based obstructions like the Bell inequalities).

The contradictions in question can be illustrated with a diffraction experiment involving not two, but three "slits". Consider then, an idealized arrangement as shown, with source S, apertures a, b, c, and a designated location d to which the particle in question might or might not travel. (The letter d is meant to suggest "detector", but modeling one explicitly would complicate our setup unnecessarily, without changing anything essential, as long as we can assume that the detector would function properly.)

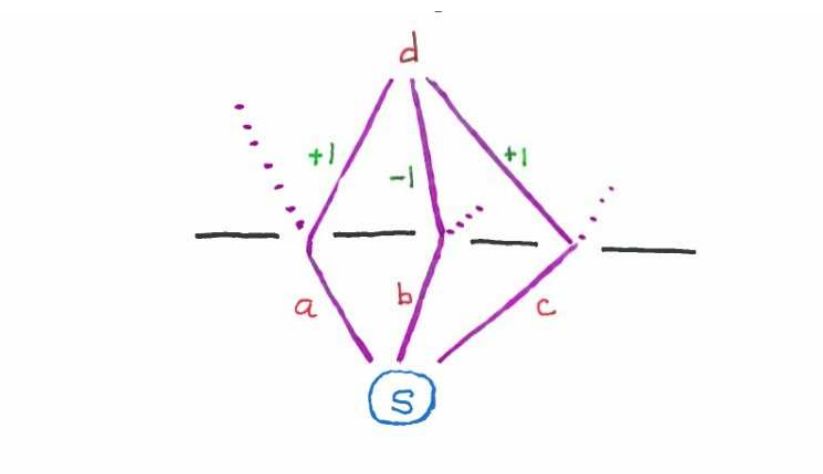

Figure 1. The three-slit paradox

To this setup belongs a history space Ω consisting of the various possible particle trajectories, and a quantal measure μ assigning a nonnegative real number to each set of trajectories. Let a be the event that the trajectory passes through slit a and similarly for b and c, and let d be the event that it arrives at d. Consider further the event A that the particle arrives at d after traversing a. (Notice here that a, b and c are all intrinsic events, not measurement events. We are not placing detectors at any of the slits, either explicitly or implicitly.) Writing the intersection $X \cap Y$ of two arbitrary events X and Y simply as their product XY, we have then that

$$A = ad,\ B = bd,\ C = cd,\quad d = A + B + C\ ,$$

where in the last equation a plus sign has been used to denote the union of disjoint subsets.

Now imagine the region d to be small enough that we can represent the path-integrals for A, B, and C by single amplitudes whose squares yield the (un-normalized) measures

of the corresponding events, and suppose further that these amplitudes are $+1$ for events A and C and -1 for B. Then $\mu(d) = \mu(A+B+C) = |1-1+1|^2 = 1$, whereas

$$\mu(A+B) = \mu(B+C) = |1-1|^2 = 0 \ .$$

Therefore, the events $A+B = d(a+b)$ and $B+C$ are precluded even though $A+B+C$ is not and can sometimes happen.

If we think classically, this is an outright contradiction. Suppose we look for the particle at d and find it there. We can then infer that since it didn't pass through a or b ($A+B$ being precluded) it must have arrived via c. But reasoning symmetrically, we can infer by the same token that it must have arrived via a. Obviously, the two conclusions contradict each other.

In the language of coevents, we can express the situation in formulas as

$$\phi(A+B) = 0, \ \phi(C+B) = 0, \ \phi(d) = \phi(A+B+C) = 1,$$

where the first two equations follow from the preclusion rule and the third expresses that the particle did arrive at d. Formally a contradiction can be derived from these three equations by Boolean manipulations following the classical rules of inference. If one asks which rules were used (see the Appendix), one comes up with the following list, where A and B represent arbitrary events and $\neg A = \Omega \backslash A$ is the complementary event to A.

From $\phi(A) = \phi(B) = 1$ conclude $\phi(AB) = 1$.

From $\phi(A) \neq 1$ conclude $\phi(A) = 0$.

From $\phi(A) = 0$ conclude $\phi(\neg A) = 1$.

From $A \subseteq B$ and $\phi(A) = 1$ conclude $\phi(B) = 1$.

These formal relationships are instructive, but one can also see the root of the inconsistency informally in a way that indicates how one might think to escape it. What the formal rules really express is the ingrained belief that reality is described by a *single* trajectory γ such that an event A happens (the corresponding predicate is true) if and only if $\gamma \in A$. We might therefore be able to extricate ourselves from the contradiction if reality were given not by a single trajectory but by some more subtle combination of

trajectories, for which — in some sense — both A and C could happen simultaneously, or alternatively for which $A + C$ could happen without either A or C happening separately. We will see that the so called "multiplicative scheme" realizes the latter alternative.

5. Freeing the coevent

Which came first, the history or the event? To the extent that an event is nothing but a set of histories, it might seem that the history came first, and this would be in accord with the classical worldview, where only a single history is in some sense actual.[†] On the other hand when we consult our experience, what we meet with are events. Individual histories we experience — if at all — only as idealized limits of events. One might argue on this basis that it is the event that should come first, and this would be consistent with a more "holistic" or "dialectical" attitude toward the history space Ω (cf. category theory, toposes, etc.)

Be that as it may, the practical needs of probabilistic theories, I think, force us to accord events an independent status, for it is only they which have nontrivial measures in general. For quantal measures this argument becomes more convincing, because the measure of an event no longer reduces, even formally to the measures of its constituent histories. (At best, it reduces to the measures of pairs of histories.) It seems clear in particular that the concept of preclusion makes no sense at all except in relation to events. Once events are dignified in this manner, the rules governing coevents also acquire a certain freedom, and I am proposing to use this freedom in order to overcome the logical conflict between preclusion and the doctrine that reality can be described fully by a single history. More concretely, I am proposing to describe reality, not by an individual history but by an individual coevent, which mathematically is a kind of "polynomial in histories". The rules governing which coevents are dynamically possible can then change in such a way as to accommodate the preclusion principle without engendering an inconsistency. In the simplest case the polynomial is just a monomial, meaning in effect just a *subset* S of the

[†] I'm leaving aside here questions of "temporality": is the past "actual", or the past and the future, or only the present, or ...? I hope that the neglect of such questions will not unduly prejudice the rest of this discussion.

history space. This simplest case, that of the *multiplicative scheme*, is the only one I will discuss herein. Other schemes are described in [16], [17] and [18].

I am tempted at this point just to present the multiplicative scheme and discuss some of its applications, but I'm afraid that without further background, it would appear far less natural than it will if its intimate connection with deductive logic is brought out. On the other hand, I know that for some people, any hint of tampering with classical logic raises a barricade between them and the slightest sympathy with whatever comes next. For them I should emphasize that the type of scheme I am proposing can stand as a self-contained framework, whether or not one accepts a logical way of looking at it. With this caveat, let me embark on some remarks relating logic to physics that will lead in a natural way to the multiplicative scheme in the next section.

For a scientist, logical inference is — or I believe should be — a special case of dynamics. Think for example of forecasting the motion of mars using Kepler's laws. Here we begin with certain events, the locations of mars at certain earlier times, and from them we infer certain other events, namely its locations at certain later times. In other cases, we may draw conclusions from the non-occurrence of an event. Thus, from the fact that the event "sighting of the new moon" did not happen last night, we might conclude that it will happen tomorrow night (and in consequence a new month of the Islamic calendar will begin). This second example illustrates, I hope, how inferences from dynamical laws can shade gradually into inferences from logic alone, for example the inference that if my keys are not in my pocket they must be in my jacket (which I left locked in my house). In the extreme case of simple abstract deductions like "if 'A' is true and 'B' is true then 'A and B' is also true", the inference feels so obvious that we hardly recognize it as an inference at all, but this feeling goes away for more complicated rules like "Peirce's law".[♭] Or think of the logical puzzles like, "the green house is to the right of the white house, coffee is drunk in the green house,...".

Notice here that the logic I'm speaking of concerns physical events, not strings of words and not propositions in a formal language. It is a "logic of nature", not a logic of language or thought or mathematical truth. This logic, I contend, is not prior to experience. Rather

[♭] $((A \to B) \to A) \to A$

it codifies certain relations among events that, until recently, have been so ubiquitous in human experience that they have been ossified and condensed into a scheme that seems as if it were unchangeable. What we do when we predict where mars will appear next week is exactly what the rules of logical inference do in a limited way, but also in an absolute and universal manner reminiscent of how geometry was once treated as prior to physics. But just as a better understanding of gravity forced geometry back into touch with physics, so also a better understanding of the microworld can do the same for logic. The resulting inferential scheme will not be universal but will depend (at least in part) on the particular physical system to which it is adapted. This, at any rate, will be true for the type of logic exemplified by the multiplicative scheme.

For present purposes, it helps to view a logic as a threefold structure or "triad", whose outlines tend to be obscured when formal logicians write about their subject. The first component of the triad is the event algebra $\mathfrak{A}$, a Boolean algebra whose members can be thought of as questions about the world, as described above. If we adopt this imagery then the second component of the triad is the space $\mathbb{Z}_2$ of possible answers (or "truth values"), and the third (and most neglected) is the "answering map" or coevent $\phi : \mathfrak{A} \to \mathbb{Z}_2$. In any given physical theory, $\mathfrak{A}$ and $\mathbb{Z}_2$ will be fixed but ϕ will vary in the same way that the solutions to Maxwell's equations vary. Each dynamically allowed ϕ describes then, a possible reality (or "quantal history" as one might term it). In this context, dynamical "laws of motion" in the traditional sense and rules of logical inference both take the form of conditions on ϕ. The classical rules of inference can be stated very simply if we view them in this manner. In fact, we can give them in three equivalent forms, one "deductive", one "algebraic" and one with a topological or order-theoretic flavor.*

Before stating these rules however, I ought to highlight another aspect of anhomomorphic logic that removes it from the more traditional milieu. Namely, it pays equal attention to both "poles" of $\mathbb{Z}_2$, both 1 (happening, truth) and 0 (not-happening, falsehood). Whether you know that an event has happened or that it has not (as with the

* I have not brought the quantifiers $\forall$ and $\exists$ into the discussion because they seem to be irrelevant. One will implicitly use them in formulating the questions in $\mathfrak{A}$, but not in inferring relations among the possible answers.

moon-sighting), you have learned something from which consequences can be drawn. Perhaps the reasons why falsehood has nevertheless tended to be ignored in favor of truth are first, that most logicians are mathematicians interested in deducing theorems from other propositions taken to be true; and second that they implicitly or explicitly adopt the rule that A is true if and only if $\neg A$ is false. In the context of physics and physical events, this rule is not as self-evident as it might appear to be in a mathematical context, because answering "no" to the question "is the particle here?" need not commit you to answering "yes" to the question "is the particle over there?". Precisely this distinction will feature prominently in the multiplicative scheme. There is also evidence that logicians in early Buddhist times took note of it. [19].

What then are the classical rules of inference expressed as conditions on ϕ? (I will assume in advance that ϕ is preclusive.) In deductive form they can, as shown by Fay Dowker [20], be condensed into three conditions (where the symbol $\Rightarrow$ indicates deducibility):

(1a) $\phi(A) = \phi(A \to B) = 1 \;\Rightarrow\; \phi(B) = 1$ ("modus ponens")

(1b) $\phi(A) = 0 \;\Rightarrow\; \phi(\neg A) = 1$

(1c) $\phi(0) = 0$

In algebraic form they are the condition:

(2) ϕ is a homomorphism of unital Boolean algebras ,

which says equivalently that ϕ preserves & (*and*) and $\neg$ (*not*). And expressed as a condition on the events $\phi^{-1}(1)$ *affirmed* by ϕ they say simply that

(3) $\phi^{-1}(1)$ is a maximal preclusive filter in Ω .

Here, following the usual definitions, a *filter* is a nonempty family Φ of events (elements of $\mathfrak{A}$, hence subsets of Ω) closed under intersection and passing to supersets. It is *preclusive* if it contains no precluded events (whence it cannot contain the empty subset 0), and it is *maximal* if it cannot be enlarged without ceasing to be preclusive. Poetically expressed, such a ϕ "maximizes being": it affirms as many events as it can, subject to fulfilling the other conditions.

In view of formulation (2), a classical coevent may be called "homomorphic", and a coevent that breaks any of the classical rules may be called *anhomomorphic*.

6. The multiplicative scheme: an example of anhomomorphic coevents

We have granted ourselves the freedom to change the rules ("of inference") governing coevents, but how to do so? Numerous avenues open up, but of the large number that have been explored by those of us working on the question (for some of them see [16]), only a handful have seemed promising, in the sense of permitting enough events to happen on one hand, but restricting the coevents sufficiently to reproduce the predictive apparatus of standard quantum theory on the other hand. The current favorite seems to be the *multiplicative scheme*, which not only is among the simplest to apply, but also represents perhaps the mildest change to the classical rules. The change is so mild, in fact, that it is non-existent if we express the classical rules in the form (3) of the previous section! The difference then springs solely from the different meaning of "preclusive", or rather (because its meaning as such has not changed) from the new patterns of preclusion (patterns of precluded events) that become possible under the influence of quantal interference.

More formally, let us make the following definitions. Recall that a coevent ϕ is *preclusive* when it honors the preclusion principle, that is when $\phi(A) = 0$ for every precluded event $A \in \mathfrak{A}$. Call such a ϕ *primitive* when it follows whatever further rules of inference we have set up. The collection of all primitive coevents, I will denote by $\widehat{\mathfrak{A}}$, since it is analogous in some ways to the spectrum of the event algebra $\mathfrak{A}$. The elements of $\widehat{\mathfrak{A}}$ are then (the descriptions of) the dynamically allowed "realities" or "possible worlds".

To arrive at the multiplicative scheme, we can retain condition (3) word for word as the definition of a *primitive preclusive multiplicative coevent*, or for short just "primitive coevent". Sorting out the definitions then shows that rules (1) and (2) do not survive intact. Of the first set, (1a) and (1c) survive but (1b) does not. Of condition (2), what survives is unitality and the preservation of the *and* operation (this being the origin of the name "multiplicative", since algebraically, & is multiplication). All of condition (3) survives, of course, but its meaning is probably easier to grasp when it is expressed in "dual" form.

To formulate it this way let us define first a map from sets F of histories to coevents $\phi = F^*$ by specifying that $F^*(A) = 1$ iff $A \subseteq F$. To put this in words, let's say that a coevent ϕ *affirms* an event A when $\phi(A) = 1$ and *denies* it when $\phi(A) = 0$. Our definition then says that F^* affirms precisely those events that contain F (as illustrated

in the diagram, where F^* affirms A but denies both B and C). When $\phi = F^*$, I will say that F is the *support* of ϕ. Now, in the case where Ω is a finite set (which we are always assuming herein), one can check that a multiplicative coevent necessarily takes the form F^* for some support $F \subseteq \Omega$. The condition (3) for primitivity then says precisely that F is as *small* as possible consistent with ϕ remaining preclusive. Given the definition of F^*, the condition for primitivity thus boils down to a rather simple criterion: the support should shrink down as much as possible without withdrawing into any precluded event.

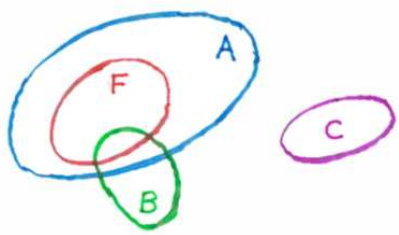

Figure 2. Three events and the coevent $\phi = F^*$

As remarked above, "truth" or "happening" is in this context a "collective property", since it pertains to events rather then to individual histories. A multiplicative coevent is also collective in nature, since it corresponds to a subset of Ω rather than an individual element.

A first test of any scheme of the present sort is that it should reproduce the classical notion of reality (namely reality as a single history) when the pattern of preclusions is itself classical. (We can take the latter to mean that an arbitrary event is precluded if and only if it is covered by precluded events. In particular, every subevent of a precluded event must be precluded.) In particular, this should happen when the quantal measure μ reduces to an ordinary measure, and also in the case of deterministic theories like classical mechanics where the dynamics reduces simply to the preclusion of an entire class of histories — those which fail to satisfy the equations of motion. It is not too difficult to verify that the multiplicative scheme passes the test in both cases. (See the theorems in Section 7.)

Resolution of the 3-slit paradox

It is a feature of the multiplicative scheme that any event E can find a primitive coevent to affirm it, as long as it is not included in some other event of zero measure. That is, there will be at least one $\phi \in \widehat{\mathfrak{A}}$ such that $\phi(E) = 1$. In our three-slit example, the events $A + C$ and $A + B + C = d$ are both of this type, so we can see already that the multiplicative scheme will avoid the false prediction that d can never occur.

To simplify things, let's imagine that there is nothing in existence but this particular experiment and let us further ignore all histories not in d and all fine structure of the histories that are in d. Then $\Omega = d$ consists of only three elements, identifiable with the three "atoms", A, B, and C, of the event algebra $\mathfrak{A}$. With such a small history space it is easy to enumerate all the possible (multiplicative) coevents, and one sees by inspection that only two are preclusive, namely $\phi = (A + C)^*$ and $\phi = (A + B + C)^*$. The latter, however, is not primitive, since the former has smaller support. There is thus a unique primitive coevent, $\phi = (A + C)^* = A^*C^*$. With respect to this coevent, two of the eight events in $\mathfrak{A}$ happen, namely $A + C$ and d itself, and the other six do not, namely A, B, C, 0, and of course $A + B$ and $B + C$ (0 being the empty subset of Ω). In particular $\phi(d) = 1$, so the paradox is removed.

This is satisfactory as far as it goes, but in one respect this 3-slit example is misleadingly simple. As the cardinality N of the history space grows, it becomes increasingly difficult in practice to work out the primitive coevents (in the multiplicative scheme there are 2^N potential supports to consider), but when the dynamics is simple enough it is possible to count them or at least to estimate their number. Typically one finds that this number also grows rapidly with N, just as one might have expected. The fact that ϕ is unique for the 3-slit setup is thus very much of an exception.

There is also another respect in which our example has been overly idealized. We have cut the experiment off at the point where the particle reaches (or does not reach, as the case may be) the location d, thereby ignoring, not only the future, but also whatever else is going on in the world besides this experiment. Both of these omissions could have serious repercussions which I'll return to briefly in the concluding section.

7. Preclusive separability and the "measurement problem"

Within the framework we have arrived at, individual histories are replaced in a certain sense by sets of histories while "laws of motion" are expressed via preclusion and the requirement of primitivity. In this way dynamics merges with logic to some extent, and we are able to speak directly about microscopic processes without succumbing to paradoxes of the Kochen Specker sort — at least in simple examples. Because of its "realistic" nature, I hope that this framework will prove useful in connection with quantum gravity, specifically in the quest for a causal set dynamics of quantal sequential growth. But a more immediate challenge is posed by the so called "measurement problem". If the multiplicative scheme cannot solve this problem, it will be hard to take it seriously as a potential basis for unifying quantum field theory with general relativity.

Of course there is no single, well posed "measurement problem". Rather this phrase refers to a complex of issues concerning the relationship of quantal processes to the macroscopic realm of classical events and "observers". Nevertheless, I think one would not be oversimplifying unduly to pose the problem as that of accounting for measurements without resorting to the notion of external agents who are not explicable in microscopic terms. In the context of the multiplicative scheme (or any of the other schemes based on anhomomorphic coevents) this problem acquires a precise formulation. One must show that the primitive coevents become classical (i.e. homomorphic) when they are restricted to a suitable subalgebra $\mathfrak{A}^{macro}$ of "instrument events".

To appreciate that this is what is needed, recall why there is a problem in the first place. Quantum mechanics as ordinarily presented either declines to describe the measurement process or it gives a manifestly false description, depending on whether or not one assumes that the state-vector "collapses" during the measurement. In the former case one is positing a phenomenon that the theory leaves in the dark, in the latter case the theory serves up a superposition of macroscopically distinct outcomes that contradicts our most elementary experiences. Now let us return to the coevent framework, where measurements are no different in principle from other quantal processes (and like other processes are to be described in terms of histories rather than evolving state-vectors.) In a measurement-like situation, the theory will yield a definite set of primitive coevents to describe the different possible outcomes. For example let events A and B be two alternative "pointer readings" in some experiment. Each of these events will correspond to a particular collection of

configurations of the "pointer molecule worldlines", and will be macroscopic in the sense that the corresponding histories will involve large numbers of particles, relatively great masses, etc. If a given coevent $\phi \in \widehat{\mathfrak{A}}$ affirms A and denies B then A is the outcome in the world described by ϕ, in the contrary case it is B. (Both types of coevent can be viable in general, since the theory is not deterministic.) [†] However, one can also construct "Schrödinger cat"-like coevents which deny both A and B, as in the 3-slit example above. Such a coevent would not be in accord with experience, which always (or almost always?) presents us with a unique outcome that does happen. Consistency with experience thus requires that no (or almost no) coevent $\phi \in \widehat{\mathfrak{A}}$ be of this ambiguous type, and this in turn is equivalent to $\phi|\mathfrak{A}^{macro}$ (the restriction of ϕ to $\mathfrak{A}^{macro}$) being classical, since when classical logic reigns, precisely one history occurs.

Formally, we can define a subalgebra $\mathfrak{A}^{macro} \subseteq \mathfrak{A}$ of macroscopic events [♭] such that disjoint elements A and B of $\mathfrak{A}^{macro}$ correspond to macroscopically distinct events in $\mathfrak{A}$. Such a subalgebra induces a partition of Ω whose equivalence classes (sets of histories distinguished by no element of $\mathfrak{A}^{macro}$) define a quotient or "coarse-graining" Ω^{macro} of Ω into "macroscopic histories". Our condition that ϕ map $\mathfrak{A}^{macro}$ homomorphically into $\mathbb{Z}_2$ is then trying to say that ϕ is supported within a single coarse-grained history (the translation being literally correct when $|\Omega| < \infty$). Put differently, the support F of such a ϕ must not overlap macroscopically distinct events. When this condition is satisfied F^* will look to $\mathfrak{A}^{macro}$ like a single coarse-grained history and will be classical in that sense.

Given the measurement problem rendered in this manner, we can solve it if we can find a sufficient condition for ϕ to behave classically and if in addition we can give reasons why the events of our macroscopic experience (almost) always fulfill the condition. To illustrate how this can work, I will quote two theorems that furnish sufficient conditions of the type we need. It should be clear from the preceding discussion that when either

[†] Having written this however, I should add that in simple examples, the theory turns out to be much closer to deterministic than one might have expected. (See the next section.)

[♭] For the events comprising $\mathfrak{A}^{macro}$ to be well defined, we might have first to condition on the happening of certain other events that are prerequisite to the existence of macroscopic objects, i.e. to the existence of what is sometimes called a "quasiclassical realm".

theorem applies, primitive coevents will behave classically as far as instrument events are concerned. One can also see the same thing by translating the conclusion of the theorems into the statement that rule (1b) above is respected. Since the multiplicative scheme validates the rest of rule (1) by construction, (1b) suffices to return us to the classical case.

The first theorem below furnishes a sufficient condition that is easy to state but more restrictive than it needs to be. The condition in the second theorem is less transparent in statement, but arguably more likely to hold in practice. (A proof of the first theorem in the finite case can be found in [16]. Proofs of the second theorem and the infinite case of the first theorem exist as well, but remain unpublished.)

THEOREM 1. Let $\Omega = \Omega' + \Omega''$ be a partition of Ω such that an arbitrary event $A \subseteq \Omega$ is precluded iff its intersections with Ω' and Ω'' are both precluded, and let ϕ be any primitive preclusive coevent in the multiplicative scheme. Then ϕ is supported within either Ω' or Ω''. That is, $\phi = F^*$ with $F \subseteq \Omega'$ or $F \subseteq \Omega''$.

THEOREM 2. The conclusion of theorem 1 persists if both Ω' and Ω'' satisfy the following weaker condition on subsets S of Ω: If any event $A \subseteq S$ lies within any precluded event B at all then it lies within a precluded event $C \subseteq S$.

As we have seen, the important consequence of these theorems is that, when either of the respective conditions is satisfied, and with respect to any primitive coevent ϕ, either Ω' or Ω'' happens but not both. The important question then becomes whether macroscopic events are in fact "preclusively separable" in this way. This would follow immediately from the still stronger condition that: No event in Ω' interferes with any event in Ω''. However this condition represents a very strict type of "decoherence" closely related to idea of a *record.*[⋆] To the extent that one is willing to posit the existence of sufficiently permanent records of macroscopic events, one can therefore regard the measurement problem as solved. To the extent that one finds this assumption implausibly strong one can still hope to prove

[⋆] In the context of unitary quantum mechanics, the condition is satisfied for records because different versions of a given record correspond to disjoint regions in configuration space. The weaker type of decoherence usually contemplated by "decoherent historians" requires only that Ω' and Ω'' (belong to $\mathfrak{A}$ and) decohere, not that their arbitrary subevents decohere as well.

that macroscopic events fulfill the conditions of one of the theorems. In this sense, the measurement problem reduces to a calculation.

8. Open questions and further work

Before the framework presented above can be considered complete, further work will be needed on some of the questions raised by the above discussion. Foremost among these is probably the question whether one can demonstrate by examples or general arguments that the events of our macroscopic experience really are preclusively separable in the sense of the above theorems. Assuming that they are, can one explain on this basis why the textbook paradigm involving the wave function and its "collapse" works as well as it does, and if so can one quantify the deviations that one should expect from this paradigm? Here much of the way forward is clear. There exists a sketch of a derivation of the collapse rule, but it needs to be followed out in more detail. In the same direction, we of course need to recover Born's rule for probabilities, either by appeal to approximate preclusion and the Cournot principle (cf. [11]) or in some better manner. And finally, the definition of primitive coevent needs to be extended to infinite event algebras, since the most important examples of quantal dynamics (atomic physics, quantum field theory, etc.) are of this type, at least in current idealizations. Here again, there is much that could be said about work already done.

Even in its partly finished state, the coevent framework, like the Bohmian version of quantum mechanics, lets us pose questions that we would not have been able to formulate from a "Copenhagen" standpoint. Thus for example, one can ask for the primitive coevents that describe the ground state of a Hydrogen atom, or of a particle in a harmonic oscillator potential. The Bohmian particle in these cases just sits still wherever it happens to find itself. One wouldn't know by following its motion what kind of force was binding it, nor could one even know its energy in many cases. It's therefore of particular interest to ask of the multiplicative scheme what sets of trajectories comprise the supports of the primitive coevents in these cases. Could one deduce from these sets of trajectories what the potential was and would the energy show up clearly? Currently such questions seem nearly beyond reach, in the first place because of the mathematical difficulties in defining the continuum path integral itself on a sufficiently large domain of events. (cf. [21])

More accessible, though no less interesting, are questions about experiments of the Kochen-Specker type, or about entangled pairs of particles passing through successive Stern-Gerlach analyzers. For a few gedankenexperiments of this type people have been able to find some or all of the primitive coevents, and in some cases to study causal relationships between the coevents at earlier and later stages of the process [22] [23]. Such examples can serve as laboratories to explore possible meanings for relativistic causality, locality, and determinism within the coevent framework. For example, in a simple extension of the Hardy experiment, one finds 286 coevents ϕ of which 280 behave deterministically in the sense that the restriction of ϕ to the subalgebra of past events uniquely determines ϕ globally (cp. a similar effect found in [24]). One can also formulate different conditions of relativistic causality ("screening off") for such systems and study to what extent, and in what circumstances they hold. If a suitable condition could be found, it could then be carried over to the causal set situation and used as a guide to formulating a quantal analog of the classical sequential growth models (cf. [14]).

If one regards the coevent schemes in logical terms, it's natural to try to bring them into relation with other non-classical logics to which anhomomorphic inference seems to bear some resemblance, such as intuitionistic, dialectical or paraconsistent logic. With dialectics, anhomomorphic logic shares a certain tolerance of contradiction, or of what classically would be regarded as self contradictory. With intuitionism it shares a non-classical understanding of negation, not however at the level of the event algebra, which remains strictly Boolean so that $\neg\neg A = A$ for all $A \in \mathfrak{A}$, but at the level of inference, where $\phi(A)$ becomes independent of $\phi(\neg A)$. On the other hand, whereas intuitionistic logic simply drops certain rules of inference like proof by contradiction, anhomomorphic logic adds crucially the new requirement of *primitivity*. Thus it cannot be characterized simply as either weaker or stronger than classical logic.

Returning to the coevent framework per se, I'd like to allude briefly some of its more radical consequences, and the risks (or possibly opportunities) they hold out for this way of conceiving quantal reality. Each of these consequences is visible in simple examples. In the Hardy example referred to just above, one potentially encounters what could be called "premonitions". To be confident of their occurrence one would have to incorporate "instrument setting events" into the model, and this has not been done. Yet it looks as if the past of the coevent can determine not only events involving the particles in question, but

also to some extent the settings themselves. Such an effect could be called a premonition on the part of the particles, but it could also be called a cause of the later setting-event, in which case there would be no suggestion of "retro-causality". In the simple example of a particle hopping unitarily between the nodes of a two-site lattice ("two site hopper"), one encounters a potential danger that also shows up in much the same form in connection with composite systems. In both cases it can happen that the restriction of the coevent to the subalgebra of early-time events (respectively events in one of the two subsystems) trivializes in the sense that its support becomes the whole space of partial histories. This means that only correlations between early and late times (resp. between one subsystem and the other) happen nontrivially. None of this is a problem unless carried to extremes. If for example, the relevant time scale for this type of trivialization in realistic systems were to be comparable with the Poincaré recurrence time, then there would be little to worry about.

Finally, let me conclude with a possibility that for now is merely a dream, but which, if it came to pass, would bring with it a striking historical irony. One might discover laws that governed the pattern of preclusions without referring, directly or indirectly, to the quantal measure μ. If that happened, it would provide a more radical revision of classical dynamics (stochastic or deterministic) than that represented by the path integral. Or in the process of working out the primitive coevents in various examples, one might even discover laws expressed directly for the coevents themselves, without even needing to derive them from preclusions. If that happened, the whole superstructure of amplitudes and generalized measures would fall away, and quantum theory would have led back to something resembling classical equations of motion, but at a higher "structural" level than occupied by our old theories that identified reality with a single history.

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Appendix. Formal deduction of the three-slit contradiction

We are given the preclusions $\phi((a+b)d) = \phi((b+c)d) = 0$ and wish to deduce that $\phi(d) = 0$, assuming that ϕ follows the classical rules of inference.

step 0: Suppose that $\phi(d) = 1$.

step 1: if also $\phi(a+b) = 1$ then $\phi((a+b)d) = 1$ contrary to what was given, hence $\phi(a+b) = 0$.

step 2: $\phi(b+c) = 0$ by symmetry.

step 3: $\phi(a+b) = 0$ implies $\phi(c) = 1$ since $c = \neg(a+b)$, the complement of $a+b$.

step 4: From $\phi(c) = 1$ follows $\phi(b+c) = 1$, contradicting step 2.

step 5: Therefore our supposition was false and $\phi(d) = 0$.

What conditions on ϕ did we use?

In step 1: If $\phi(A) = \phi(B) = 1$ then $\phi(AB) = 1$; if $\phi(A) \neq 1$ then $\phi(A) = 0$.

In step 3: If $\phi(A) = 0$ then $\phi(\neg A) = 1$

In step 4: If $A \subseteq B$ and $\phi(A) = 1$ then $\phi(B) = 1$

In the multiplicative scheme, only step 3 would fail. Notice that in reasoning *about* ϕ we have also employed classical logic, in particular proof by contradiction in steps 1 and 5.

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